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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

INVESTIGATION OF TRACES OF SOME DIFFERENTIAL EQUATIONS WITH OPERATOR COEFFICIENTS

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Field of science:	Mathematics
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GENERAL CHARACTERISTICS OF THE WORK

Rationale and development degree of the theme topic.

In the dissertation work are considered spectral problems for differential equations with unbounded operator coefficients These problems cover many boundary value problems for partial differential equations. The self-adjointness conditions, the nature of the spectrum, the asymptotics of the operators corresponding to the problems which include the spectral parameter in the boundary condition as well as in equation are studied and their regularized traces are calculated. The properties of the self-adjoint operator are used both in investigation of the spectrum and obtaining of trace formulas. Note that the self-adjoint operator in the problems involving the spectral parameter in the boundary condition for partial derivative differential equations with exit from space was previously shown in the works of J. Odnoff and S. Ercolano, M. Schechter. In this direction for unbounded operator equations which we know the works of V.I. Gorbachuk, M.A. Rybak, B.A. Aliyev, M. Bayramoglu, N.M. Aslanova. However there are many unexplored issues, some of which are the subject of the dissertation. Many problems in mechanics, mathematical physics and the theory of partial differential equations lead to the study of boundary problems for differential operator equations in different spaces. The asymptotic distribution of eigenvalues of boundary value problems with unbounded operator coefficient was first investigated by A.G. Kostyuchenko and B.M. Levitan. After that, many works appeared devoted to the investigation of the spectrum of differential operators with operator coefficients. The asymptotic distribution of the eigenvalues of operators having a discrete spectrum is problem of quantum mechanics. The theory of differential operator equations with unbounded operator coefficients is a common tool for the study of infinite systems of ordinary differential equations, partial differential equations and integro-differential equations. The main task in this theory is to determine the behavior of the eigenvalues and eigenfunctions of the associated differential operator. Calculation of regularized traces of this type problems is one of the most important problems of spectral analysis.

Trace formulas are applied for the approximation of the first eigenvalues, in mechanics, in the index theory of linear operators and in solving inverse problems. In the algebra, there are a lot of works devoted to the calculation of the traces of scalar differential operators, which is analogous to the concept of the trace of a matrix. In this direction, the first work for the operator with discrete spectrum belongs to I.M.Gelfand and B.M.Levitan. Trace formulas for scalar differential operators were investigated by I.M.Gelfand and B.M.Levitan, L.A.Dikiy, C.J.Halberg and V.A. Kramer, A.M. Savchuk and A.A. Shkalikov and many other scientists. A broad review of the works on this topic is given in detail in doctor's dissertation of N.M. Aslanova (Baku, 2013).The trace formula of self-adjoint abstract operators with continuous spectrum was first investigated by I.M.Lifshitz, then was developed by S.G.Krein, L.D.Faddeyev and others. At first the regularized trace for differential operator with a bounded operator coefficient was found by R.Z.Khalilova, and the trace formula for the Sturm-Liouville equation with an unbounded operator coefficient was found by F.G.Magsudov, M.Bayramoglu and A.A.Adigozalov. For the first time, the definition of the regularized trace for differential operators with unbounded operator coefficient was given in this work.

Object and subject of the research The main object of the dissertation work is the spectral analysis of differential equations with an unbounded operator coefficient. The subject of the study is the investigation of the spectrum of boundary value problems with spectral parameter in the boundary conditions, with unbounded operator coefficient and the calculation of regularized traces.

Goal and tasks of the research. The main goal of the dissertation work is to solve the following main issues:

1. The study of the nature of the spectrum of the Sturm-Liouville equation with unbounded operator coefficient, where the linear function of the spectral parameter appears in the boundary condition both in front of the function and its derivative. Obtaining of

the asymptotic formula for the eigenvalues of corresponding operator;

2. Calculation of the regularized trace for one particular case of considered problem;

3. Obtaining the asymptotic distribution formula for the eigenvalues of the Sturm-Liouville equation with unbounded operator coefficient with the boundary condition including the rational function of the spectral parameter and proving the trace formula;

4. Study of the spectral properties of the problem corresponding to the Bessel equation with unbounded operator coefficient when spectral parameter in the boundary condition involved in front of the derivative of the function;

5. Setting the first regularized trace formula for the Bessel equation with unbounded operator coefficients;

6. Finding the second regularized trace formula for the second order differential equations with unbounded operator coefficient and spectral parameter dependent boundary condition (two different problems);

Research methods. In the work, the methods of differential equations, functional analysis, spectral theory of self-adjoint operators, the theory of functions of complex variables and perturbation were applied.

The basic aspects to be defended.

1. The spectral properties of the Sturm-Liouville operator equation with the boundary condition in which spectral parameter exists in front of the function and its derivative were investigated and a regularized trace was calculated in a particular case;

2. The asymptotic distribution formula for the eigenvalues of the second order differential equation and boundary condition depending on a rational function of spectral parameter is found and trace formula is calculated;

3. The spectral properties are studied of the problem according to the operator Bessel equation where spectral parameter involved in front of the derivative in the boundary condition, the self-adjoint operator is determined corresponding to the considered problem by widing from the space and the asymptotic formula for eigenvalues is

obtained;

4. The first regularized trace of the Bessel operator equation is found;

5. The second regularized trace is calculated for the Sturm-Liouville equation with unbounded operator coefficient with spectral parameter in the boundary condition (two different problems);

Scientific novelty of the research. In the dissertation work:

1) The asymptotic behavior of the eigenvalues for the Sturm-Liouville operator equation is investigated where the spectral parameter is included in the boundary condition as a linear function and a regularized trace is found in a particular case;

2) The asymptotic nature of the eigenvalues of the Sturm-Liouville operator equation with unbounded operator coefficient is studied, where a rational function of the spectral parameter involved to boundary condition and regularized trace formula is obtained;

3) The formula for asymptotic distribution of the eigenvalues for the Bessel equation with unbounded operator coefficient is proved in which the spectral parameter involved in front of the derivative in the boundary condition and the first regularized trace is found;

4) The second order regularized traces of the second order differential-operator equations with the spectral parameter dependent are calculated;

Theoretical and practical value. The results of the work are of theoretical character. They can be used in approximate finding of the first eigenvalues of differential operators, in mechanics and also when studying inverse problems of spectral analysis.

Approbation and application. The main results of the dissertation were presented and discussed at the following republican and international conferences, scientific seminar and vebinar:
1st International Scientific Conference of young scientists and specialists on "The role of multidisciplinary approach in solution of actual problems of fundamental and applied sciences (earth, technical and chemical) (Baku, 2014), International Scientific Conference dedicated to the 85th of Y.Mammadov (Baku, 2015), “Actual problems of mathematics and mechanics” dedicated to the 93th

anniversary of the national leader of Azerbaijan H.Aliyev (Baku, 2016), International Conference on “Modern problems of innovative technologies in oil and gas production and applied mathematics" dedicated to the 90th anniversary of academician A. Mirzajanzade (Baku, 2018), III International Scientific Conference of young researchers dedicated to the 96th Anniversary of the national leader of Azerbaijan, H.Aliyev, (Baku 2019), at the webinar of Scientific Research Institute of Applied Mathematics of Baku State University (Baku, 2021), 6th International HYBRID Conference on Mathematical Advances and Applications (Turkey, 2023), seminar of Department “High mathematics” of Azerbaijan Architecture and Construction University (Baku, 2023).

The personal contribution of the author is to state the purpose of the study and to choose a direction. In addition, all the obtained results and research methods belong to the author herself.

Author's publications. According to the research (in total 12 scientific works), 6 articles (4 of them without co-authors, 1 is included in the SCIE list of Web of Science, 1 is included in SCOPUS, 4 are included in Zentralblatt MATH databases), 3 conference material and 3 theses (5 international and 1 republican, as well as 1 of them abroad) were published in the publishing houses recommended by the HAC under the President of the Republic of Azerbaijan. The list of works is given at the end of the abstract.

Name of the organization where the dissertation work was performed. The work was performed in the department of “Differential equations” of the Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan.

Structure and volume of the dissertation (in signs, indicating the volume of each structural subsection separately). General volume of the dissertation work consists of - 213207 sings (title page - 333 signs, table of contents - 2516 signs, introduction - 45218 signs, the first chapter - 62000 signs, the second chapter – 46000 signs, the third chapter - 56000 signs, conclusion - 1140). The list of references consist of 84 titles.

THE MAIN CONTENT OF THE DISSERTATION

The dissertation work consists of introduction, three chapters and references. In the introduction, the rationale of the research work is justified and its development degree is shown, goals and tasks of the study are formulated, scientific novelty is given, theoretical and practical importance of the study is noted, information on approbation of the work is represented and the information about the approbation of the work is given. Chapter I is dedicated to the investigation of the eigenvalues of boundary value problems for Sturm-Liouville equations with unbounded operator coefficients in which the spectral parameter included in the boundary condition and to find the first regularized trace and consists of 6 subchapters.

In 1.1 was considered the spectral problem where has a linear fuction of spectral parameter in boundary candition in the space $L_2(H,(0,1))$:

$$l[y] \equiv -y''(t) + Ay(t) + q(t)y(t) = \lambda y(t) \tag{1}$$

$$y'(0) = 0 \tag{2}$$

$$ay(1) + by'(1) = \lambda(cy(1) - dy'(1)), \quad a,b,c,d \in R. \tag{3}$$

$L_2(H,(0,1))$ is a space of vector functions $y(t)$ such that $\int_0^1 \|y(t)\|^2 dt < \infty$. H is a separable Hilbert . Denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the scalar product and the norm in H, respectively. A is a self-adjoint positive definite operator in H. $A > E$, E is the identity operator in H, $A^{-1} \in \sigma_\infty$. Under these conditions, the spectrum of the operator A is discrete. Denote the eigenvalues and eigen-vectors of the operator A by $\gamma_1 \le \gamma_2 \le ...$ and $\varphi_1, \varphi_2,..,$ respectively.

Suppose that the operator-valued function $q(t)$ is weakly measurable and $\|q(t)\|$ is bounded on $[0,1]$, moreover the following conditions are satisfied:

1) There exists the second order weak derivative of $q(t)$ for

$\forall t \in [0,1]$ and $\left[q^{(k)}(t)\right]^* = q^{(k)}(t),\ k = 0,1,2;$

2) $\sum_{j=1}^{\infty} \left|(q^{(k)}(t)\varphi_j, \varphi_j)\right| < const;$

3) $q'(0) = q'(1) = 0$

4) $\int_0^1 (q(t)f, f)dt = 0$ for $\forall f \in H$.

Let us introduce the direct sum space $\mathcal{L}_2 = L_2(H,(0,1)) \oplus H$. Define the scalar product in this space as:

$$(Y,Z)_{\mathcal{L}_2} = \int_0^1 (y(t), z(t))dt + \frac{1}{\rho}(y_1, z_1),\ \rho = \begin{vmatrix} c & -a \\ d & b \end{vmatrix} = bc + ad > 0,$$

where $Y = \{y(t), y_1\} \in \mathcal{L}_2$, $Z = \{z(t), z_1\} \in \mathcal{L}_2$, $y(t), z(t) \in L_2(H,(0,1))$ $y_1, z_1 \in H$.

For $q(t) \equiv 0$ one can associate with problem (1)-(3) in space $\mathcal{L}_2$ a self-adjoint operator L_0 defined as

$D(L_0) = \{Y : Y = \{y(t), y_1\} / \ y(t) \in D(A), -y''(t) + Ay(t) \in L_2(H,(0,1)),$
$y'(0) = 0, y_1 = cy(1) - dy'(1)\},\quad L_0 Y = \{-y''(t) + Ay(t),\ ay(1) + by'(1)\}$
($y'(t)$ is obsolutely continuous in norm of H) .

The operator corresponding to the case $q(t) \not\equiv 0$ is denoted by $L = L_0 + Q$, where $Q: Q\{y(t), cy(1) - dy'(1)\} = \{q(t)y(t), 0\}$ is a bounded self-adjoint operator in $\mathcal{L}_2$. Obviously L is a bounded self-adjoint operator in $\mathcal{L}_2$. Let R_λ^0 and R_λ be the resolvents of the operators L_0 and L .

The following lemma is proved in 1.1 .

Lemma 1. The operator L is simmetric in $\mathcal{L}_2$.

The spectrum of L_0 is discrete. The operator L has a discrete spectrum, because Q is bounded in $\mathcal{L}_2$.

In 1.2 are investigated the asymptotics of the eigenvalues of the operators L_0 və L. Suppose that the eigenvalues of A behave like

$$\gamma_k \sim gk^{\alpha}, k \to \infty,\ g > 0, \alpha > 0 \tag{4}$$

Theorem 1. The eigenvalues of the operator L_0 form two sequences:

$$\lambda_{k,n} \sim \gamma_k + \alpha_n^2, \quad \alpha_n \sim \pi n,\ n \to \infty,$$

$$\lambda_k \sim -\frac{b}{d} + \frac{-c^2 \pm c\sqrt{c^2 + 4d(b + d\gamma_k)}}{2d^2}.$$

The following relation is true for the resolvents of the operators L_0 and L: $\quad R_\lambda(L) = R_\lambda(L_0) - R_\lambda(L)QR_\lambda(L_0)$

By virtue of theorem1 and the properties that hold for s numbers of compact operators we prove the following theorem

Theorem 2. Let $A = A^* > E$, in H, A^{-1} be compact, moreover holds the relation (4). Then

$$\lambda_n(L_0) \sim \mu_n(L) \sim dn^{\delta}, \qquad \delta = \begin{cases} \dfrac{2\alpha}{\alpha+2}, \alpha > 2, \\ \dfrac{\alpha}{2}, \alpha < 2, \\ 1, \alpha = 2. \end{cases}$$

where $\{\mu_n\}_{n=1}^{\infty}$ are eigenvalues of the operator L.

In 1.3 the first regularized trace of the operator generated by particular case of boundary condition (3) is calculated. If we take in (3) $c = 0, b = d = 1$ then this condition takes form:

$$ay(1) + y'(1) = -\lambda y'(1), \qquad a > 0. \tag{5}$$

Then the scalar product in L_2 becomes as:

$$(Y, Z)_{L_2} = \int_0^1 (y(t), z(t))dt + \frac{1}{a}(y_1, z_1) \quad .$$

For $q(t)\equiv 0$ in space L_2 one can associate with problem (1),(2),(5) a selfadjoint operator L_0 defined as

$$D(L_0)=\{Y:Y=\{y(t),y_1\}/\,y(t)\in D(A),\,-y''(t)+Ay(t)\in L_2(H,(0,1)),$$

$$y'(0)=0,\,y_1=-y'(1)\}$$, ($y'(t)$ is obsolutely continuous in norm of H)

The following lemma is true[1].

Lemma 3. If at $k\to\infty,\gamma_k\sim gk^{\alpha},0<g<\infty,2<\alpha<\infty$ then there exists a subsequence $\lambda_{n_1}<\lambda_{n_2}<\ldots$ of the sequence $\lambda_1,\lambda_2,\ldots$ such that:

$$\lambda_p-\lambda_{n_m}\geq d_0\left(p^{\frac{2\alpha}{2+\alpha}}-n_m^{\frac{2\alpha}{2+\alpha}}\right),\ p=n_m,n_m+1,\ldots\ ,\quad d_0>0\,.$$

Introduce the following notations:

$$\lambda^{(i)}=\sum_{k=n_{i-1}+1}^{n_i}\lambda_k\,,\quad \mu^{(i)}=\sum_{k=n_{i-1}+1}^{n_i}\mu_k\,,\ i=1,2,\ldots,\ n_0=0 \tag{6}$$

Call the sum $\sum_{i=1}^{\infty}\left(\mu^{(i)}-\lambda^{(i)}\right)$ a regularized trace of the operator L, since the sum of this series, as it will be shown below, doesn't depend on what way there has been chosen a subsequence $n_1,n_2,\ldots$ which satisfies the statement of lemma 3. In the present paragraph the formula for the sum of this series has been obtained

Lemma 4. Let $\|q(t)\|$ be bounded on the segment $[0,1]$ and the conditions of lemma 3 be fulfilled. Then at large m the following equality holds:

$$\sum_{n=1}^{n_m}(\mu_n-\lambda_n)=\sum_{j=i}^{N}(-1)^j M_m^j+\frac{(-1)^N}{2\pi i}\int_{|\lambda|<l_m}\lambda\, tr\left[R_\lambda\left(QR_\lambda^0\right)^{N+1}\right]d\lambda,$$

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[1] Максудов, М.Г., Байрамоглы, М., Адыгезалов, А.А. О регуляризованном следе оператора Штурма-Лиувилля на конечном отрезке с неограниченным операторным коэффициентом // Доклады Академии наук СССР, -1984. 277 (4), – с. 795–799.

$\mu_{n_m}, m = 1,2,3,\dots$ is a subsequence, that satisfies the statement of lemma 3 (N is an arbitarary natural number).

$$l_m = \frac{1}{2}\left(\mu_{n_m+1} - \mu_{n_m}\right),\ M_m^j = \frac{1}{2\pi i}\int_{|\lambda|=l_m} tr\left[\left(QR_\lambda^0\right)^j\right] d\lambda ,$$

Denote the orthonormal eigenvectors of the operator L_0 by $\{\Psi_n\}$, $n = 1,2,\dots$. We get,

$$\psi_n = \sqrt{\frac{4ax_{k,n}}{2ax_{k,n} + a\sin 2x_{k,n} + 4x_{k,n}^3 \sin^2 x_{k,n}}} \times$$

$$\times \left\{\cos\left(x_{k,n}t\right)\varphi_k, x_{k,n}\sin x_{k,n}\varphi_k\right\},\ \left(\begin{matrix} n = \overline{0,\infty}, k = \overline{N,\infty} \\ n = \overline{1,\infty}, k = \overline{1,\infty}\end{matrix}\right)$$

where $x_{k,n} \sim \pi n$ asymptotics hold and $x_{k,n}$ are the roots of the equation $a\cos z - z\sin z - \left(z^2 + \gamma_k\right)z\sin z = 0$, $z = \sqrt{\lambda - \gamma_k}$.

Lemma 5. Provided that for operator-valued function $q(t)$ hold the conditions 1)-3), then:

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{2ax_{k,n}\int_0^1 \cos\left(2x_{k,n}t\right)q_k(t)dt}{2ax_{k,n} + a\sin 2x_{k,n} + 4x_{k,n}^3\sin^2 x_{k,n}}\right| +$$

$$+\sum_{k=N}^{\infty}\left|\frac{2ax_{k,0}\int_0^1 \cos\left(2x_{k,0}t\right)q_k(t)dt}{2ax_{k,0} + a\sin 2x_{k,0} + 4x_{k,0}^3\sin^2 x_{k,0}}\right| < \infty ,\ q_k(t) = \left(q(t)\varphi_k, \varphi_k\right).$$

Theorem 3. Let the conditions of lemma 3 be satisfied. Provided operator-valued $q(t)$ satisfies conditions 1)-3),then the following formula is true:

$$\lim_{m\to\infty} M_m^1 = -\frac{tr\, q(0) + tr\, q(1)}{4}.$$

Theorem 4. Let the condition of lemma 3 be fulfilled. If the operator function $q(t)$ satisfies condition 1)- 3), then at $n \geq 2$

$$\lim_{m \to \infty} M_m^n = 0.$$

From lemma 3, theorems 3, 4 it follows that:

$$\sum_{i=1}^{\infty} \left(\mu^{(i)} - \lambda^{(i)}\right) = \frac{tr\, q(0) + tr\, q(1)}{4} \qquad (7)$$

Thus,the following theorem is proved.

Theorem 5. Let for the operator function $q(t)$ holds requirement 1)-4). Then under the conditions of lemma 3, for the regularized trace formula (7) is valid.

In 1.4 is studied the spectral problem for the Sturm-Liouville equation with unbounded operator coefficient, where a rational function of the spectral parameter is included in the boundary condition. There is considered the operator generated by differential expression (1) and boundary conditions

$$y'(0) = 0, \qquad y(1)(1+\lambda) = y'(1)(1+h+\lambda), \; h > 0.$$

Suppose that for $q(t)$ holds the above conditions. Define the scalar product in L_2 as $(Y, Z)_{L_2} = \int_0^1 (y(t), z(t))dt + \frac{1}{h}(y_1, z_1)$.

For $q(t) \equiv 0$ in L_2 one can associate with considered problem the selfadjoind operator L_0 defined as:

$$D(L_0) = \{Y : Y = \{y(t), y_1\} / \, y(t) \in D(A), -y''(t) + Ay(t) \in L_2(H, (0,1)),$$
$$y'(0) = 0, \; y_1 = -y(1) + y'(1)\} ,$$

($y'(t)$ is obsolutely continuous in norm of H)

$$L_0 Y = \{-y''(t) + Ay(t), \; y(1) - (1+h)y'(1)\}$$

The operator corresponding to the case $q(t) \not\equiv 0$ is denoted by $L = L_0 + Q$ where $Q : Q\{y(t), -y(1) + y'(1)\} = \{q(t)y(t), 0\}$ is a bounded self-adjoint operator in L_2.

Lemma 6. Operator L_0 is symmetric in L_2.

Using that lemma it might be shown that L_0 is self-adjoint positive definite operator. According to Relliche theorem L_0 has a discrete spectrum. Since $q(t)$ is bounded then L has also a discrete spectrum due to relation

$$R_\lambda(L) - R_\lambda(L_0) = R_\lambda(L) Q R_\lambda(L_0)$$

In 1.5 the asymptotic nature of the eigenvalues of the operators L_0 and L is studied.

Theorem 6. The eigenvalues of the operator L_0 form two sequences: $\lambda_k = \gamma_k + \alpha_{k,0}^2$, $|\alpha_{k,0}| < M = const$;

$$\lambda_{k,n} \sim \gamma_k + \alpha_n^2 \ , \ \alpha_n \sim \pi n + O\left(\frac{1}{n}\right), \ n \to \infty .$$

Lemma 7. Suppose that $A = A^* > E$ in H , A^{-1} is compact and holds relation (4). Then for the eigenvalues of L_0 we have

$$\lambda_n(L_0) \sim d_1 n^{\frac{2\alpha}{\alpha+2}}, \ d_1 = const .$$

In 1.6 the first regularized trace formula for the operator defined in 1.4 is obtained. We have found the orthonormal eigenvectors of the operator L_0 as

$$\psi_n = \sqrt{\frac{4hx_{k,n}}{\mathrm{B}_{k,n}}} \left\{ \cos(x_{k,n} t)\varphi_k , (- x_{k,n} \sin x_{k,n} - \cos x_{k,n})\varphi_k \right\}$$

$$\mathrm{B}_{k,n} = 2hx_{k,n} + h \sin 2x_{k,n} + 4x_{k,n}^3 \sin^2 x_{k,n} + 4x_{k,n}^2 \sin 2x_{k,n} +$$

$$+ 4x_{k,n} \cos^2 x_{k,n} \qquad \left(\begin{matrix} n = \overline{0,\infty}, k = \overline{N,\infty} \\ n = \overline{1,\infty}, k = \overline{1, N-1} \end{matrix} \right)$$

where $x_{k,n}$ are the roots of the characteristic equation

$$y_k(1) - (1+h) y_k'(1) = \lambda(-y_k(1) + y_k'(1)) , \ z = \sqrt{\lambda - \gamma_k} \ , \ y_k(t) = (y(t), \varphi_k)$$

Lemma 8. Provided that for operator-valued function $q(t)$ hold the conditions 1)-3), then

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\left|\frac{2hx_{k,n}\int_0^1 \cos(2x_{k,n}t)(q(t)\varphi_k,\varphi_k)dt}{\mathrm{B}_{k,n}}\right|<\infty .$$

Theorem 7. If the operator function $q(t)$ satisfy conditions 1)-4). Then under the conditions of lemma 7, we get the following regularized trace formula :

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}(\mu_n-\lambda_n)=\frac{tr\,q(0)+tr\,q(1)}{4}$$

Chapter II consists of 3 subchapters. This chapter is devoted to the investigation of the eigenvalues of the spectral problem for the Bessel equation with unbounded operator coefficient and spectral parameter dependent boundary condition. The regularized trace of the same problem is also found.

In 2.1 in $L_2(H,(0,1))$ is defined the operator generated by differential expression

$$l[y]\equiv -y''(t)+\frac{\nu^2-\frac{1}{4}}{t^2}y(t)+Ay(t)+q(t)y(t)=\lambda y(t),\nu\geq 1 \qquad (8)$$

and boundary condition

$$-y(1)=\lambda y'(1) \qquad (9)$$

where for the operator A holds the above conditions. Suppose that the operator-valued function $q(t)$ is weakly measurable, $\|q(t)\|$ is bounded on $[0,1]$ and the following conditions are satisfied:

1) There exist weak derivatives of $q(t)$ till second order on $[0,1]$ and $q^{(j)}(t)$ $(j=0,1,2)$ are selfadjoint operators in H for each $t\in[0,1]$ $\left[q^{(j)}(t)\right]^*=q^{(j)}(t)$, $q^{(j)}(t)\in\sigma_1(H)$.

where $\sigma_1(H)$ is a trace class, i.e., a class of compact operators in separable Hilbert space H, whose s-numbers form a convergent series. Let denote the norm in $\sigma_1(H)$ by $\|\cdot\|_1$.

2) The functions $\left\|q^{(j)}(t)\right\|_1$ $(j=0,1,2)$ are bounded on $[0,1]$.

3) $\int_0^1 (q(t)f, f)dt = 0$ for each $f \in H$.

4) There exists the operator T which satisfies the following conditions: $T = T^* \in \sigma_1(H)$ and for each $f \in H$ on the vicinity of zero the next inequality holds $\left|\left(q^{(j)}(t)f, f\right)\right| \leq |(Tf, f)|$ $(j=0,1,2)$.

The scalar product in L_2 defined as

$$(Y,Z)_{L_2} = \int_0^1 (y(t), z(t))dt + (y_1, z_1)$$

For $q(t) \equiv 0$ in direct sum L_2 associate with problem (8),(9) the selfadjoint operator L_0 defined as:

$$D(L_0) = \{Y \in L_2, l[y] \in L_2(H,(0,1)),\ y_1 = y'(1)\}$$

$$L_0 Y = \{-y''(t) + \frac{\nu^2 - \frac{1}{4}}{t^2} y(t) + Ay(t), -y(1)\}$$

The operator L_0 has a discrete spectrum. The self-adjoint expansions of the minimal operator are determined by the boundary conditions at point 1. The operator corresponding to the case $q(t) \not\equiv 0$ denote by $L = L_0 + Q$ and this operator is selfadjoint in L_2, where Q is selfadjoint bounded operator and $Q\{y(t), y'(1)\} = \{q(t)y(t), 0\}$.

Call the sum $\sum_{i=1}^{\infty}\left(\mu^{(i)} - \lambda^{(i)}\right)$ a regularized trace of the operator L. In 2.2 an asymptotic formula for the eigenvalues of the operator L_0 is found. By virtue of the spectral expansion of the operator A we obtain the following transcendental equation

$$\left(z^2 + \gamma_k\right) z J_{\nu-1}(z) + \left(1 + \frac{z^2 + \gamma_k}{2}(1 - 2\nu)\right) J_\nu(z) = 0,\ z = \sqrt{\lambda - \gamma_k} \quad (10)$$

Lemma 9. For the eigenvalues of L_0 the following asymptotic is true: $\lambda_{m,k} = \gamma_k + \alpha_m^2$, $\alpha_m \sim \left(\pi m + \frac{\nu\pi}{2} + \frac{\pi}{4} \right), m \to \infty$.

Lemma 10. Equation (10) has only real roots.

Let's denote the real roots of this equation by $x_{m,k}$ $\left(k = \overline{1,\infty}\right)$.

Take a rectangular contour C with vertices at the points $\pm iB, A_m \pm iB$, which bypasses the origin along small semicircle on the right side of imaginary axis. Here B is a large positive number and $A_m = \pi m + \frac{\nu\pi}{2} - \frac{\pi}{4}$

Lemma 11. For a sufficiently large integer m , the number of zeros of the function $z^{-\nu}\left(\left(z^2 + \gamma_k\right) z J_\nu'(z) + \left(1 + \frac{z^2 + \gamma_k}{2}\right) J_\nu(z) \right)$ inside of C is equal to m .

Theorem 8. Let $A = A^* > E$ in H, A^{-1} be compact and eigenvalues of A satisfy the relation $\gamma_k \sim ak^\alpha, k \to \infty, a > 0, \alpha > 0$. Then

$$\lambda_n(L_0) \sim \mu_n(L) \sim d_1 n^\delta \quad , \quad \delta = \begin{cases} \frac{2\alpha}{\alpha + 2}, \alpha > 2, \\ \frac{\alpha}{2}, \alpha < 2, \\ 1, \alpha = 2. \end{cases}$$

In 2.3 the regularized trace of the operator L_0 is calculated. The orthonormal eigenvectors are found as:

$$\frac{1}{J_\nu\left(x_{m,k}\right)} \sqrt{\frac{8x_{m,k}^2\left(x_{m,k}^2 + \gamma_k\right)^2}{H\left(x_{m,k}\right)}} \times$$

$$\times\left\{\sqrt{t}J_\nu(x_{m,k}t)\varphi_k,\left(\frac{1}{2}J_\nu(x_{m,k})+x_{m,k}J'_\nu(x_{m,k})\right)\varphi_k\right\},$$

$$H(x_{m,k})=4x_{m,k}^6+8x_{m,k}^4\gamma_k+x_{m,k}^4-4x_{m,k}^4\nu^2+4x_{m,k}^2\gamma_k^2-$$
$$-8x_{m,k}^2\gamma_k\nu^2+2x_{m,k}^2\gamma_k+12x_{m,k}^2-4\nu^2\gamma_k^2+4\gamma_k+4+\gamma_k^2.$$

The following lemma is proved.

Lemma 12. Suppose that for operator-valued function $q(t)$ holds the conditions 1)-4). If $\alpha>0$, then

$$\sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\left|\int_0^1\frac{8x_{m,k}^2\left(x_{m,k}^2+\gamma_k\right)^2 tJ_\nu^2(x_{m,k}t)(q(t)\varphi_k,\varphi_k)}{H(x_{m,k})J_\nu^2(x_{m,k})}dt\right|<\infty.$$

The main result of this chapter is given in the next theorem.

Theorem 9. Let the conditions of Theorem 8 be held. If the operator-valued function $q(t)$ satisfies the conditions 1-4, then

$$\lim_{m\to\infty}\sum_{n=1}^{n_m}(\mu_n-\lambda_n)=\frac{2\nu\, tr\, q(0)+3\, tr\, q(1)}{4}.$$

In chapter III, the second regularized trace is calculated for the Sturm-Liouville equation with unbounded operator coefficient with spectral parameter dependent boundary condition. This chapter consists of 4 subchapters. There was considered the operator generated by differential expression (1) and boundary conditions

$$y(0)=0\quad,\quad y(1)=-\lambda y'(1)$$

in Hilbert space $L_2(H,(0,1))$. Assume that operator-valued function $q(t)$ is weakly measurable and the following conditions hold:

1) There exist weak derivatives till fourth order on $[0,1]$ denoted by $q^{(k)}(t)$ which are from $\sigma_1(H)$. $\left\|q^{(k)}(t)\right\|_{\sigma_1(H)}\le const$ $\left(k=\overline{0,4}\right)$ are bounded for each $t\in[0,1]$, also $Aq^{(k)}(t)\in\sigma_1(H)$, $\left\|Aq^{(k)}(t)\right\|_{\sigma_1(H)}\le const$, $k=\overline{0,2}$.

2) $q'(0)=q'(1)=q(1)=0;$

3) $\int_0^1 (q(t)f, f)dt = 0$ is true for each $f \in H$.

In 3.1 the operators are generated by differential expression (1) and boundary conditions where the spectral parameter involved. Associate with this problem for $q(t) \equiv 0$ the operator L_0 defined as

$$D(L_0) = \{Y : Y = \{y(t), y_1\} / \, y(t) \in D(A), - y''(t) + Ay(t) \in L_2(H,(0,1)),$$
$$y(0) = 0,\ y_1 = -y'(1)\},\quad L_0 Y = \{-y''(t) + Ay(t),\ y(1)\}.$$

($y'(t)$ is obsolutely continuous in norm of H)

Denote the operator corresponding to the case $q(t) \not\equiv 0$ by $L = L_0 + Q$. Here $Q\{y(t), -y'(1)\} = \{q(t)y(t), 0\}$). The scalar product in L_2 defined as: $(Y,Z)_{L_2} = \int_0^1 (y(t), z(t))dt + (y_1, z_1)$,

$$Y = \{y(t), y_1\},\ Z = \{z(t), z_1\},\ y(t), z(t) \in L_2(H,(0,1))\ y_1, z_1 \in H\ .$$

Obviously, the operators L and L_0 have discrete spectrum. Let the eigenvalues of these operators be $\lambda_1 \le \lambda_2 \le ...$ and $\mu_1 \le \mu_2 \le ...$ Denote the resolvent of the operator L_0^2 by R_λ^0. Provided $L_0 Q L_0^{-1}$ is bounded, Q is defined as above and for $N > \frac{1}{2\omega}$ (where $\omega \in [0;1)$, $\omega < \frac{1}{2} - \frac{2+\alpha}{4\alpha}$) we have[2]

$$\lim_{m\to\infty}\left(\sum_{n=1}^{n_m}\left(\lambda_n^2 - \mu_n^2\right) + \right.$$
$$\left. + \frac{1}{2\pi i}\int_{\Gamma_m}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k} tr\left[\left(L_0 Q + Q L_0 + Q^2\right)R_0(\lambda)\right]^k d\lambda\right) = 0$$

The following is called the second regularized trace of the operator L

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[2] Садовничий, В.А., Подольский, В.Е. Следы операторов с относительно компактным возмущением // – Москва: Математический сборник, -2002. 193 (2), –с. 129–152.

$$\lim_{m\to\infty}\left(\sum_{n=1}^{n_m}\left(\lambda_n^2-\mu_n^2-\int_0^1 trq^2(t)dt\right)+\right.$$
$$\left.+\frac{1}{2\pi i}\int_{\Gamma_m}\sum_{k=2}^{N}\frac{(-1)^{k-1}}{k}tr\left[\left(L_0Q+QL_0+Q^2\right)R_0(\lambda)\right]^k d\lambda\right)$$

Denote it by $\sum_{n=1}^{\infty}\left(\lambda_n^{(2)}-\mu_n^{(2)}\right)$. In 3.2 is given some auxiliary facts. In 3.3 is found the second regularized trace formula for the considered problem.

Theorem 10. Let $q(t)$ be an operator-function with properties 1-3, $L_0^{-1}QL_0$ be bounded operator in L_2 and $\gamma_k \sim gk^{\alpha}$, $g>0, \alpha>2$ then

$$\sum_{n=1}^{\infty}\left(\lambda_n^{(2)}-\mu_n^{(2)}\right)=-\frac{trq^2(0)}{4}-\frac{trAq(0)-trAq(1)}{2}+\frac{trq''(0)-trq''(1)}{8}.$$

The following lemma is proved.

Lemma 13. If hold properties 1,2, and $\gamma_k \sim gk^{\alpha}$, $0<g<\infty$, $2<\alpha<\infty$, then the following series is absolutely convergent

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{\left(x_{k,n}^2+\gamma_k\right)2x_{k,n}\int_0^1\cos(2x_{k,n}t)(q(t)\varphi_k,\varphi_k)dt}{2x_{k,n}-\sin 2x_{k,n}+4x_{k,n}^3\cos^2 x_{k,n}}\right|+$$
$$+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{4x_{k,n}\int_0^1\sin^2(x_{k,n}t)\left(q^2(t)\varphi_k,\varphi_k\right)dt}{2x_{k,n}-\sin 2x_{k,n}+4x_{k,n}^3\cos^2 x_{k,n}}-\int_0^1\left(q^2(t)\varphi_k,\varphi_k\right)dt\right|<\infty.$$

In 3.4, the second regularized trace is calculated for problem (1),(2),(5) .

Lemma 14. Provided that hold the conditions 1,2 and $\gamma_k \sim gk^{\alpha}$, $0<g<\infty$, $2<\alpha<\infty$, then (here a is a constant in (5)):

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{4ax_{k,n}\int_0^1\cos^2(x_{k,n}t)g_k(t)dt}{2ax_{k,n}+a\sin 2x_{k,n}+4x_{k,n}^3\sin^2 x_{k,n}}-\int_0^1 g_k(t)dt\right|+$$

$$+\sum_{k=N}^{\infty}\left|\frac{4ax_{k,0}\int_0^1\cos^2(x_{k,0}t)g_k(t)dt}{2ax_{k,0}+a\sin 2x_{k,0}+4x_{k,0}^3\sin^2 x_{k,0}}-\int_0^1 g_k(t)dt\right|+$$

$$+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{(x_{k,n}^2+\gamma_k)2ax_{k,n}\int_0^1\cos(2x_{k,n}t)f_k(t)dt}{2ax_{k,n}+a\sin 2x_{k,n}+4x_{k,n}^3\sin^2 x_{k,n}}\right|+$$

$$+\sum_{k=N}^{\infty}\left|\frac{(x_{k,0}^2+\gamma_k)2ax_{k,0}\int_0^1\cos(2x_{k,0}t)f_k(t)dt}{2ax_{k,0}+a\sin 2x_{k,0}+4x_{k,0}^3\sin^2 x_{k,0}}\right|<\infty.$$

Theorem 11. Let $L_0^{-1}QL_0$ be bounded operator in L_2 and $\gamma_k \sim gk^{\alpha}, g>0, \alpha>2$ Provided operator-valued function $q(t)$ satisfies conditions 1-3, then the following is true:

$$\sum_{n=1}^{\infty}\left(\lambda_n^{(2)}-\mu_n^{(2)}\right)=\frac{trq^2(0)}{4}+\frac{trAq(0)+trAq(1)}{2}-$$

$$-\frac{trq''(0)+trq''(1)}{8}-\int_0^1 trq^2(t)dt.$$

CONCLUSIONS

The dissertation work is devoted to the investigation of spectral properties and finding the regularized traces for differential equations with unbounded operator coefficients. The following results were obtained in the dissertation work:

- ✓ The spectral properties of the Sturm-Liouville operator equation with spectral parameter dependent are investigated, the asymptotic formula for the eigenvalues is obtained.
- ✓ The first and second regularized traces are calculated for a special case of that problem.
- ✓ The asymptotic nature of the eigenvalues was studied for the differential operator with unbounded operator coefficient, where spectral parameter is included in the boundary condition as a rational function and a regularized trace formula was found.
- ✓ The spectral problem for the Bessel equation with unbounded operator coefficient was investigated, where the spectral parameter involved in front of the derivative of the function. Also the asymptotic distribution formula was obtained for the eigenvalues.
- ✓ .The formula for the regularized trace of the operator Bessel equation is obtained.
- ✓ The second regularized trace formula for the Sturm-Liouville equation with unbounded operator coefficient, with spectral parameter dependent boundary condition was obtained.

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2. Aslanova, N.M., Movsumova, H.F. On asymptotics of eigenvalues for second order differential operator equation //- Baku: Caspian Journal of Applied Mathematics, Ecology and Economics, -2015. 3 (2), -p. 96-105.
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condition // - Baku: Journal of Contemporary Applied Mathematics, -2016. 6 (2) , -p. 33-47.
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11. Aslanova, N.M., Movsumova, H.F. On some spectral properties of differential operator with unbounded operator coefficient // - Baku: Journal of Contemporary Applied Mathematics , -2022. 12 (1), -p. 3-16.
12. Movsumova, H.F. Trace formula for differential equation with coefficient containing first degree polynomial depending on spectral parameter // 6 th İnternational HYBRID Conference on Mathematical Advances and Applications, - İstanbul: -10-13 MAY, - 2023, -p.110

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