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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

COMPARATIVE STUDY OF EXACTLY SOLVABLE OSCILLATOR -TYPE CONFINED MODELS OF QUANTUM SYSTEMS WITH POSITION-DEPENDENT MASS

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COMMON CHARACTERISTIC OF THE DISSERTATION

Relevance of the topic and degree of development. In recent years, the combined development of solid-state physics and other related scientific fields has led to discoveries that can be considered a revolution in the submicron and nanoscale, the dimensions of which are smaller than a micron. These discoveries, in turn, have been successfully applied in industry and technological processes, giving impetus to the development of modern nanotechnology. As is known, classical laws of physics lose their fundamentality at the nanoscale, and observed physical phenomena cannot be explained using these laws. In this case, physical systems behave like quantum systems, and their explanation becomes possible within the framework of the fundamental laws of quantum physics. In other words, classical systems, determined at the macroscopic level, give way to quantum systems, the dynamics and evolution of which are based on probability theory at the nanoscale and angstrom level.

In addition, many scientific and research works are being conducted in the direction of studying the influence of the gravitational field on various quantum systems. The increased interest in this kind of research is due, first of all, to the important role that the gravitational field can reveal in quantum measurements, many "hidden" properties. For example, the discovery of shifts in the energy levels of the hydrogen atom as a result of studying the influence of the gravitational field on a quantum system aroused great interest. The interaction of the gravitational field with matter and the possible quantization of this field are also important objects of research. Various approaches and theories are used to quantize the gravitational field and construct its quantum theory. For example, the perturbative approach based on small excitations in the classical background, the ring quantization approach based on the rearrangement of dynamic variables, string theory based on the excitation of natural components of the gravitational field by new methods, and other similar theories are aimed at explaining gravitational effects using quantum theory.

One of the widely used formalisms in modern scientific research is the position-dependent band structure and its generalization to the case of effective mass. However, some difficulties may arise when using this formalism. The main reason for the difficulty is that, although many problems in quantum physics can be solved mathematically exactly using the reduced mass approach introduced in the two-body problems, replacing the fixed mass or band structure with variable analogues makes the problem mathematically difficult and it is very likely that the exact solution obtained for a homogeneous band structure or mass will turn into an approximate solution in the case of a mass that varies with position. However, the inclusion of this formalism led to the explanation of the phenomenon of the tunnel effect in superconductors, considered one of the ingenious discoveries in physics and experimentally discovered by I. Giaever in 1960 and awarded the Nobel Prize in Physics in 1973. It also made possible the exact solution of many problems in quantum physics.

In the dissertation work, precisely solvable oscillator-type confined models of quantum systems with position-dependent mass are comparatively studied. When at least one of the geometric dimensions of the crystal is in the de Broglie wavelength configuration of the electron, its thermodynamic, kinetic, and optical properties undergo unique changes. In other words, if the movement of charge carriers is limited in one, two or three directions, their energy is quantized. To ensure energy quantization, i.e. generation of a discrete energy spectrum, the infinite crystal is limited by barriers, i.e. boundaries are created, and exactly solvable models are constructed that obey the effect of quantum confinement in the form of a quantum well, which behaves as a limited quantum system with a non-rectangular profile of a harmonic oscillator.

In short, structures in solid state physics are quantum systems of very small dimensions, subject to confinement effects. It is clear that when we carry out theoretical calculations for these systems, we must take into account that the problem of the wave function or the joint distribution function of momentum and position, which will characterize the system, must be solved in finite values, not in infinite values in the region of infinite coordinate values.

Object and subject of the study. *The object* of the study is exactly solvable models of oscillation-type confined quantum systems with position-dependent mass. *The subject* of the study is the search, comparison and study of visible images of the nonlinear energy spectrum and wave functions of stationary states by exactly solving the Schrödinger equation expressed by various kinetic energy operators within the framework of the position-dependent mass formalism for harmonic oscillators-type infinite and finite deep quantum wells models.

Research goals and objectives. The main goal of the dissertation work is to comprehensively study exactly solvable oscillator-type confinement models of quantum systems with position-dependent mass.

To achieve this goal, the following tasks have been performed.

-Finding exact solutions of the Schrödinger equation expressed by the Zhu-Kroemer kinetic energy operator within the position-dependent mass formalism for harmonic oscillator-type infinite and finite deep quantum well models;

-Determination of the exact solutions of the Schrödinger equation expressed by the Gora-Williams kinetic energy operator within the position-dependent mass formalism for a harmonic oscillator-type infinite deep quantum well both under the influence of an external gravitational field and in the absence of such influence;

-Finding exact solutions of the Schrödinger equation expressed by the kinetic energy operator compatible with Galilean invariance within the position-dependent mass formalism for a harmonic oscillator-type infinite deep quantum well both under the influence of an external gravitational field and in the absence of such influence;

-Determination of exact limit relations restored by the disappearance of confinement between exact solutions of harmonic oscillatory quantum well models of the Schrödinger equation expressed by Zhu-Kroemer, Gora-Williams and Galilean invariance kinetic energy operators within the position-dependent mass formalism and the known non-relativistic canonical quantum harmonic oscillator;

-Finding and proving a new limit relation that shows the existence of a direct connection between pseudo-Jacobi and

Hermitian polynomials.

Research methods. Within the scope of the objectives and tasks set in the dissertation work, researches were carried out with the help of various methods and approaches of theoretical physics, theory of special functions, differential equations, orthogonal polynomials and mathematical analysis course.

The basic provisions for defense:

1. Exact solutions of the Schrödinger equation expressed by the Zhu-Kroemer kinetic energy operator within the position-dependent mass formalism for harmonic oscillator-type infinite and finite deep quantum well models;

2. Exact solutions of the Schrödinger equation expressed by the Gora-Williams kinetic energy operator within the position-dependent mass formalism for a harmonic oscillator-type infinitely deep quantum well both under the influence of an external gravitational field and in the absence of such influence;

3. Exact solutions of the Schrödinger equation expressed by the kinetic energy operator compatible with Galilean invariance within the position-dependent mass formalism for a harmonic oscillator-type infinitely deep quantum well both under the influence of an external gravitational field and in the absence of such influence;

4. Exact limit relations restored by the disappearance of confinement between exact solutions of harmonic oscillatory quantum well models of the Schrödinger equation expressed by Zhu-Kroemer, Gora-Williams and Galilean invariance kinetic energy operators within the position-dependent mass formalism and the known non-relativistic canonical quantum harmonic oscillator;

5. A new limit relation showing the existence of a direct connection between pseudo-Jacobi and Hermitian polynomials.

Scientific novelties of the dissertation:

-Within the position-dependent mass formalism, the Schrödinger equation expressed by the Zhu-Kroemer kinetic energy operator is exactly solved for harmonic oscillator-type infinite and finite deep quantum well models, and the nonlinear energy spectrum and the wave of stationary states expressed by Gegenbauer, Jacobi and pseudo-Jacobi polynomials obvious images of the function have been found[3];

-It was observed that the reduced frequency of the nonlinear energy spectrum, which was found by exactly solving Schrödinger equation expressed by the Zhu-Kroemer kinetic energy operator for the model of harmonic oscillator-type infinite deep quantum wells, depends on the confinement parameter and increases sharply when this parameter approaches zero [8], [9];

-The Schrödinger equation, expressed by the kinetic energy operator of Gore-Williams and the kinetic energy compatible with Galilean invariance within the position-dependent mass formalism, has been exactly solved for a harmonic oscillator-type infinite deep quantum well both under the influence of an external gravitational field and in the absence of such influence and exact expressions of both the nonlinear energy spectrum and the wave functions of stationary states expressed by Gegenbauer and Jacobi polynomials have been found [1], [7];

-From the comparison of the energy spectra of the Schrödinger equations expressed with the Gore-Williams kinetic energy operator and the Galilean invariance kinetic energy operator within the position-dependent mass formalism, it was observed that the energy spectrum of the model expressed with the Galilean invariance kinetic energy operator always takes larger values [2], [5];

-Within the position-dependent mass formalism, the exact solutions of the Schrödinger equation for the harmonic oscillator-type quantum well models, expressed by Zhu-Kroemer, Gora-Williams kinetic energy operators and the kinetic energy operator compatible with Galilean invariance has been exactly solved and restored by the disappearance of the confinement between the exact solutions of the known non-relativistic canonical quantum harmonic oscillator limit relations are calculated exactly [4];

-A new limit relation which shows the existence of a direct relationship between pseudo-Jacobi and Hermitian polynomials, was found, and the correctness of this relation was proved mathematically [6], [10].

Theoretical and practical significance of the research.

One-dimensional quantum harmonic oscillator models whose wave functions is zero at a finite value of the position, that is, with the confinement effect, if they are exactly solved, are very important for the development of modern nano-scale physics and technologies in various directions. In the current dissertation work, the nonlinear energy spectra and wave functions of stationary states found by exactly solving for infinite and finite deep quantum wells whose profile is in the form of a harmonic oscillator are of great practical importance in explaining the various characteristics of modern nano-scale structures with a non-rectangular profile obtained experimentally. On the other hand, the exact solutions of the mentioned confinement effect problems found in the external gravitational field may play a very important role in the estimation of nonlinear optical parameters of many heterostructures with confinement effect in the future. The direct limit relationship found between pseudo-Jacobi and Hermitian polynomials and proved to be correct by us while solving the problems of quantum mechanics is of great interest from the point of view of mathematics for updating and expanding the known characteristics for special functions and orthogonal polynomials.

Approbation and application.

The main provisions of the dissertation work and the obtained scientific results were widely covered and discussed at the following scientific meetings, seminars and conferences:

• 7th International Scientific and Practical Conference "Science and Practice: Introduction to Modern Society" (Manchester, UK, 6-8.10.2020)

• 13th International Scientific and Practical Conference "Science and Practice: Implementation in Modern Society" (October 16-18, 2022, Manchester, UK)

• II Republican Scientific Conference "Fundamental Problems of Mathematics and the Application of Intelligent Technologies in Education." (Sumgait State University, December 15-16, 2022, Sumgait, Azerbaijan)

The name of the organization in which the dissertation work is completed:

The dissertation work was completed in the "Quantum

Informatics" laboratory of the Institute of Physics of the Ministry of Science and Education of the Republic of Azerbaijan.

Volume, structure and main content of the dissertation.

The dissertation consists of an introduction, 4 chapters, a conclusion and a list of references, occupies 151 pages, written in A4 format. The main part of the work (without figures, tables, graphs and bibliography) 234,493 (including Introduction - 1467, Chapter I - 42,000, Chapter II - 90,000, Chapter III - 64,000, Chapter IV - 32,000, Conclusions - 5026) - sign. The list of references includes 106 named sources cited in the dissertation. The dissertation contains 6 photographs reflecting the results obtained.

THE CONTENT OF THE WORK

The introduction indicates the relevance of the topic and the degree of development, the goals and objectives of the research, research methods, the main defended provisions, scientific novelty, theoretical and practical significance of the research, a review of the work, and provides extensive information about research methods.

The first chapter of the dissertation is mainly of an overview nature and talks about what approaches exist in quantum mechanics for quantizing between momentum and coordinate, and then, based on these approaches, how to build models of the problem nonrelativistic harmonic oscillator with exact solutions, discusses the formalism of theposition-dependent band structure in various types of solids and its generalization to the case of effective mass, as well as the limitation effect in nonrelativistic quantum mechanics.

The first paragraph briefly explains the differences between canonical and non-canonical approaches in non-relativistic quantum mechanics. The next paragraph explains how exactly solvable models of the linear harmonic oscillator problem are constructed in canonical and non-canonical approaches of nonrelativistic quantum mechanics. The third paragraph of this chapter reviews some well-known experiments that can be explained by the position-dependent variable zone structure formalism in various types of solids, and explains under what conditions the variable zone structure formalism generalizes to the case of effective mass. The last paragraph explains how the confinement effect occurs in non-relativistic quantum mechanics and how important this effect is in solid state physics and nanotechnology, and then presents some calculations of how the measurements manifest themselves quantitatively.

The second chapter of the dissertation is devoted to the construction of an exact solvable model of the Zhu-Kroemer confinement of a nonrelativistic linear harmonic oscillator in the canonical approach. Considering that the presented model has a confinement effect, i.e. limited, then first of all, ways to achieve the confinement effect are shown. To this end, the first paragraph explains some of the mathematical details of the kinetic energy generalization for the position-dependent effective mass, and then, given the kinetic energy generalization, the position-dependent effective mass is also considered in the quantum harmonic oscillator potential, and in the second point, both infinitely high and finite for a high-potential well, this problem is solved exactly in terms of Gegenbauer and pseudo-Jacobian polynomials [3].

The effective mass M(x) varies depending on the coordinate and was proposed by Zhu-Kroemer[1].

$$\hat{H}_0^{JK} = -\frac{\hbar^2}{2}\frac{1}{\sqrt{M(x)}}\frac{d^2}{dx^2}\frac{1}{\sqrt{M(x)}} \tag{1}$$

Using the kinetic energy operator, the Hamilton operator for a nonrelativistic linear harmonic oscillator is written as:

$$\hat{H}^{JK} = -\frac{\hbar^2}{2}\frac{1}{\sqrt{M(x)}}\frac{d^2}{dx^2}\frac{1}{\sqrt{M(x)}} + \frac{M(x)\omega^2x^2}{2}. \tag{2}$$

Here $V(x)$ is the potential of the nonrelativistic linear harmonic oscillator under study, defined as:

$$V(x) = \begin{cases} \frac{M(x)\omega^2x^2}{2}, & -a < x < a \\ \infty, & x = \pm a. \end{cases} \tag{3}$$

[1] Lima, R. F., Vieira, M., Furtado, C. Yet another position dependent mass quantum model // Journal of Mathematical Physics, -2012. 53, - p. 072101

From this statement it is clear that a quantum system behaving as a linear harmonic oscillator is completely limited by two infinitely high fences at coordinate values x=±a, and this limitation should be slightly reflected in the analytical expression effective mass M(x) depending on the coordinate . Therefore, in order to determine the type of manifestation of the function M≡M(x), we first impose the following conditions on it:

●The effective mass M(x), depending on the coordinate, must coincide with the constant mass m_0 in the case of x=0;

●The effective mass M(x), depending on the coordinate, should restore the constant mass m_0 in the limiting case a→∞;

● In connection with the determination of the effective mass depending on the coordinate M(x), the confinement effect should be observed at values of the coordinate x=±a.

● The Schrödinger equation with the Zhu-Kroemer kinetic energy operator must be solved exactly, and its solution must coincide with the solution of the non-relativistic model of an infinite harmonic oscillator in the limiting case a→∞.

Given the above conditions, we can write the following expression for the coordinate-dependent effective mass M(x) satisfying these conditions:

$$M \equiv M(x) = \frac{a^2 m_0}{a^2 - x^2}. \tag{4}$$

After taking into account expressions (3) and (4) in expression (2) and making the necessary simplifications, the Hamilton operator will take the following form:

$$\widehat{H}^{JK} = -\frac{\hbar^2}{2M}\left[\frac{d^2}{dx^2} - \frac{2x}{a^2 - x^2}\frac{d}{dx} - \frac{1}{a^2 - x^2} - \frac{x^2}{(a^2 - x^2)^2}\right] + \frac{m_0 \omega^2 a^2 x^2}{2(a^2 - x^2)}. \tag{5}$$

Given this statement, we can write the Schrödinger equation, which is a second order differential equation, as follows:

$$\left[\frac{d^2}{dx^2} - \frac{2x}{a^2 - x^2}\frac{d}{dx} - \frac{1}{a^2 - x^2} - \frac{x^2}{(a^2 - x^2)^2}\right]\psi$$

$$+\left(\frac{2ma^2E}{\hbar^2(a^2-x^2)}-\frac{m\omega^2a^2x^2}{\hbar^2(a^2-x^2)^2}\right)\psi=0. \quad (6)$$

To exactly solve the equation, we will use the Nikiforov-Uvarov method, which is used when solving second-order differential equations. Finally, the following nonequidistant energy spectrum for the Zhu-Kroemer limitation model of a nonrelativistic linear harmonic oscillator:

$$E\equiv E_n^{JK}=\hbar\sqrt{\omega^2+\frac{\hbar^2}{m^2a^4}}\left(n+\frac{1}{2}\right)+\frac{\hbar^2}{2ma^2}(n^2+n+1) \quad (7)$$

and we obtain the following expression for the wave functions of stationary states [3]:

$$\psi\equiv \psi_n^{JK}(x)=C_n^{JK}\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}\sqrt{\frac{m^2\omega^2a^4}{\hbar^2}+1}}C_n^{\left(\sqrt{\frac{m^2\omega^2a^4}{\hbar^2}+1}+\frac{1}{2}\right)}\left(\frac{x}{a}\right), \quad (8)$$

where the hypergeometry $C_n^{(\bar{\lambda})}(x)$ is Gegenbauer polynomials, expressed by the functions ${}_2F_1$ as follows:

$$C_n^{(\bar{\lambda})}(x)=\frac{(2\bar{\lambda})_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,n+2\bar{\lambda}\\ \bar{\lambda}+\frac{1}{2}\end{matrix};\frac{1-x}{2}\right),\bar{\lambda}\neq 0 \quad (9)$$

the normalization coefficient c_n^{JK} is defined as:

$$c_n^{JK}=2^{\sqrt{\frac{m^2\omega^2a^4}{\hbar^2}+1}}\Gamma\left(\sqrt{\frac{m^2\omega^2a^4}{\hbar^2}+1}+\frac{1}{2}\right)\sqrt{\frac{\left(n+\sqrt{\frac{m^2\omega^2a^4}{\hbar^2}+1}+\frac{1}{2}\right)n!}{\pi a\Gamma\left(n+2\sqrt{\frac{m^2\omega^2a^4}{\hbar^2}+1}+1\right)}}, \quad (10)$$

and is found from the condition of orthogonality of the Gegenbauer polynomials[2] $C_n^{(\bar{\lambda})}(x)$, satisfied for $\bar{\lambda}>-\frac{1}{2}$ and $\bar{\lambda}\neq 0$.

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[2] Koekoek, R. Hypergeometric Orthogonal Polynomials and Their q-Analogues/ R.

$$\int_{-1}^{1}(1-x^2)^{\bar{\lambda}-\frac{1}{2}}C_m^{(\bar{\lambda})}(x)C_n^{(\bar{\lambda})}(x)dx=\frac{\pi\Gamma(n+2\bar{\lambda})2^{1-2\bar{\lambda}}}{\{\Gamma(\bar{\lambda})\}^2(n+\bar{\lambda})n!}\delta_{mn}. \quad (11)$$

Therefore, in the coordinate representation, the wave functions of stationary states (8) are also orthogonal in the interval $-a<x<a$:

$$\int_{-a}^{a}[\psi_m(x)]^*\,\psi_n^{JK}(x)dx=\delta_{mn.} \quad (12)$$

Now let us partially change the requirement for the potential $V(x)$ of the quantum system under study as follows. Consider that our potential is the potential of a harmonic oscillator, the mass of which varies depending on the coordinate:

$$V(x)=\frac{M(x)\omega^2x^2}{2}, \quad (13)$$

however, in the case of $x\to\pm\infty$ the potential behaves as $\frac{m_0\omega^2a^2}{2}$. That is, in fact, the dependence of the mass on the position should be introduced in such a way that this time, instead of infinitely high walls, two finite barriers arise and our potential becomes a finite quantum hole with a non-rectangular profile. The fact that the potential behaves differently than before requires the definition of new conditions for the mass $M(x)$, different from the previous ones:

-The position-dependent effective mass $M(x)$ must be equal to the constant effective mass m_0 at the origin $x=0$;

-The position-dependent effective mass $M(x)$ should become zero at coordinate values $x=\pm\infty$.

-The Schrödinger equation corresponding to the free Hamiton operator and potential (13) must be solved exactly to analytically express the position-dependent effective mass $M(x)$.

Let us write the analytical expression for the position-dependent effective mass $M(x)$, satisfying the above conditions, as follows:

$$M\equiv M(x)=\frac{a^2m_0}{a^2+x^2}. \quad (14)$$

Koekoek, P.A Lesky, R.F Swarttouw // - Springer-Verlag Berlin Heidelberg, - 2010, 578 p

Let us find obvious expressions for the wave functions and energy spectrum of stationary states by exactly solving Schrödinger equation corresponding to the free Hamilton operator(1), the potential(3) and the analytical expression for the position-dependent effective mass $M(x)$ (14). If we write the corresponding Schrödinger function for this,

$$-\frac{\hbar^2}{2M}\left[\frac{d^2}{dx^2}+\frac{2x}{a^2+x^2}\frac{d}{dx}+\frac{1}{a^2+x^2}-\frac{x^2}{(a^2+x^2)^2}\right]\psi^{JK}(x)+\frac{a^2m\omega_0^2x^2}{2(a^2+x^2)}\psi^{JK}(x)= E^{JK}\psi^{JK}(x) \qquad (15)$$

By solving this equation by the Nikiforov-Uvarov method[3], the resulting second-order differential equation can be exactly solved by direct comparison with the following well-known second-order differential equation for pseudo-Jacobi polynomials $P_n(\xi,v,N)$:

$$(1+\xi^2)\bar{y}''-2(v-N\xi)\bar{y}'+n(2N-n+1)\bar{y}=0, \qquad (16)$$

$$\bar{y}=P_n(\xi,v,N). \qquad (16.a)$$

From this comparison we find that the energy spectrum of E^{JK} is nonequidistant and finite:

$$E^{JK}=\hbar\omega_0\frac{(N+1)\left(n+\frac{1}{2}\right)-\left(\frac{n+1}{2}\right)-\frac{1}{2}}{\sqrt{(N+1)^2-1}},$$

$$n=0,1,2,3,\dots,N. \qquad (17)$$

And $\psi^{JK}(x)$, which are wave functions of stationary states, are expressed through pseudo-Jacobi polynomials in the following order:

$$\tilde{\psi}^{JK}(x)=c_n^{JK}\left(1+\frac{\lambda_0{}^2x^2}{\sqrt{(N+1)^2-1}}\right)^{-\frac{N+1}{2}}P_n\left(\frac{\lambda_0x}{[(N+1)^2-1]^{\frac{1}{4}}},0,N\right).$$

Here $P_n(\xi,v,N)$ are pseudo-Jacobi polynomials defined by

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[3] Nikiforov, A.F. Special Functions of Mathematical Physics: A Unified Introduction with Applications / A.F. Nikiforov, V.B. Uvarov, - Springer Basel AG, - 1988. 427 p.

the hypergeometric function $_2F_1$ as follows:

$$P_n(x,v,N) = \frac{(-2i)^n(-N+iv)_n}{(n-2N-1)_n}\,{}_2F_1\left(\begin{matrix}-n, n-2N\\ -N+iv\end{matrix}; \frac{1-ix}{2}\right), \quad (18)$$
$$n = 0, 1, 2, \dots, N.$$

$\tilde{\psi}^{JK}(x)$ is orthonormalized, and the normalization parameter c_n^{JK} is found from the following orthogonality condition for pseudo-Jacobian polynomials:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}(1+x^2)^{-N-1}\,e^{2v\cdot arctanx}P_m(x,v,N)P_n(x,v,N)dx = \frac{\Gamma(2N+1-2n)\Gamma(2N+2-2n)2^{2n-2N-1}n!}{\Gamma(2N+2-n)|\Gamma(N+1-n+iv)|^2}\delta_{mn}. \quad (19)$$

Its obvious picture is this:

$$c_n^{JK} = \frac{\Gamma(N-n+1)}{2^{n-N}\Gamma(2N-2n+1)}\sqrt{\frac{\Gamma(2N-n+1)}{\pi a_N(2N-2n+1)n!}}. \quad (20)$$

The third paragraph of this chapter is more about the recently discovered property of orthogonal polynomials, which mathematically arises after the exact solution in the second paragraph. The new result mentioned is to show the existence of a hitherto unknown direct limit connection between the orthogonal pseudo-Jacobi and Hermite polynomials and to prove its correctness.

Hermite polynomials are considered the simplest polynomials among the polynomials included in the Askey scheme of orthogonal polynomials, and are expressed by hypergeometric functions of type $_2F_0$ as follows:

$$H_n(x) = (2x)^n\,{}_2F_0\left(-\frac{n}{2}; -\frac{n-1}{2}; -\frac{1}{x^2}\right), n = 0,1,2, \dots. \quad (21)$$

Here $_rF_s$ is a hypergeometric function of type, defined as follows:

$${}_rF_s\left(\begin{matrix}a_1, \cdots, a_r\\ b_1, \cdots, b_s\end{matrix}; x\right) = \sum_{k=0}^{\infty}\frac{(a_1,\cdots,a_r)_k}{(b_1,\cdots,b_s)_k}\frac{x^k}{k!}, \quad (22)$$

and by appointment

$$(a_1, \cdots, a_r)_k = (a_1)_k \cdots (a_r)_k,$$

$(a)_k$ is the Pohammer symbol and has the following analytical expression:

$$(a)_0 = 1 \text{ və } (a)_k = a(a+1)(a+2)\cdots(a+k-1),\ k = 1,2,3,\ldots. \quad (23)$$

It is well known from the theory of special functions that if at least one of the numbers a_i defining the products of the Pohammer symbols in the copy of the expression (22) is equal to $-n$, where n is a positive integer, then define the hypergeometric function ${}_rF_s$. The infinite series will turn into a finite sum by truncating at the positive integer n instead of infinity, and the hypergeometric function ${}_rF_s$ will be a polynomial of the variable x.

The Hermite polynomials defined by the formula (21) are exact solutions of the following differential equation of second order:in dəqiq həllidirlər:

$$y''(x) - 2x \cdot y'(x) + 2n \cdot y(x) = 0,\ y(x) = H_n(x). \quad (24)$$

Hermite polynomials satisfy the following orthogonality condition for all values of the variable x in the real domain:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x) dx = 2^n n!\, \delta_{mn}. \quad (25)$$

Pseudo-Jacobi polynomials, as we mentioned, are orthogonal polynomials expressed by ${}_2F_1$ type hypergeometric functions and are defined as follows:

$$P_n(x; \nu, N) = \frac{(-2i)^n (-N + i\nu)_n}{(n - 2N - 1)_n}\, {}_2F_1\left(\begin{matrix} -n; n - 2N - 1 \\ -N + i\nu \end{matrix}; \frac{1 - ix}{2} \right),$$

$$n = 0,1,2, \ldots, N. (26)$$

Here, ν is an arbitrary real number and N is an arbitrary natural number. The mentioned polynomials are exact solutions of the following second order differential equation:

$$(1 + x^2) y''(x) + 2(\nu - Nx) y'(x) - n(n - 2N - 1) y(x) = 0,$$

$$y(x) = P_n(x; \nu, N). \quad (27)$$

It can be seen from expression (26) that the main difference between pseudo-Jacobi polynomials and Hermitian polynomials is that the number n, which determines the general formulation of the polynomial, is finite. However, pseudo-Jacobi polynomials, like Hermitian polynomials, satisfy the following orthogonality condition for all values of the variable x in the real domain:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(1+x^2)^{-N-1}e^{2\nu\arctan x}P_m(x;\nu,N)P_n(x;\nu,N)dx =$$

$$\frac{\Gamma(2N+1-2n)\Gamma(2N+2-2n)2^{2n-2N-1}n!}{\Gamma(2N+2-n)|\Gamma(N+1-n+i\nu)|^2}\delta_{mn}. \quad (28)$$

As mentioned above, he introduced a new limit relation between pseudo-Jacobi polynomials and Hermitian polynomials. The mathematical expression of this relationship is as follows:

$$\lim_{N\to\infty} N^{\frac{n}{2}}P_n\left(\frac{x}{\sqrt{N}};\frac{\nu}{N},N\right)=\frac{1}{2^n}H_n(x). \quad (29)$$

Later, showed the existence of two different limit relations between pseudo-Jacobi polynomials and Hermite polynomials, which have the following expressions:

$$\lim_{N\to\infty} N^{\frac{n}{2}}P_n\left(\frac{x}{\sqrt{N}};\nu,N\right)=\frac{1}{2^n}H_n(x), \quad (30)$$

$$\lim_{N\to\infty} N^{\frac{n}{2}}P_n\left(\frac{x}{\sqrt{N}};\nu\sqrt{N},N\right)=\frac{1}{2^n}H_n(x-\nu). \quad (31)$$

Let's analyze the limit relations (30) and (31) mutually. First, consider that

$$P_0(x;\nu,N)=1,\ P_1(x;\nu,N)=x-\frac{\nu}{N},\ \ldots, \quad (32)$$

$$H_0(x)=1,\ H_1(x)=2x,\ \ldots. \quad (33)$$

In the above expressions, although the expressions of both polynomials coincide in the case of $n=0$, in the case of $n=1$ these values differ and the pseudo-Jacobi polynomial already depends on both ν and N. Let's look at the comparative analysis of the limits (30) and (31) at the value $n=1$. From expression (30) we get that:

$$\lim_{N\to\infty}\sqrt{N}\left(\frac{x}{\sqrt{N}}-\frac{\nu}{N^2}\right)=x. \quad (34)$$

If we take into account that here ν is an arbitrary real number, as mentioned above, then the truth of expression (34) will not be violated even under the condition $\nu\to\pm\infty$. This means that the limit expression (30) is true for any values of the real parameter ν [6].

Let us now apply the same approach to (31). At this time we get that

$$\lim_{N\to\infty}\sqrt{N}\left(\frac{x}{\sqrt{N}}-\frac{\nu}{N}\right)=x. \tag{35}$$

Another writting of this expression

$$\lim_{N\to\infty}\frac{\nu}{\sqrt{N}}=0. \tag{36}$$

From here, we clearly see that the truth of expression (36) is violated at values of $\nu\to\pm\infty$. That is, from the comparative analysis, we conclude that the limit given by the expression (36) is true only for finite values of the real parameter ν in a special case [10].

It should be noted that for the $n=1$ case we have already obtained the correct answer as a result of the comparative analysis, so we do not investigate the more complex limits for the highly ordered $n>1$ cases. However, it is clear that this type of analysis will lead to the same result.

The third chapter of the dissertation is dedicated to the exactly solvable Gora-Williams confinement model of the non-relativistic linear harmonic oscillator in the canonical approach [1].

The Gora-Williams kinetic energy operator is the Hermite operator and is defined as follows:

$$\hat{H}_0^{QU}=-\frac{\hbar^2}{4}\left[\frac{1}{M(x)}\frac{d^2}{dx^2}+\frac{d^2}{dx^2}\frac{1}{M(x)}\right]. \tag{37}$$

If we write the potential of the considered non-relativistic linear harmonic oscillator as follows:

$$V(x)=\begin{cases}\dfrac{M(x)\omega^2x^2}{2} & |x|\le a,\\ \infty, & |x|>a,\end{cases} \tag{38}$$

in this case, we will actually get an oscillator model subject to the confinement effect at $x=\pm a$ values and whose effective mass varies depending on the coordinate. In order to achieve the confinement effect, we first choose the following analytical expression for the function $M(x)$, which is the position-dependent effective mass of the oscillator:

$$M\equiv M(x)=\frac{a^2m_0}{a^2-x^2}\,. \tag{39}$$

It can be checked that expression (3.3) satisfies the following conditions:

$$M(\pm a) = \infty, \qquad M(0) = m_0 \qquad \lim_{a\to\infty} M(x) = m_0.$$

Then, taking into account the expression (37) and the fact that the momentum operator in the canonical approach is defined as $\hat{p}_x = -i\hbar\frac{d}{dx}$, for the Hamiltonian operator of the considered quantum oscillator model we get the expression:

$$\widehat{H}^{QU} = -\frac{\hbar^2}{4}\left[\frac{1}{M(x)}\frac{d^2}{dx^2} + \frac{d^2}{dx^2}\frac{1}{M(x)} + \frac{M(x)\omega^2 x^2}{2}\right] \quad (40)$$

After simple calculations, we can write the corresponding Schrödinger equation as:

$$\left[-\frac{\hbar^2}{2M}\left(\frac{d^2}{dx^2} - \frac{M'}{M}\frac{d}{dx} + \left(\frac{M'}{M}\right)^2 - \frac{1}{2}\frac{M''}{M}\right) + \frac{M\omega^2x^2}{2}\right]\psi = E\psi. \quad (41)$$

Let's compare the second-order differential equation obtained by performing a series of mathematical calculations with the Nikiforov-Uvarov method with the second-order differential equation of the Gegenbauer polynomial:

$$(1 - x^2)\bar{y}'' - (2\bar{\lambda} - 1)x\bar{y}' + n(n + 2\bar{\lambda})\bar{y} = 0. \qquad (42)$$

Here, $\bar{y} = C_n^{\bar{\lambda}}(x)$ are Gegenbauer polynomials. After simple calculations, we obtain the following expression for the energy spectrum of the Gore-Williams confinement model of the considered non-relativistic linear harmonic oscillator:

$$E \equiv E_n^{QU} = \hbar\omega\left(n + \frac{1}{2}\right) + \frac{\hbar^2}{2m_0 a^2}(n^2 + n + 1). \qquad (43)$$

The expression for the wave functions of the stationary states of the oscillator will be as follows:

$$\psi \equiv \psi_n^{QU}(x) = c_n^{QU}\left(1 - \frac{x^2}{a^2}\right)^{\frac{m_0\omega a^2}{2\hbar}} C_n^{\left(\frac{m_0\omega a^2}{\hbar} + \frac{1}{2}\right)}\left(\frac{x}{a}\right). \qquad (44)$$

Here c_n^{QU} is the normalization factor, defined as:

$$c_n^{QU} = 2^{\frac{m_0\omega a^2}{\hbar}}\Gamma\left(\frac{m_0\omega a^2}{\hbar}+\frac{1}{2}\right)\sqrt{\frac{\left(n+\frac{m_0\omega a^2}{\hbar}+\frac{1}{2}\right)n!}{\pi a\Gamma\left(n+\frac{2m_0\omega a^2}{\hbar}+1\right)}}, \qquad (45)$$

$C_n^{(\bar{\lambda})}(x)$ are Gegenbauer polynomials defined by ${}_2F_1$ hypergeometric functions as follows:

$$C_n^{(\bar{\lambda})}(x) = \frac{(2\bar{\lambda})_n}{n!}\ {}_2F_1\left(\begin{matrix}-n, n+2\bar{\lambda} \\ \bar{\lambda}+1/2\end{matrix};\frac{1-x}{2}\right), \quad \bar{\lambda} \neq 0. \qquad (46)$$

It can be easily shown that in the limit $a \to \infty$ that is, when the confinement effect disappears, the expressions (43) and (44) exactly restore the expressions of the energy spectrum of the infinite harmonic oscillator and the wave functions of the stationary states, respectively.

Now, let's put different requirements on the potential $V(x)$ of the investigated quantum system. Let's say that the harmonic oscillator potential, whose mass varies depending on the position, is given as follows:

$$V(x) = \frac{M(x)\omega^2 x^2}{2}, \qquad (47)$$

however, in the case of $x \to \pm\infty$ the potential behaves as $\frac{m_0\omega^2 a^2}{2}$.

The fact that the potential behaves differently than before requires that we write the analytical expression of the coordinate-dependent effective mass $M(x)$ as:

$$M \equiv M(x) = \frac{a^2 m_0}{a^2 + x^2}, \qquad (48)$$

For simplicity, we will assume that $a > 0$. Then, we can easily show that

$$M(0) = m_0 \qquad \lim_{x=\pm\infty}\frac{a^2 m_0}{a^2+x^2} = 0.$$

Given these conditions, we can write the corresponding Schrödinger equation as:

$$-\frac{\hbar^2}{4}\left[\frac{1}{M(x)}\frac{d^2}{dx^2}+\frac{d^2}{dx^2}\frac{1}{M(x)}\right]\psi^{QU}(x) + \frac{M(x)\omega_0^2 x^2}{2}\psi^{QU}(x) = E^{QU}\psi^{QU}(x). \quad (49)$$

After a series of mathematical calculations, the equation obtained from the solution of the Schrödinger equation, which we are looking at, can be solved exactly by directly comparing it with the following well-known second differential equation for pseudo-Jacobi polynomials $P_n(\xi, v, N)$.

$$(1+\xi^2)\bar{y}'' - 2(\nu - N\xi)\bar{y}' + n(2N-n+1)\bar{y} = 0,$$
$$\bar{y} = P_n(\xi, v, N). \quad (50)$$

From this comparison, we can easily see that the E^{QU} energy spectrum is non-equidistant and finite:

$$E^{QU} = \hbar\omega_0 \frac{(N+1)\left(n+\frac{1}{2}\right) - \left(\frac{n+1}{2}\right) - \frac{1}{2}}{N+1},$$
$$n = 0, 1, 2, 3, \dots, N. \quad (51)$$

The wave functions of stationary states $\psi^{QU}(x)$, are expressed by means of pseudo-Jacobi polynomials in the following order:

$$\tilde{\psi}^{QU}(x) = c_n^{QU}\left(1+\frac{\lambda_0{}^2x^2}{N+1}\right)^{-\frac{N+1}{2}} P_n\left(\frac{\lambda_0 x}{(N+1)^{\frac{1}{2}}}, 0, N\right). \quad (52)$$

Here, $P_n(\xi, v, N)$ are pseudo-Jacobi polynomials, defined by the hypergeometric function ${}_2F_1$ as follows:

$$P_n(\xi, v, N) = \frac{(-2i)^n(-N+iv)_n}{(n-2N-1)_n}\ {}_2F_1\left(\begin{matrix}-n, n-2N\\ -N+iv\end{matrix}; \frac{1-ix}{2}\right), \quad (53)$$
$$n = 0, 1, 2, \dots, N.$$

$\tilde{\psi}^{QU}(x)$ is orthonormal and the normalization parameter c_n^{QU} is found from the following orthogonality condition for pseudo-Jacobi polynomials:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}(1+x^2)^{-N-1}\, e^{2v\cdot arctanx} P_m(x, v, N) P_n(x, v, N) dx$$
$$= \frac{\Gamma(2N+1-2n)\Gamma(2N+2-2n)2^{2n-2N-1}n!}{\Gamma(2N+2-n)|\Gamma(N+1-n+iv)|^2}\delta_{mn}. \quad (54)$$

The apparent form of the c_n^{QU} normalization parameter is as follows:

$$c_n^{QU} = \frac{\Gamma(N-n+1)}{2^{n-N}\Gamma(2N-2n+1)}\sqrt{\frac{\Gamma(2N-n+1)}{\pi a_N(2N-2n+1)n!}} \tag{55}$$

Also, the position-dependent effective mass is quantized by the quantum number N of $M(x)$-in the following way:

$$M_N(x) = \frac{N+1}{N+1+\lambda_0^2 x^2} m_0. \tag{56}$$

The graph for N=3 values of the potential of the Gora-Williams kinetic energy operator oscillator model with coordinate-dependent mass and the energy spectra found from the exact solutions corresponding to this potential ($m_0 = \omega_0 = \hbar = 1$) is depicted.

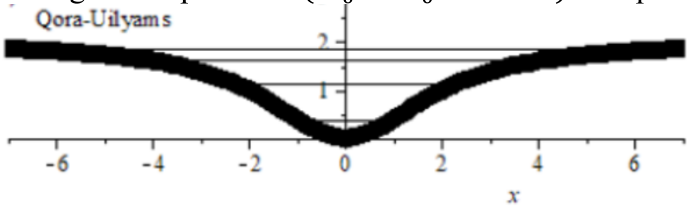

Fig.1 Graphic representation of the Gora-Williams oscillator model with coordinate-dependent mass (47) potential and the energy spectra found from exact solutions corresponding to this potential for N=3 value ($m_0 = \omega_0 = \hbar = 1$).

Let's expand these types of oscillator problems with exact solutions and when the non-relativistic harmonic oscillator, whose effective mass varies depending on the coordinate, is suddenly affected by an external gravitational field, find the exact solutions of the problem and observe how the external gravitational field changes the quantum harmonic oscillator we are studying.

Before examining the exact solution of a non-relativistic quantum harmonic oscillator under the influence of the Gora-Williams kinetic energy operator in a gravitational field, let us consider the solution of a constant effective mass harmonic oscillator in an external gravitational field. Considering the effect of an external uniform gravitational field on a non-relativistic linear-harmonic quantum oscillator of constant mass, we can write the potential of the oscillator as:

$$V(x)=\frac{m_0\omega^2x^2}{2}+m_0gx. \tag{57}$$

Here, m_0 and ω are the constant effective mass and periodic frequency of the non-relativistic quantum harmonic oscillator, respectively. In this case, the corresponding Schrödinger equation is as follows:

$$\left[\frac{\hat{p}_x^2}{2m_0}+\frac{m_0\omega^2x^2}{2}+m_0gx\right]\psi(x)=E\psi(x). \tag{58}$$

The one-dimensional momentum operator in the canonical approach defined as

$$\hat{p}_x=-i\hbar\frac{d}{dx},$$

equation (58) takes the following form:

$$-\frac{\hbar^2}{2m_0}\frac{d^2\psi(x)}{dx^2}+\left[\frac{m_0\omega^2x^2}{2}+m_0gx\right]\psi(x)=E\psi(x). \tag{59}$$

After some mathematical calculations, we get the following expression for the energy spectrum of the harmonic oscillator:

$$E=E_n^g=\hbar\omega\left(n+\frac{1}{2}\right)-\frac{m_0g^2}{2\omega^2}, \qquad n=0,1,2,\ldots \tag{60}$$

Accordingly, the wave function of stationary states is defined by the following expression:

$$\psi\equiv\psi_n^g(x)=\frac{1}{\sqrt{2^nn!}}\left(\frac{m_0\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{m_0\omega\left(x+\frac{g}{\omega^2}\right)^2}{2\hbar}}H_n\left(\sqrt{\frac{m_0\omega}{\hbar}}\left(x+\frac{g}{\omega^2}\right)\right). \tag{61}$$

Now let's examine the effect of the gravitational field on the non-relativistic quantum harmonic oscillator whose effective mass depends on the coordinate. In this case, the potential of the harmonic oscillator can be written as:

$$V(x)=\begin{cases}\dfrac{M(x)\omega^2x^2}{2}+M(x)gx, & |x|\le a,\\ \infty, & |x|>a.\end{cases} \tag{62}$$

Given the expression of the Gora-Williams kinetic energy operator and the potential, we can write the complete Hamiltonian operator as follows [7]:

$$\widehat{H}^{QU}=-\frac{\hbar^2}{2M}\left[\frac{d^2}{dx^2}-\frac{M'}{M}\frac{d}{dx}-\frac{1}{2}\frac{M''}{M}+\left(\frac{M'}{M}\right)^2\right]+\frac{M\omega^2x^2}{2}+M(x)gx. \quad (63)$$

After simple calculations, we get the following expression for the energy spectrum of the harmonic oscillator:

$$E\equiv E_n^{gQU}=\sqrt{\frac{1}{2}+\sqrt{\frac{1}{4}-\frac{g^2}{a^2\omega^4}}}\hbar\omega\left(n+\frac{1}{2}\right)+\frac{\hbar^2}{2m_0a^2}(n^2+n+1)-\frac{m_0\omega^2a^2}{2}\left(1-\sqrt{1-\frac{4g^2}{\omega^4a^2}}\right). \quad (64)$$

And the wave function of the stationary states of the considered harmonic oscillator is a

$$\psi\equiv\psi_n^{gQU}(x)=c_n^{gQU}\left(1-\frac{x}{a}\right)^{-\kappa_1}\left(1+\frac{x}{a}\right)^{-\kappa_2}P_n^{(-2\kappa_1,-2\kappa_2)}\left(\frac{x}{a}\right). \quad (65)$$

Here, c_n^{QU}

$$\boldsymbol{c_n^{gQU}=\frac{1}{2^{\sqrt{\kappa}+\frac{1}{2}}}\sqrt{\frac{(2n+2\sqrt{\kappa}+1)\Gamma(n+2\sqrt{\kappa}+1)n!}{a\Gamma(n-2\kappa_1+1)\Gamma(n-2\kappa_2+1)}}} \quad (\mathbf{66})$$

is the normalization factor and is determined from the orthogonality condition of Jacobian polynomials as follows:

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)dx=\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}\delta_{mn}. \quad (67)$$

The fourth chapter of the dissertation work is devoted to how the confinement model of the non-relativistic linear harmonic oscillator obeying the Galilean invariance of the kinetic energy operator is established in the canonical approach. Therefore, in the first paragraph of the present chapter, some properties of the case of

Galilean invariance for position-dependent effective mass are first explained [8].

In the next paragraph, we investigate how to construct different kinetic energy operators using a coordinate-dependent mass approach compatible with Galilean invariance and find the exact solution of a new non-relativistic linear harmonic oscillator model compatible with Galilean invariance [2].

The analytical expression of the kinetic energy operator for the confinement model of a quantum harmonic oscillator compatibble with Galilean invariance is chosen as follows based on the extensive discussions in the previous paragraph of this chapter[44]:

$$\widehat{H}_0^{Q\dot{\text{I}}} = \frac{1}{6}\left[\frac{1}{M(x)}\hat{p}^2 + \hat{p}\frac{1}{M(x)}\hat{p} + \hat{p}^2\frac{1}{M(x)}\right]. \tag{68}$$

Here, M(x) is the pozition-dependent effective mass of the oscillator.

In this case, we will get an oscillator model that is actually subjected to the confinement effect and whose effective mass is coordinate dependent, and in this case, the effective mass dependent on the coordinate to achieve the confinement effect

$$M \equiv M(x) = \frac{a^2 m_0}{a^2 - x^2} \tag{69}$$

and in this case the Hamiltonian operator of the oscillator can be described as follows [5]:

$$\widehat{H}^{Q\dot{\text{I}}} = -\frac{\hbar^2}{6}\left[\frac{1}{M(x)}\frac{d^2}{dx^2} + \frac{d}{dx}\frac{1}{M(x)}\frac{d}{dx} + \frac{d^2}{dx^2}\frac{1}{M(x)}\right] + \frac{M(x)\omega^2 x^2}{2}. \tag{70}$$

After the necessary calculations, for the energy spectrum of the harmonic oscillator

$$E_n^{Q\dot{\text{I}}} = \hbar\omega\left(n + \frac{1}{2}\right) + \frac{\hbar^2}{2m_0 a^2}n(n+1) + \frac{\hbar^2}{3m_0 a^2}, \tag{71}$$

while for wave functions of stationary states we have received their

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[4] Levy-Leblond, J.M. Position-dependent effective mass and Galilean invariance // Physical Review A, -1995. 52, - p. 1845-1849.

statements:

$$\psi_n^{Q\dot{I}}(x) = c_n\left(1 - \frac{x^2}{a^2}\right)^{\frac{m_0\omega a^2}{2\hbar}} C_n^{\left(\frac{m_0\omega a^2}{\hbar}+\frac{1}{2}\right)}\left(\frac{x}{a}\right). \qquad (72)$$

In the last paragraph of the chapter, the influence of the external gravitational field on the considered harmonic oscillator model is investigated within the formalism of the variable mass depending on the position [5]

$$\widehat{H}^{gQ\dot{I}} = -\frac{\hbar^2}{6}\left[\frac{1}{M(x)}\frac{d^2}{dx^2} + \frac{d}{dx}\frac{1}{M(x)}\frac{d}{dx} + \frac{d^2}{dx^2}\frac{1}{M(x)}\right] + \frac{M(x)\omega^2x^2}{2} + M(x)gx. \quad (73)$$

By solving the Schrödinger equation according to the Hamiltonian operator, the exact expressions for the energy spectrum and the wave functions of the stationary states of the considered model are presented [4]:

$$E \equiv E_n^{gQI} = \hbar\omega\sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{g^2}{a^2\omega^4}}\left(n + \frac{1}{2}\right) + \frac{\hbar^2}{2m_0a^2}n(n+1) + \frac{\hbar^2}{3m_0a^2}} - m_0\omega^2a^2\left(\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{g^2}{a^2\omega^4}}\right). \quad (74)$$

$$\psi \equiv \psi_n^{gQI}(x) = c_n^{gL}\left(1 - \frac{x}{a}\right)^{-\kappa_1}\left(1 + \frac{x}{a}\right)^{-\kappa_2} P_n^{(-2\kappa_1,-2\kappa_2)}\left(\frac{x}{a}\right). \quad (75)$$

Normalization factor c_n^{gGI}:

$$c_n^{gQI} = \frac{1}{2^{\sqrt{\kappa}+\frac{1}{2}}}\sqrt{\frac{\left(2n + 2\sqrt{\kappa} + 1\right)\Gamma\left(n + 2\sqrt{\kappa} + 1\right)n!}{a\Gamma(n - 2\kappa_1 + 1)\Gamma(n - 2\kappa_2 + 1)}}, \quad (76)$$

$P_n^{(\alpha,\beta)}(x)$ It is determined from the orthogonality relation of Jacobian polynomials [9].

As a result, we can say that the considered model of the harmonic oscillator is very interesting and important and differs from the simple harmonic oscillator in terms of its characteristics. The confinement effect and the non-linearity of the energy spectrum allow us to say that the model can be widely used in the future.

MAIN RESULTS

In the current dissertation work, for the first time, the Schrödinger equation describing quantum wells with a non-rectangular profile for three different kinetic energy operators within the framework of the position-dependent mass formalism was comparatively exactly solved, and exact expressions of both the energy spectrum and the wave function of the stationary states were found. The non-rectangular profiles of the quantum wells were obtained thanks to the analytical expression of the known harmonic oscillator potential. The behavior of the walls of the quantum wells with finite and infinite deep quantum wells, that is, the appearance of the confinement effect applied to the model, was obtained through the selection of analytical expressions of the position-dependent mass in a unique way. In general, the following important results were found in the dissertation work:

- Within the position-dependent mass formalism, the Schrödinger equation expressed by the Ju-Kroemer kinetic energy operator is exactly solved for harmonic oscillator-type infinite and finite deep quantum well models, and the nonlinear energy spectrum and the wave functions of stationary states expressed by Gegenbauer, Jacobi and pseudo-Jacobi polynomials obvious expressions have been found;

- It was observed that the reduced frequency of the nonlinear energy spectrum, found by exactly solving the Schrödinger equation expressed by the Zhu-Kroemer kinetic energy operator for the model of harmonic oscillator-type infinite deep quantum wells, depends on the confinement parameter and increases sharply when this parameter approaches zero;

- The Schrödinger equation expressed by the Gore-Williams kinetic energy operator and the kinetic energy operator compatible with Galilean invariance within the position-dependent mass formalism for a harmonic oscillator-type infinite deep quantum well is exactly solved both under the influence of an external gravitational field and in the absence of such influence, and nonlinear exact expressions of the energy spectrum and the wave functions of stationary states expressed by Gegenbauer and Jacobi polynomials were found;

- From the comparison of the energy spectra of the Schrödinger equations expressed with the Gore-Williams kinetic energy operator and the Galilean invariance kinetic energy operator within the position-dependent mass formalism, it was observed that the energy spectrum of the model expressed with the Galilean invariance kinetic energy operator always takes larger values;

- Within the framework of the position-dependent mass formalism, the limit relations between the exact solutions of the harmonic oscillatory quantum well models of the Schrödinger equation expressed by Zhu-Kroemer, Gora-Williams and Galilean invariance kinetic energy operators restored with the disappearance of the confinement and the exact solutions of the known non-relativistic canonical quantum harmonic oscillator are calculated exactly;

- A new limit relation was found, which shows the existence of a direct relationship between pseudo-Jacobi and Hermitian polynomials, and the correctness of this relation was proved mathematically.

In the near future, the results listed above may play a major role both in the explanation of various characteristics of modern nanostructures to the experimentally obtained non-rectangular complex profile within the framework of quantum physics, as well as in finding direct limits between many orthogonal polynomials that have not been known so far.

List of scientific works published on the thesis.

1. Jafarov, E.I., Nagiyev, S.M., Seyidova, A.M. Explicit solution of the position-dependent mass Schrodinger equation with Gora-Williams kinetic energy operator: confined harmonic oscillator model // U.P.B. Sci. Bull., - 2020. A (82),1, - p. 327-336.
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