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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

STUDY OF SOME ILL-POSED PROBLEMS FOR SECOND ORDER HYPERBOLIC EQUATIONS BY OPTIMAL CONTROL THEORY METHODS

Specialty: 3338.01-System analysis, control and information processing

Field of science: Mathematics

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GENERAL CHARACTERISTICS OF THE WORK

Rationale and development degree of the dissertation. Beginning from the middle of the past century, because of their applied importance, ill-posed and inverse problems began to be studied and the results were applied in physics, geophysics, medicine, ecology and in other fields of science. At present, ill-posed problems can be found in such fields of mathematics as differential equations, mathematical physics, computational mathematics, etc. It is known that every ill-posed problem can be interpreted as a problem inverse to a certain well-posed problem. In the well-posed the functions that describe various physical phenomen and processes are found. For solving a well-posed problem in mathematical physics, the domain where the process occurs, the coefficients of the equation, boundary conditions, and in non-stationary process the initial conditions also are given. But in many cases, these quantities are unknown. Then there appear such inverse or ill-posed problems that according to information on the solution of the well-posed problem it becomes necessary to find these unknown quantities. So, along with the solution of a boundary value problem considered in inverse or ill-posed problems, some functions contained in the well-posed problem also become unknown.

A.N.Tikhonov, M.M.Lavrentyev, V.G.Romanov, V.K.Ivanov, V.V.Vasin, S.I.Kabanikhin, R.Lattes, J.L.Lions, A.D.Iskenderov, A.Y.Akhundov, M.I.Belishev, A.S.Blagoveshensky, K.T.Iskakov, A.G.Yagola, F.P.Vasiliev, A.Hasanov and others have been studied in inverse and ill-posed problems.

There exist various methods for solving ill-posed problems: regularization method, quasiinversion method, quasisolution method, gradient methods and others. The last years, one of these methods, the variation or optimization method is studied more intensively and applied in many problems. Variational statement of certain inverse and ill-posed problems was given in the works of K.R.Ayda-zadeh, V.M.Abdullayev, O.M.Alifanov, Y.A.Artyukhin, S.V.Rumyantsev, S.I.Kabanikhin, K.T.Iskakov, A.D.Iskandarov, R.G.Tagiyev and others for partial equations and these problems were studied.

One of these method is that the considered ill-posed or inverse problems are reduced to some optimal control problems and these problems are studied by the methods of optimal control theory. This time, in the considered boundary value problem, the coefficients of the equation, boundary or initial functions play as a controll and the function whose minimum is sought, is constructed by means of additional information. This functional is called an incompatibility functional as well. If the minimum value of this functional is zero, then an additional condition in the ill-posed or inverse problem is satisfied.

Variational statements of inverse problems for parabolic type equations were exstensively researched and studied in the works of V.M. Abdullayev, O.M. Alifanov, E.A. Artyukhin, S.V.Rumyantsev, A.D.İskenderov, R.K.Tagıyev, R.A.Gasymov, İ.K.Shakenov. Variational statements of inverse problems for elliptic equation were studied for a problem of finding the coefficient of the small term in the equations under study in the works of A.D. Iskenderov, R.S. Gasymova, the problem of finding the coefficient in the main part of the equation in the works of R.K. Tagiyev, R.S. Gasymov, the problem of finding the coefficient of the small term in the Helmholts equation in the works of A.B. Rahimov, G. Ferrandin. Reducing the inverse problem of finding the coefficient of the small term for hyperbolic type equation to an operator equation in the Hilbert space, construction of quadratic functional by means of this operator equation, the problem of minimization of the constructed functional were extensively studied in the works of S.K. Kabanikhin, the variational statements for the considered equation for the problem of finding the right hand side of the nonlocal boundary condition wave equation, in the works of H.F. Guliyev, Y.S. Gasymov, H.T. Tagiyev, the problem of finding the high coefficient of the equation and finding the coefficient in the acoustics problem for a string vibration equation, in the works of H.F. Guliyev, V.N. Nasibzade.

It should be noted that different and important optimal control problems for partial equations were studied by K.R.Ayda-zadeh, J.L.R.Arman, S.S.Nakhiyev, G.T.Ahmedov, A.G.Butkovsky, F.P.Vasilev, K.G.Hasanov, A.I.Egorov, Y.V.Egorov, A.D.Iskenderov, A.Z.Ishmuhammedov, Y.S.Gasimov, V.Komkov, H.F.Guliyev, J.L.Lions, K.A.Lourie, K.B.Mansimov, T.G.Melikov, M.J.Mardanov, B.I.Plotnikov, M.A.Sadygov, S.Y.Seravayski, T.G.Sirazetdinov, V.I.Sumin, R.G.Tagiyev, M.H.Yagubov, Y.A.Sharifov, Sh.Sh.Yusubov, Z.I.Khalilov, Y.Sokolovsky, M.B.Suryanarayana, T.Zollezzi, E.Zuazua and others.

In the submitted dissertation work, the problem of finding the coefficients of boundary value problems for second order nonlinear hyperbolic equations and the problem of finding initial functions in boundary value problems for second order linear hyperbolic equations were reduced to optimal control problems and the obtained problems were studied by the methods of optimal control theory. At the end of the dissertation work two concrete problems were solved by the numerical method. Taking into account was has been said above, the topic of the dissertation work is urgent.

Object and subject of the study. The main object of the presanted dissertation work are boundary value problems for second order hyperbolic equations, inverse problem and optimal control problems. The subject of the work are approaches based on reduction of finding the coefficient of the equations and initial functions to an optimal control problem and methods for solving optimal control problems.

Goals and objectives of the study. To reduce the problem of finding some coefficients in boundary value problems for second order nonlinear hyperbolic equations and the problem of finding initial functions in boundary value problems for linear hyperbolic equations to appropriate optimal control problems, to study the obtained problems by means of the methods of optimal control theory, to derive optimality conditions, to solve the problem numerically composing solution algorithms by means of optimality conditions.

Research methods. In the dissertation work, the methods of mathematical theory of optimal control and optimization, the methods of mathematical physics, functional analysis and computational mathematics are used.

Main results to be defended.

- to find the coefficients of nonlinear hyperbolic second order equations and to reduce the problems of definition of initial functions in boundary value problems for linear hyperbolic equations to optimal control problems;
- to study the obtained optimal control problems;
- to prove that the aim functionals are differentiable and to obtain expressions for their gradients;
- to derive conditions for optimality in the form of variational inequalities;
- to give an algorithm for solving optimal control problems in two cases by means of derived optimality conditions and to carry out experiments on their numerical solution.

Scientific novelty of the research.

- ➢ Finding the coefficients of second order nonlinear hyperbolic equations and reducing the problems of definition of initial functions in boundary value problems for linear hyperbolic equations to optimal control problems;
- ➢ The obtained optimal control problems were studied;
- ➢ Differentiability of aim functionals was proved and expressions for their gradients were obtained;
- ➢ Optimality conditions were find in the form of variational inequalities;
- ➢ Conditions for optimality in the form of variational inequlities were derived;
- ➢ By means of the derived optimality conditions an algorithm for solving optimal control problem in two cases was given and experiments an their numerical solution were carried out.

Theoretical and practical importance of the research.

The obtained results are mainly of theoretical character. The methods of the work can be applied for other partial equations. The practical importance of the work is that the obtained results can be used in approximate solution of different inverse and ill-posed problems and control problems in wave and vibration processes.

Approbation of the work. The main results of the dissertation work were reported in the following scientific seminars and conferences: in the seminars of the chair of "Electronics and information technologies" (head: ass. prof. M.E.Aliyev), of the chair "Informatics" (head: assos. prof. G.A.Rahimova) of Nakhchivan State University, in the seminars of the chair of "Mathematical methods of control theory" of Baku State University (head: prof. H.F.Guliyev) at the republican scientific conference "Actual problems of mathematics and mechanics" devoted to 100-th anniversary of corr-member of ANAS, the known scientist, doctor of physical-mathematical sciences, prof. Goshgar Teymur oglu Ahmedov (Baku-2017), at the IV International scientific conference "Actual problems of applied mathematics" (Nalchik-2018), at the V International scientific conference "Nonlocal boundary value problems and related problems of mathematical biology, informatics and physics" (Nalchik-2018), the 6th International conference on "Control and Optimization with Industrial Applications" (Baku-2018), "III actual problems of Physics, Mathematics and Astronomy" Republican Scientific Conference (Nakhchivan 2023).

Authors personal contribution. All the obtained results and suggestions belong to the author.

Authors publications. The results of the dissertation work were published in 14 works, the list of publications is at the end of the dissertation work.

The name of the institution where the dissertation work was performed. The work was performed at the chair of "Electronics and information technologies" of Nakhchivan State University.

Total volume of the dissertation work indicating separately the volume of each structural units. The title page - 395 signs, contents-2797, introduction-31746, chapter I- 80527, chapter II – 72107, chapter III – 20670, total volume of the work consists of 208242 signs.

THE CONTENT OF THE WORK

The dissertation work consists of introduction, 3 chapters, the list of used references and appendix.

The rationale of the topic is justified and short content of the work is presented in the introduction.

Chapter I consists of four sections and deals with reducing the problems of finding the coefficients of second order nonlinear hyperbolic equations to optimal control problems and their study.

In section **1.1.** we consider the finding of the pair $(u(x,t), v(x)) \in U \times V$ from the relations

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + |u|u + vu = f(x,t), \quad (x,t) \in Q, \tag{1}$$

$$u|_s = 0, \quad u|_{t=0} = u_0(x), \quad \frac{\partial u}{\partial t}|_{t=0} = u_1(x), \quad x \in \Omega, \tag{2}$$

$$u(x,T) = \varphi(x), \quad x \in \Omega \tag{3}$$

ın these relations Δ is a Laplace operator with respect to the variable x, $f(x,t) \in L_2(Q)$, $u_0(x) \in \overset{0}{W}{}_2^1(\Omega)$, $u_1(x) \in L_2(\Omega)$, $\varphi(x) \in \overset{0}{W}{}_2^1(\Omega)$ are the given functions, Q=Ω×(0,T) is a cylinder in R^{n+1}, $\mathrm{T} > 0$ is a given number. Ω is a bounded domain in R^n with rather smooth boundary in $\partial\Omega$ (n = 3,4), $S = \partial\Omega \times (0,T)$ is the lateral side of the cylinder Q ,

$$U = \left\{u: \quad u \in L_\infty(0,T; \overset{0}{W}{}_2^1(\Omega)), \frac{\partial u}{\partial t} \in L_\infty\left(0,T;\ L_2(\Omega)\right)\right\},$$

$$V = \{v: v \in L_2(\Omega),\ a \leq v(x) \leq b \ almost\ everywhere\ on\ \Omega\}, \tag{4}$$

$a,\ b$ are the given numbers, $a < b$.

It should be noted that problem (1)-(3) is a problem inverse to the direct problem (1), (2) and the direct problem has a unique generalized solution from the class U. Let us reduce this problem to an optimal control problem: find the function that affords minimum to the functional

$$J_0(v) = \frac{1}{2}\int_\Omega [u(x,T;v) - \varphi(x)]^2 dx \tag{5}$$

in the class of functions V, here the function $u(x,t;v)$∈U is the solution of problem (1), (2) corresponding to $v = v(x)$. We call the function $v(x)$ a control, the class V a class of admissible controls.

There is a close relation between the problems (1)-(3) and (1), (2), (4), (5). If the minimum value of the functional (5) is 0, then the condition (3) is satisfied. For aviding degeneration in the obtained necessary condition of optimality, we consider the following problem. Find a control from the class V affording a minimum value to the functional

$$J_\alpha(v) = J_0(v) + \frac{\alpha}{2}\int_\Omega |v(x)|^2\, dx\,,\ \ \alpha > 0 \qquad (6)$$

within the conditions (1), (2), here α is a given number.

Theorem 1. Let in the problem (1),(2),(4),(6) $f(x,t) \in L_2(Q)$, $u_0(x) \in \overset{0}{W}{}_2^1(\Omega)$, $u_1(x) \in L_2(\Omega)$, $\varphi(x) \in \overset{0}{W}{}_2^1(\Omega)$ are the given functions. Then the optimal controls set

$$V_* = \left\{ v_* \in V : J_\alpha(v_*) = \inf_{v\in V} J_\alpha(v) \right\}$$

of this problem is not empty, is weakly compact in $L_2(\Omega)$ and arbitrary minimizing sequence $\{v_k(x)\}$ weakly converges to V_* in $L_2(\Omega)$.

Theorem 2. Assume that in the problem, (1), (2), (4), (6) the conditions of theorem 1 are satisfied. Then the functional (6) is Frechet continuously differentiable in the set V and its differential with the increment $\delta v \in L_4(\Omega)$ at the point $v \in V$ is determined by the expression

$$\langle J'_\alpha(v), \delta v\rangle = \int_\Omega [\alpha v - \int_0^T u\psi dt]\delta v dx\,,$$

here the function $\psi = \psi(x,t;v)$ is the solution of the adjoint problem

$$\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + 2|u|\psi + v\psi = 0\,, \qquad (x,t) \in Q, \qquad (7)$$

$$\psi|_s = 0, \psi|_{t=T} = 0,\ \ \frac{\partial\psi}{\partial t}|_{t=T} = -[u(x,T;v) - \varphi(x)],\ x \in \Omega \quad (8)$$

from the space U.

Theorem 3. Assume that in the problem, (1), (2), (4), (6) the condition of theorem 1 are satisfied. Then necessary condition for the optimality of the control $v = v_*(x) \in V$ in this problem is satisfaction of the inequality

$$\int_\Omega \left[\alpha\, v_*(x) - \int_0^T u_*(x,t)\ \psi_*(x,t)dt\right] \left(v(x) - v_*(x)\right)dx \geq 0$$

for arbitrary control $v \in V$, here $u_*(x,t) = u(x,t;v_*)$ and $\psi_*(x,t) = \psi(x,t;\ v_*)$ are the solutions of the boundary value problems (1), (2) and (7), (8), respectively.

In section **1.2** the problem on finding the pair $(v(x), u(x,t)) \in V \times U$ from the relations

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - u^3 + v\frac{\partial u}{\partial x} = f(x,t), (x,t) \in Q = (0,l) \times (0,T), \quad (9)$$

$$u(0,t) = 0, \quad u(l,t) = 0, \quad 0 \le t \le T, \quad (10)$$

$$u|_{t=0} = u_0(x), \quad \frac{\partial u}{\partial t}|_{t=0} = u_1(x), \quad 0 \le x \le l, \quad (11)$$

$$u(x,T) = \varphi(x), \quad 0 \le x \le l, \quad (12)$$

$$u \in L_6(Q) \quad (13)$$

is considered, here

$$V=\left\{v \in W_2^1(0,l): \alpha \le v(x) \le \beta, \ \left|\frac{dv(x)}{dx}\right| \le \mu \text{ almost everywhere on } (0,l)\right\} \quad (14)$$

$$U=\left\{u: u \in L_\infty(0,T; \overset{0}{W}{}_2^1(0,l)), \ \frac{\partial u}{\partial t} \in L_\infty(0,T; L_2(0,l))\right\}$$

$l > 0, T > 0, \alpha, \beta, \mu > 0$ are the given numbers, $f \in L_2(Q), u_0 \in \overset{0}{W}{}_2^1(0,l)$, $u_1 \in L_2(0,l)$, $\varphi \in \overset{0}{W}{}_2^1(0,l)$ –are the given functions.

Note that for the functions $v(x), f(x,t), u_0(x), u_1(x)$ of the problem (9)-(11) in general, there is no global solution with respect to t[1]. Therefore we should a priori look for the set of pairs $\{v,u\}$ satisfying the relations (9)-(11), (13), (14). The pair $\{v,u\}$ is called a pair of control-state[1]. If the relations (9)-(11), (13), (14) are satisfied for the pair $\{v,u\}$, we call it an admissible pair. For each $v(x) \in V$ the function $u(x,t)$ satisfying the relations (9)-(11), (13) is a generalized solution of the boundary value problem (9)-(11) from the class U.

Assume that the set of admissible pairs is not empty.

Now, let us consider the following problem: find the minimum of the functional

$$J(v,u) = \frac{1}{2}\int_0^l [u(x,T) - \varphi(x)]^2 dx + \frac{1}{6}\int_Q (u - u_d)^6 dxdt \quad (15)$$

in the set of admissible pairs, here $u_d \in L_6(Q)$ is a given function.

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[1] Lions J.L. Control of singular distributed systems. M.: Nauka 1987, -368 p.

Theorem 4. Assume that in the problem (9)–(11), (13), (14), (15) $f \in L_2(Q)$, $u_0 \in \overset{0}{W}{}_2^1(0,l)$, $u_1 \in L_2(0,l)$, $\varphi \in \overset{0}{W}{}_2^1(0,l)$, $u_d \in L_6(Q)$ are the given functions. Then this problem has an optimal pair.

Theorem 5. Assume that the conditions of theorem 4 are satisfied and $\{v^0, u^0\}$ is an optimal pair. Then there exists such a function $\psi^0(x,t)$ that for the triple $\{v^0, u^0, \psi^0\}$ the following relations are valid:

$$\psi^0 \in L_\infty(0,T; W_2^{\frac{2}{3}}(0,l)), \quad \frac{\partial \psi^0}{\partial t} \in L_\infty(0,T; W_2^{-\frac{1}{3}}(0,l))$$

and

$$\frac{\partial^2 u^0}{\partial t^2} - \frac{\partial^2 u^0}{\partial x^2} - (u^0)^3 + v^0 \frac{\partial u^0}{\partial x} = f, \quad (x,t) \in Q,$$

$$u^0(0,t) = 0, u^0(l,t) = 0, 0 \le t \le T,$$

$$u^0(x,0) = u_0(x), \frac{\partial u^0(x,0)}{\partial t} = u_1(x), 0 \le x \le l,$$

$$\frac{\partial^2 \psi^0}{\partial t^2} - \frac{\partial^2 \psi^0}{\partial x^2} - 3(u^0)^2\psi^0 - \frac{\partial}{\partial x}(v^0\psi^0) = (u^0 - u_d)^5, (x,t) \in Q,$$

$$\psi^0(0,t) = 0, \quad \psi^0(l,t) = 0, 0 \le t \le T,$$

$$\psi^0(x,T) = 0, \frac{\partial \psi^0(x,T)}{\partial t} = -[u^0(x,T) - \varphi(x)], \ 0 \le x \le l,$$

$$\int_0^l \left(\int_0^T \psi^0(x,t) \frac{\partial u^0(x,t)}{\partial x} dt\right)\left(v(x) - v^0(x)\right)dx \le 0 \ \forall v \in V,$$

here $W_2^{\frac{2}{3}}(0,l)$, $W_2^{-\frac{1}{3}}(0,l)$ are fractional order Sobolev spaces[2].

In section **1.3** we consider a problem optimal of control of the coefficients of the first order derivatives in the equation of vibrations of the membrane with discontinuity of solution.

Assume that the process is described by the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u - u^2 + v_1(x)\frac{\partial u}{\partial x_1} + v_2(x)\frac{\partial u}{\partial x_2} = f(x,t), \ (x,t) \in Q \quad (16)$$

satisfying initial and boundary conditions

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[2] Lions L.J., Magenes E. Nonehomogeneous boundary value problems and their applications. M.: Nauka 1971, -371 p.

$$u|_S = 0,\ u|_{t=0} = u_0(x),\ \left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1(x),\ x \in \Omega \qquad (17)$$

in the cylinder $Q = \Omega \times (0, T)$, here $\Omega \subset R^2$ is a domain with smooth boundary $\partial\Omega$, $S = \partial\Omega \times (0, \mathrm{T})$, is lateral surface of the cylinder Q, $T > 0$ is a given number $x = (x_1, x_2) \in \Omega, f(x,t) \in L_2(Q)$, $u_0(x) \in \overset{0}{W}{}_2^1(\Omega), u_1(x) \in L_2(\Omega)$ are the given functions, $\Delta u \equiv \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}$ is a Laplace operator, $v(x) = \big(v_1(x), v_2(x)\big)$ is a control function.

Assume that,

$$\mathrm{V} = \left\{ v(x) = \big(v_1(x), v_2(x)\big) : v_i(x) \in \mathrm{C}^1\big(\overline{\Omega}\big), |v_i(x)| \le \mu_i, \left|\frac{\partial v_i}{\partial x_j}\right| \le \mu_{ij} \right\} \quad (18)$$

$$v(x) \in V,\ i, j = 1, 2$$

and the set V is closed in the of the metrics $C^1\big(\overline{\Omega}\big)$ here $\mu_i, \mu_{ij}, i, j = 1, 2$ are the given positive numbers.

Note that the problem (16), (17) has no global solution for the functions $v(x)$, $f(x,t)$, $u_0(x)$, $u_1(x)$ with respect to t[1]. Therefore, we must consider the set of pairs $\{v, u\}$

$$u \in L_4(Q) \qquad (19)$$

a priori satisfying (16)-(18) and the relations. The pair $\{v, u\}$ is said to be a control-state pair[1]. If the relations (16), (17), (18), (19) are satisfied, then the pair $\{v, u\}$ is called an admissible pair.

Assume that the set of pairs $\{v, u\}$ is not empty. (20)

Note that if the condition (19) is satisfied, then the boundary value problem (16), (17) has a generalized solution[2,3], from the space

$$U = \left\{ u : u \in L_\infty(0, T; \overset{0}{W}{}_2^1(\Omega)), \frac{\partial u}{\partial t} \in L_\infty\big(0, T;\ L_2(\Omega)\big) \right\},$$

i.e. at $t = 0$ for the condition $u(x, 0) = u_0(x)$ arbitrary function $\eta \in U, \eta(x, T) = 0$ the integral identity

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[3]Ladyzhenskaya O.A. Boundary value problem of mathematical physics. M.: Nauka, 1973, -371 p.

$$\int_Q \Big[-\frac{\partial u}{\partial t}\frac{\partial \eta}{\partial t}+\frac{\partial u}{\partial x_1}\frac{\partial \eta}{\partial x_1}+\frac{\partial u}{\partial x_2}\frac{\partial \eta}{\partial x_2}-u^2\eta+v_1(x)\frac{\partial u}{\partial x_1}\eta+$$

$$+v_2(x)\frac{\partial u}{\partial x_2}\eta\Big]dxdt-\int_\Omega u_1(x)\eta\,(x,0)dx=\int_Q f\eta dxdt$$

is satisfied.

Let us consider the following problem: find an admissible pair $\{v,u\}$ affording a minimum value to the functional

$$J(v,u)=\frac{1}{4}\|u-u_d\|^4_{L_4(\mathrm{Q})}+\frac{1}{2}\|u(x,T)-\varphi(x)\|^2_{L_2(\Omega)} \quad (21)$$

in the set of admissible pairs, here $u_d\in L_4(\mathrm{Q})$, $\varphi(x)\in L_2(\Omega)$ are the given functions.

Theorem 6. Assume that the data of the problem (16), (17), (21) satisfy the conditions (18)-(20) and the conditions $u_d\in L_4(\mathrm{Q})$, $\varphi(x)\in L_2(\Omega)$. Then problem (16), (17), (21) has an optimal pair, i.e. $\mathrm{J}(v^0,u^0)=min\,\mathrm{J}(v,u)$, here $\{v,u\}$ are admissible pairs.

In the considered problem we introduce adapted penalty functional to the optimal pair $\{v^0,u^0\}$:

$$J^a_\varepsilon(v,u)=J(v,u)+$$

$$+\frac{1}{2\varepsilon}\left\|\frac{\partial^2 u}{\partial t^2}-\Delta u-u^2+v_1\frac{\partial u}{\partial x_1}+v_2\frac{\partial u}{\partial x_2}-f\right\|^2_{L_2(Q)}+$$

$$+\frac{1}{2}\|u-u^0\|^2_{L_2(Q)}+\frac{1}{2}\left[\|v_1-v_1^0\|^2_{L_2(\Omega)}+\|v_2-v_2^0\|^2_{L_2(\Omega)}\right], \quad (22)$$

here the functions v,u satisfy the conditions

$$v\in V,\;\; u\in L_4(Q),\;\; \frac{\partial^2 u}{\partial t^2}-\Delta u\in L_2(Q),$$

$$u|_S=0,\;\; u(x,0)=u_0(x),\;\; \frac{\partial u(x,0)}{\partial t}=u_1(x),\;\; x\in\Omega, \quad (23)$$

$\varepsilon>0$ *is a* penalty parameter.

Theorem 7. Assume that the above conditions imposed on the data of problem (22), (23) are satisfied. Then for each $\varepsilon>0$ the problem (22), (23) has on optimal pair, i.e.

$$J^a_\varepsilon(v^0_\varepsilon,u^0_\varepsilon)=\min_{\{v,u\}}J^a_\varepsilon(v,u).$$

Theorem 8. Assume that the above conditions are fulfilled in the problem (16)-(19) and in this problem $\{v^0, u^0\}$ is an optimal pair. Then there exist such a triple $\{v^0, u^0, \psi\}$ that the following relations are satisfied:

$$\frac{\partial^2 u^0}{\partial t^2} - \Delta u^0 - (u^0)^2 + v_1^0 \frac{\partial u^0}{\partial x_1} + v_2^0 \frac{\partial u^0}{\partial x_2} = f(x,t), (x,t) \in Q, \quad (24)$$

$$\frac{\partial^2 \psi}{\partial t^2} - \Delta \psi - 2u^0 \psi - \frac{\partial}{\partial x_1}\left(v_1^0 \psi\right) - \frac{\partial}{\partial x_2}\left(v_2^0 \psi\right) = (u - u_d)^3, (x,t) \in Q, (25)$$

$$u^0|_S = 0,\ u^0(x,0) = u_0(x),\ \frac{\partial u^0(x,0)}{\partial t} = u_1(x), \quad x \in \Omega\ , \quad (26)$$

$$\psi|_S = 0,\ \psi(x,T) = 0,\ \frac{\partial \psi(x,T)}{\partial t} = -[u(x,T) - \varphi(x)]\ , x \in \Omega\ , \quad (27)$$

$$u^0 \in L_\infty(0,T; \overset{0}{W}{}_2^1(\Omega)),\ \frac{\partial u^0}{\partial t} \in L_\infty\left(0,T; L_2(\Omega)\right), \quad (28)$$

$$\psi \in L_\infty(0,T; W_2^{\frac{1}{3}}(\Omega)),\ \frac{\partial \psi}{\partial t} \in L_\infty(0,T; W_2^{-\frac{2}{3}}(\Omega)), \quad (29)$$

here $W_2^{\frac{1}{3}}(\Omega), W_2^{-\frac{2}{3}}(\Omega)$ are fractional order Sobolev spaces,

$$\int_Q \left[\psi \frac{\partial u^0}{\partial x_1}(v_1 - v_1^0) + \psi \frac{\partial u^0}{\partial x_2}(v_2 - v_2^0)\right] dxdt \geq 0 \quad \forall v \in V. \quad (30)$$

Note that the problem (25)-(27) is called a problem adjoint to the problem (16), (17), (21).

In section **1.4** we consider a speed action problem for a second order nonlinear hyperbolic equation. Let us consider the system whose state is described by the equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + |u|u + vu = f(x,t), \quad (x,t) \in Q \quad (31)$$

and boundary and initial conditions

$$u|_s = 0,\ u|_{t=0} = u_0(x),\ \frac{\partial u}{\partial t}|_{\,t=0} = u_1(x), \quad x \in \Omega\ , \quad (32)$$

here the conditions imposed on the data are the same as in section 1.1, the control function $v{=}v(x,t)$ is taken from the class of admissible controls

$V{=}\{v(x,t){:}\ v \in L_4(Q),\ a{\le}v(x,t){\le}b$ Q almost everywhere$\}$,

here a, b are the given numbers.

The generalized solution of the boundary value (31), (32) is contained in the class U in section 1.1.

Assume that the data of the problem (31), (32) satisfy the conditions $f(x,t) \in L_2(Q), u_0(x) \in \overset{0}{W}{}_2^1(\Omega), u_1(x) \in L_2(\Omega), Q = \Omega \times (0,T)$ is a cylinder in $R^{n+1}, T > 0$ is a given number. Ω is a bounded domain R^n (n= 3,4), its boundary $\partial\Omega$ is rather smooth.

Let us consider the following problem: find a pair $(v,\tau) \in V \times (0,T)$ that can reduce the system (31), (32) from the initial state $(u_0(x), u_1(x))$ to the given set K as soon as possible, here the set

$$K \text{ is a weak closed set in } \overset{0}{W}{}_2^1(\Omega) \times L_2(\Omega). \tag{33}$$

Assume there exits such a pair $(v,\tau) \in V \times (0,T)$ the condition

$$\left\{u(x,\tau;v), \frac{\partial u(x,\tau;v)}{\partial t}\right\} \in K \tag{34}$$

is fulfilled for the appropriate solution $u(x,t;v)$ of the problem (31), (32).

In the considered problem, optimal time is determined from the condition

$$\tau_0 = \inf\{\tau\}\ , \tag{35}$$

i.e. the moment τ_0 is the exact lower bound of all τ satisfying the condition (34).

Theorem 9. Assume that in problem (31), (32) the functions $f(x,t) \in L_2(Q), u_0(x) \in \overset{0}{W}{}_2^1(\Omega), u_1(x) \in L_2(\Omega)$ and the conditions (33),(34) are satisfied. Then there exists such a pair $(v_0,\tau_0) \in V \times (0,T)$ that $\left\{u(x,\tau_0;v_0), \frac{\partial u(x,\tau_0;v_0)}{\partial t}\right\} \in K$ and condition (35) is satisfied.

Theorem 10. Assume that in problem (31)-(34) the conditions of theorem 9 are satisfied. In addition, assume that in the special case, the set K is in the form $\{0, \chi_1(x)\}$, here $\chi_1(x) \in L_2(\Omega)$. Then a necessary condition for the pair $(v_*,\tau_*) \in V \times (0,T)$ to be optimal in the speed -in-action problem is to satisfy the inequality

$$\int_0^{\tau_*}\int_\Omega u_*(x,t)\psi_*(x,t)\big(v(x,t) - v_*(x,t)\big)dxdt \ +$$
$$+ \left(1 - \int_\Omega \frac{\partial u_*(x,t)}{\partial t}\frac{\partial \psi_*(x,t)}{\partial t}dx\right)(\tau - \tau_*) \geq 0 \tag{36}$$

for arbitrary pair $(v,\tau) \in V \times (0,T)$. Here $u_*(x,t)$ is the solution of problem (31), (32) for (v_*, τ_*), while $\psi_*(x,t)$ is any nontivial generalized solution of the adjoint problem

$$\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + 2|u|\psi + v\psi = 0, \ \ (x,t) \in (0,\tau), \quad (37)$$

$$\psi|_s = 0, \ \psi(x,T) = 0, \ \ x \in \Omega \quad (38)$$

is any generalized non-trivial solution of the adjoint problem.

Note that as for $t = \tau$ there is no condition on $\frac{\partial\psi\,(x,t)}{\partial t}$, the problem (37), (38) has infinitely many solutions. In addition, the expression for the gradient of the functional $J(v,\tau) = \tau(v)$ is in the form

$$J'(v,\tau) = \left(u(x,t)\psi\,(x,t), (1 - \int_\Omega \frac{\partial u\,(x,t)}{\partial t}\frac{\partial\psi\,(x,t)}{\partial t}dx)\right) \in L_2(\mathrm{Q}) \times L_\infty(0,T).$$

Chapter 2 consisting of 4 sections was devoted to defining initial functions in a mixed problem for a second order linear hyperbolic equation.

In section **2.1** we study definition of initial functions with respect to the observed value of boundary functions of a second order hyperbolic equation.

In section **2.2** we study initial control problem with respect to two intermediate observation moment in a mixed problem stated for a linear hyperbolic equation.

Assume that it is required to find the minimum of the functional

$$J(v) = \frac{1}{2}\int_\Omega\{[u(x,t_1;v) - z_1(x)]^2 + [u(x,t_2;v) - z_2(x)]^2\}dx \quad (39)$$

within the conditions

$$\frac{\partial^2 u}{\partial t^2} + Au = f(x,t), (x,t) \in Q, \quad (40)$$

$$u\,|_s = 0, u\,(x,0) = \varphi_0(x), \ \frac{\partial u\,(x,0)}{\partial t} = \varphi_1(x), x \in \Omega\,, \quad (41)$$

here $T > 0$ is a given number, $t_1, t_2 \in (0,T)$, $t_1 < t_2$ are arbitrary moments of time, $z_1(x)$, $z_2(x) \in \overset{0}{W}{}_2^1(\Omega)$ are the given functions, $\varphi_0 \in \overset{0}{W}{}_2^1(\Omega)$, $\varphi_1 \in L_2(\Omega)$ – are the unknown functions that determine the initial stage of the process; $\Omega \subset R^n$, $Q = \Omega \times (0,\mathrm{T})$, $S = \partial\Omega \times (0,T)$ are the domains from section 2.1.

$Au \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u$ is a differential expression, so that $a_{ij} \in C^1(\overline{\Omega})$, $a_0 \in C(\overline{\Omega})$ are the given functions, $a_{ij}(x)=a_{ji}(x), x \in \Omega,\ i,j = 1,\dots,n$ and for all $x \in \overline{\Omega}$, $\forall \xi \in R^n$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{n} \xi_i^2, \quad \alpha = const > 0,\ a_0 \geq 0.$$

Assume that at arbitrary moments of time $t_1, t_2 \in (0,T)$, $t_1 < t_2$ we observe the state $u(x,t)$ of the process:

$$u(x,t_1) = z_1(x),\ u(x,t_2) = z_2(x),\ \ x \in \Omega\,. \qquad (42)$$

According to these observations, it is required to restore the initial state $v(x) = (\varphi_0(x), \varphi_1(x))$ of the process.

İn theory of inverse problems this problem is called a retrospective inverse problem.There is a close relation between the retrospective problem and problem (39)-(41). If in the problem (39)-(41) $\min_{v\in V} J(v) = 0$ then in retrospective problem the additional condition (42) is satisfied.

Note that for the given control $v \in V \equiv \overset{0}{W}{}_2^1(\Omega) \times L_2(\Omega)$ a unique generalized solution of problem (40), (41) is in the class U.

Theorem 11. Assume that the above conditions imposed on the data of problem (39)-(41) are satisfied and $\sqrt{\lambda_k}(t_2 - t_1) \neq \pi m, m,k \in N$, here $\lambda_k > 0$ are eigenvalues of the spectral problem $AX = \lambda X,\ X|_{\partial\Omega} = 0$. Then $\inf_{v\in V} J(v) = 0$.

In what follows, for avoding admissible degeneration on the optimality condition to be obtained, instead of the functional (39) we take the functional

$$J_\beta(v) = J(v) + \frac{\beta}{2}\|v\|^2_{\overset{0}{W}{}_2^1(\Omega)\times L_{2(\Omega)}}, \quad \beta = const > 0 \qquad (43)$$

and consider the following optimal control problem: to minimize the functional (43) within the conditions (40), (41) in the convex, closed set $V_m \subset H = \overset{0}{W}{}_2^1(\Omega) \times L_2(\Omega)$.

Theorem 12. Assume that the above conditions imposed on the data of problem, (40), (41), (43) are satisfied. Then the functional (43) is continuously Frechet differentiable in the set V, and its differential with the increment $\delta v \in H$ at the point $v \in V_m$ is defined by the following expression:

$$\langle J'_\beta(v),\delta v\rangle=\int_\Omega\left\{\left[-\frac{\partial\psi(x,0;v)}{\partial \mathrm{t}}+\beta\varphi_0(x)\right]\delta\varphi_0(x)+\right.$$

$$\left.+[\psi(x,0;v)\ +\beta\varphi_1(x)]\delta\varphi_1(x)+\beta\sum_{i=1}^{k}\frac{\partial\varphi_0}{\partial x_i}\frac{\partial\delta\varphi_0}{\partial x_i}\right\}dx.$$

Theorem 13. Assume that the conditions imposed on the data of the problem, (40), (41), (43) are satisfied. Then in this problem the necessary and sufficient condition for the control $v_*=v_*(x)=\left(\overset{*}{\varphi}_0(x),\overset{*}{\varphi}_1(x)\right)\in V_m$ to be optimal is the fulfillement of the inequality for the control $\forall v(x)=\left(\varphi_0(x),\varphi_1(x)\right)\in V_m$

$$\int_\Omega\left\{\left[-\frac{\partial\psi_*(x,0)}{\partial \mathrm{t}}+\beta\overset{*}{\varphi}_0(x)\right]\left(\varphi_0(x)-\overset{*}{\varphi}_0(x)\right)+\right.$$

$$\left.+\left[\psi_*(x,0)+\beta\overset{*}{\varphi}_1(x)\right]\left(\varphi_1(x)-\overset{*}{\varphi}_1(x)\right)+\beta\sum_{\mathrm{i}=1}\frac{\partial\overset{*}{\varphi}_0(x)}{\partial x_i}\left(\frac{\partial\varphi_0(x)}{\partial x_i}-\frac{\partial\overset{*}{\varphi}_0(x)}{\partial x_i}\right)\right\}dx\geq 0,$$

here $\psi_*(x,t)=\psi(x,t;v_*)$ is a generalized solution of the adjoint problem

$$\frac{\partial^2\psi}{\partial t^2}+A\psi=0,(x,t)\in Q,$$

$$\psi(x,T)=0,\frac{\partial\psi(x,T)}{\partial t}=0,x\in\Omega,$$

$$\psi(x,t_i+0)-\psi(x,t_i-0)=0,\ x\in\Omega,\ i=1,2,$$

$$\frac{\partial\psi(x,t_i+0)}{\partial t}-\frac{\partial\psi(x,t_i-0)}{\partial t}=-[u(x,t_i;v)-z_i(x)],\quad x\in\Omega,\quad i=1,2,$$

$$\psi\mid_S=0$$

from $W^1_{2,0}(Q)$.

In section **2.3** an optimization method is considered for a second order linear hyperbolic equation in the Dirichlet problem. Let us consider the Dirichlet problem

$$\frac{\partial^2 u}{\partial t^2}+A(t)u=f(x,t),\quad (x,t)\in Q, \tag{44}$$

$$u\mid_S=0,\ \ u(x,0)=u_0(x),\ \ u(x,T)=g(x),\ \ x\in\Omega, \tag{45}$$

here $Q=\Omega\times(0,T)$ is a cylinder in R^{n+1} , Ω is a bounded domain with a smooth boundary $\partial\Omega$ in R^n, $S=\partial\Omega\times(0,\mathrm{T})$ is a lateral surface of the cylinder Q, $T>0$ is a given number,

$$A(t)u = -\sum_{i.j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) + a_0(x,t)u$$

is differential expression, the functions $a_{ij}(x,t)$, $i,j = 1, \dots, n$, $a_0(x,t)$ are measurable bounded in Q and for almost all $(x,t) \in Q$ the conditions $a_{ij}(x,t)=a_{ji}(x,t)$, $i,j=1,\dots,n$, $\left|\frac{\partial a_{ij}(x,t)}{\partial t}, a_0(x,t)\right| \le \mu$ and for $\forall \xi \in R^n$ the conditions $\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \ge \nu \sum_{i=1}^{n} \xi_i^2$ are satisfied, μ, ν are positive constants, $f \in L_2(Q)$, $u_0 \in \overset{0}{W}{}_2^1(\Omega)$, $g \in \overset{0}{W}{}_2^1(\Omega)$ are the given functions.

It is known,[4,5] that the Dirichlet problem (44), (45) is an ill-posed problem. This ill-podsed problem can be expressed as inverse problem to a well-posed initial boundary value problem. Indeed, in the problem

$$\frac{\partial^2 u}{\partial t^2} + A(t)u = f(x,t), \ \ (x,t) \in Q, \qquad (44)$$

$$u|_{s} = 0, \ u(x,0) = u_0(x), \ \ \frac{\partial u(x,0)}{\partial t} = v(x), \ \ x \in \Omega \qquad (46)$$

if $v(x)$ is given, this problem is well-posed. Now, let us assume that in (46) $v(x)$ is unknown. Assume that for defining the function $v(x)$ the additional information

$$u(x,T) = g(x), \ x \in \Omega \qquad (47)$$

is known.

Here the inverse problem consists of finding the function $v(x)$ from the relations (44), (46), (47) with respect to the given function $f(x,t)$, $u_0(x), g(x)$.

Let us consider the operator L acting from the space $L_2(\Omega)$ to the space $L_2(\Omega)$:

$$L: v(x) = \frac{\partial u(x,0)}{\partial t} \to g(x) = u(x,T).$$

Then we can write the inverse problem in the operator form as

$$Lv = g \ . \qquad (48)$$

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[4] Kabanikhin S.I. Inverse and ill-posed problems. Novosibirsk, 2009,-457 p.

[5] Lattes R., Lions J.L. Quasilineerinversion method and its applications. M.: Mir, 1970,-336 p.

We can solve the problem (48) by minimizing the aim functional

$$J(v) = \frac{1}{2}\|Lv - g\|^2_{L_{2(\Omega)}}. \qquad (49)$$

If in the problem (44), (46), (49) there is such a control $v_* = v_*(x) \in L_2(\Omega)$ that, $J_* = J(v_*) = \inf_{v \in L_2(\Omega)} J(v) = 0$ then the controll $v_* = v_*(x)$ and the solution $u_* = u(x,t;v_*)$ of appropriate problem (44), (46) is the solution of the inverse problem (44), (46), (47).

Theorem 14. Assume that the conditions imposed on the data of the problem (44), (46), (49) satisfy the above conditions. Then the functional (49) is Frechet continuously $L_2(\Omega)$ and its differentiable at the point $v \in L_2(\Omega)$ is determined by the expression

$$\langle J'(v), \delta v\rangle = -\int_\Omega \psi(x,0;v)\delta v(x)\,dx,$$

here

$$J'(v) = -\psi(x,0;v)$$

is the gradient of the functional and the function $\psi(x,t;v)$
is the solution of the adjoint problem

$$\frac{\partial^2\psi}{\partial t^2} + A(t)\psi = 0, \qquad (x,t) \in Q\ , \qquad (50)$$

$$\psi|_S = 0\ , \psi(x,T;\ v) = 0\ , \frac{\partial\psi(x,T;v)}{\partial t} = [u(x,T;v) - g(x)], x \in \Omega\,. \qquad (51)$$

Theorem 15. Suppose that the conditions of theorem 14 are satisfied. Then necessary and sufficient condition for the optimality of the control $v_* = v_*(x) \in L_2(\Omega)$ is the fulfillment of the equality

$$\psi(x,0;v_*) = 0\ , \qquad (52)$$

here the function $\psi(x,t;v_*)$ is the solution of the adjoint problem (50), (51) for $v = v_*(x)$.

It is shown that the norm of the above introduced operator L is bounded.

The method of the speedest descent can be applied to the minimization problem (44), (46), (49). This method consists of calculation of the approximations $v_k(x)$ by the rule[6]

$$v_{k+1}(x) = v_k(x) - \alpha J'(v_k) = v_k(x) + \alpha\psi(x,0,v_k),\ k = 0,1,\dots\ , \qquad (53)$$

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[6]Васильев Ф.П. Методы рещения эксремальных задач. М.: Наука, 1981,-400 с.

here $v_0(x)$ is an initial approximation. $J'(v)$ is the gradient of the functional (49), the descent parameter α_k is determined from the condition

$$\varphi_k(\alpha_k) = \inf_{\alpha \geq 0} \varphi_k(\alpha), \varphi_k(\alpha) = J(v_k - \alpha J'(v_k)).$$

Theorem 16. Assume that the conditions of theorem 14 are satisfied. Then the following statements for $\{v_k(x)\}$ the sequence determined by the rule (53) is valid: the sequence $\{J(v_k)\}$ is monoton descreasing, $\lim_{k\to\infty} \|J'(v_k)\| = \lim_{k\to\infty} \|\psi_k(x, 0;\ v_k)\|_{L_2(\Omega)} = 0$, the sequence $\{v_k(x)\}$ minimizes the functional (49) in $L_2(\Omega)$, in $L_2(\Omega)$ converges weakly to the set $V_* = \left\{v_* \in L_2(\Omega) :\ J_* = \inf_{v \in L_2(\Omega)} J(v)\right\}$ and the estimation $0 \leq J(v_k) - J_* \leq \frac{C}{k}$, k=1,2,... is valid.

If in the problem (44), (46), (49) a control $v(x)$ is sought in the convex closed set V contained in $L_2(\Omega)$, then the condition for optimality takes the following form

$$\int_\Omega \psi(x, 0; v_*)\big(v(x) - v_*(x)\big)dx \leq 0 \quad \forall v \in V \qquad (54)$$

the formula (53) is replaced by the formula

$$v_{k+1}(x) = P_V\big(v_k(x) - \alpha_k J'(v_k)\big),$$

here P_V is the operator that projects the point $v \in L_2(\Omega)$ on the set V, as α_k we can take the number satisfying the condition $0 < \varepsilon_0 \leq \alpha_k \leq \frac{2}{\mathcal{L} + 2\varepsilon}$, $\mathcal{L}$ is a Lipshits constant of $J'(v)$, the pozitive numbers ε_0, ε –are the parameters of the method.

If in the problem (44), (46), (49) there is a control $v_* = v_*(x) \in V \subseteq L_2(\Omega)$ (V-is a convex, closed set) that $J_* = J(v_*) = 0$, then since in the problem (50), (51) $u(x, T; v_*) = g(x)$, the solution of this problem is $\psi(x, t) = 0$. This case is called a degenarated case. In this case, the condition (52) and (54) are satisfied trivially. For non trivial case, instead of the functional $J(v)$ we can consider the functional

$$J_\beta(v) = J(v) + \frac{\beta}{2}\|v\|^2_{L_2(\Omega)},\ \ \beta = const > 0\,. \qquad (55)$$

İf we consider the problem on finding the minimum in the convex closed set V contained in $L_2(\Omega)$, then the optimality condition (54) is replaced by the condition

$$\int_\Omega [-\psi(x,0;v_*) + \beta v_*(x)](v(x) - v_*(x))dx \geq 0 \ \ \forall v \in V,$$

here $v_*(x) \in V$ is a control that affords a minimum to the functional (55), $\psi(x,t;v_*)$ is the solution corresponding to the control $v = v_*(x)$ of the adjoint problem (50), (51).

In section **2.4** we consider a problem on reducing mixed condition boundary value problem for a wave equation to an optimal control problem and study it.

Here it is assumed that in the domain $Q = \Omega \times (0,T)$ the process is described by the following boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}, \quad (x_1, x_2, t) \in Q, \tag{56}$$

$$u|_{t=0} = u_0(x_1,x_2), \ \ u|_{t=T} = u_1(x_1,x_2), \ \ (x_1,x_2) \in \Omega, \tag{57}$$

$$\frac{\partial u}{\partial x_1}|_{x_1=0} = \frac{\partial u}{\partial x_1}|_{x_1=l_1} = 0, \quad (x_2,t) \in (0,l_2) \times (0,T), \tag{58}$$

$$\frac{\partial u}{\partial x_2}|_{x_2=0} = \frac{\partial u}{\partial x_2}|_{x_2=l_2} = 0, \quad (x_1,t) \in (0,l_1) \times (0,T), \tag{59}$$

here $\Omega=(0,l_1)\times(0,l_2)$ are rectangles, $l_1 > 0$, $l_2 > 0$, $T > 0$ are the given numbers, $u_0 \in W_2^1(\Omega)$, $u_1 \in W_2^1(\Omega)$ are the given functions. It is clear that, (56)-(59) is an ill-posed problem[4]. The second one from the conditions (57) is replaced by the condition

$$\frac{\partial u}{\partial t}|_{t=0} = v(x_1,x_2), \ \ (x_1,x_2) \in \Omega$$

and we consider the following optimal control problem: find a function $v(x_1,x_2) \in L_2(\Omega)$ that together with (56), (58), (59) and the function $u(x_1,x_2,t;v)$ satisfying the conditions

$$u|_{t=0} = u_0(x_1,x_2), \ \frac{\partial u}{\partial t}|_{t=0} = v(x_1,x_2), \ \ (x_1,x_2) \in \Omega \tag{60}$$

affords minimum to the functional

$$J_0(v) = \frac{1}{2}\int_\Omega [u(x_1,x_2,T;v) - u_1(x_1,x_2)]^2 \, dx_1 dx_2.$$

In this section, at first it is proved that $\inf_{v \in L_2(\Omega)} J_0(v) = 0$. Then for the optimality condition to be nondegenerate, we consider the following problem: find the minimum of the functional

$$J_\alpha(v) = J_0(v) + \frac{\alpha}{2}\|v\|^2_{L_2(\Omega)}, \ \alpha>0 \text{ (here } \alpha \text{ is given number)} \tag{61}$$

in the closed, convex set $V_m \subset L_2(\Omega)$ within the conditions (56), (58), (59), (60).

Theorem 17. Assume that the conditions imposed on the data of the problem, (56), (58)-(61) are satisfied. Then the function (61) is Frechet differentiable in $L_2(\Omega)$ and its differential is determined by the formula

$$\langle J'_\alpha(v), \delta v\rangle = \int_\Omega [-\psi(x_1,x_2,0;v) + \alpha v(x_1,x_2)]\ \delta v(x_1,x_2)dx_1dx_2,$$

here $\psi(x_1,x_2,t;v)$ is the solution of the following adjoint problem:

$$\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial {x_1}^2} + \frac{\partial^2\psi}{\partial {x_2}^2}, (x_1,x_2,t) \in Q,$$

$$\psi|_{t=T} = 0, \left.\frac{\partial\psi}{\partial t}\right|_{t=T} = [u(x_1,x_2,T;v) - u_1(x_1,x_2)], (x_1,x_2) \in \Omega,$$

$$\left.\frac{\partial\psi}{\partial x_1}\right|_{x_1=0} = \left.\frac{\partial\psi}{\partial x_1}\right|_{x_1=l_1} = 0, (x_2,t) \in (0,l_2)\times(0,T),$$

$$\left.\frac{\partial\psi}{\partial x_2}\right|_{x_2=0} = \left.\frac{\partial\psi}{\partial x_2}\right|_{x_2=l_2} (x_1,t) \in (0,l_1)\times(0,T).$$

Theorem 18. Assume that the conditions of theorem 17 are fulfilled. Then the necessary and sufficient condition for the control $v_*=v_*(x_1,x_2)\in V_m$ to be an optimal control in the problem (56), (58)-(61) is the fulfillement of the inequality

$$\int_\Omega [-\psi(x_1,x_2,0;v_*) + \alpha v_*(x_1,x_2)]\big(v(x_1,x_2) - $$
$$ - v_*(x_1,x_2)\big)dx_1dx_2 \geq 0 \quad \forall v \in V,$$

here $\psi(x_1,x_2,t;v_*)$ is the solution for the above adjoint problem for $v = v_*(x_1,x_2)$.

Chapter III of the dissertation work consists of 2 sections and is deals with to numerical solutions of some model problems.

In section **3.1** we consider a problem of finding the pair $\big(u(x,t), v(x)\big) \in U \times V$ from the relations

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + |u|u + vu = f(x,t),\ \ 0 < x < l,\ \ 0 < t < T, \quad (62)$$

$$u(0,t) = 0,\quad u(l,t) = 0,\ \ 0 \leq t \leq T, \quad (63)$$

$$u(x,0) = u_0(x),\ \frac{\partial u(x,0)}{\partial t} = u_1(x),\quad 0 \leq x \leq l, \quad (64)$$

$$u(x,T) = \varphi(x),\quad 0 \leq x \leq l, \quad (65)$$

here $f(x,t), u_0(x),\ u_1(x),\ \varphi(x)$ are the given functions, $l > 0$, $T > 0$ are the given numbers,

$$U = \left\{u(x,t):\ u \in L_\infty(0,T;\overset{0}{W}{}_2^1(0,l)), \frac{\partial u}{\partial t} \in L_\infty(0,T;L_2(0,l))\right\},$$

$$V = \{v(x):\ v \in L_2(0,l),\ \ a \le v(x) \le b, x \in (0,l)\},$$

a, b –are the given numbers.

Note that, the problem (62)-(65) is an problem inverse to the well-posed problem (62)-(64). We reduce this problem the following optimal control problem: to find the function affording a minimum to the functional $J_0(v) = \frac{1}{2}\int_0^l [u(x,T;v) - \varphi(x)]^2\,dx$ in the class V, here the function $u(x,t;v)$ is a solution of the problem (62)-(64) corresponding to $v = v(x)$. We call $v(x)$ a control function, the class V a class of admissible functions. To avoid admissible degeneration in the optimality conditions, we consider the following problem: find a control from the class V giving a minimum value to the functional

$$J_\beta(v) = J_0(v) + \frac{\beta}{2}\int_0^l |v(x)|^2 dx \qquad \beta > 0$$

within the conditions (62)-(64), here β is a given number.

To solve this problem numerically, we assume that the data of the problem are rather smooth, give an algorithm for solving the problem and apply the gradient projection method.

Furthermore, by solving the main and adjoint boundary value problems by the grid method, the considered problems were solved numerically.

Results of numerical exsperiments, graphs and appropriate tables were given.

In section **3.2** we consider the following problem.

From the class of admissible control

$$V = \left\{v(x): \alpha \le v(x) \le \beta, \left|\frac{dv}{dx}\right| \le \mu\right\}$$

we must find a control, that together with the solution of the boundary value problem

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - u^3 + v(x)\frac{\partial u}{\partial x} = f(x,t), 0 < x < l, 0 < t < T,$$

$$u(0,t) = 0, u(l,t) = 0, 0 \le t \le T,$$

$$u(x,0) = u_0(x), \frac{\partial u(x,0)}{\partial t} = u_1(x), 0 \le x \le l$$

affords a minimum value to the functional

$$J(v,u) = \frac{1}{2}\int_0^l [u(x,T;v) - \varphi(x)]^2 dx + \frac{1}{6}\int_0^l \int_0^T [u(x,t;v) - u_d(x,t)]^6 dxdt,$$

here l, T, α, β, μ are the given numbers $f(x,t), u_0(x), u_1(x), \varphi(x)$, $u_d(x,t)$ −are the given functions.

To solve this problem numerically, it is assumed that the data of the problem are rather smooth, an algorithm for solving the problem is given. Using the gradient projection method, by solving the main and adjoint problems by the grid methods, the considered problem is solved numerically.

The results of the numerically experiment, the graphs and appropriate tables were given.

At the end I express my deep gratitute to my supervisor doctor of physical-mathematical sciences prof. H.F.Guliyev for the problem statement and his constant attention to the work.

CONCLUSIONS

In the dissertation work the problems of finding the coefficients in boundary value problems for some second order nonlinear hyperbolic equations and the problem of finding initial functions in boundary value problems for linear hyperbolic equations were reduced to optimal control problems and the obtained problems were solved by the methods of optimal control theory.

The following main results were obtained:

- the problems of finding the coefficients of second order nonlinear hyperbolic equations and problems of finding initial functions in boundary value problems for linear hyperbolic equations were reduced to optimal control problems;
- the obtained optimal control problems were studied;
- differentiability of the aim functions was proved and expressions for their gradients were obtained;
- optimality conditions were find in the form of variational inequalities;

➢ conditions for optimality in the form of variational inequlities were derived;

➢ by means of the derived optimality conditions an algorithm for solving optimal control problem in two cases was given and experiments on their numerical solution were carried out.

The main results of the dissertation work were published in the following works:

1. **Quliyev H.F., Səfərova Z.R.** Dalğa tənliyi üçün qarışıq sərhəd şərtləri olan məsələnin optimal idarəetmə məsələsinə gətirilməsi və onun tədqiqi // AMEA-nın müxbir üzvü, tanınmış alim və görkəmli riyaziyyatçı, fizika-riyaziyyat elmləri doktoru, professor Qoşqar Teymur oğlu Əhmədovun anadan olmasının 100 illik yubileyinə həsr olunmuş "Riyaziyyat və mexanikanın aktual problemləri" adlı Respublika elmi konfransı, -Bakı: 02- 03 noyabr, -2017, -s. 74-75.
2. **Кулиев Г.Ф., Сафарова З.Р**. Метод оптимального управления в задаче Дирихле для гиперболического уравнения второго порядка // Bakı Universitetinin xəbərləri, Fizika-riyaziyyat elmləri seriyası, № 4,-2017, -s.21-28.
3. **Quliyev H.F., Səfərova Z.R.** Dalğa tənliyi üçün qarışıq şərtli sərhəd məsələnin optimal idarəetmə məsələsinə gətirilməsi və onun tədqiqi //Naxçıvan Dövlət Universiteti, Elmi əsərlər, Fizika-riyaziyyat və texnika elmləri seriyası, №8 (89), -2017, -s.45-51.
4. **Кулиев Г.Ф., Сафарова З.Р.** Вариационный подход к решению одной коэффициентно–обратной задачи для гиперболического уравнения// IV Международная научная конференция «Актуальные проблемы прикладной математики», -Нальчик-Эльбрус: 22-26 мая 2018 г., с.146.
5. **Guliyev H.F., Safarova Z.R**. On a determination of the initial functions from the observed values of the boundary functions for the second-order hyperbolic equation // Jomard Publishing, Advanced Mathematical Models and Applications, vol. 3, №3, -

2018, -p. 215-222.

6. **Guliyev H.F., Safarova Z.R.** About the problem of finding the coefficients of the derivative in the string oscillation equations with the solution which has discontinuity // The 6th International Conference on “Control and Optimization with Industrial Applications”, -Baku: 11-13 July, -2018, -p. 125-127. **(WoS)**
7. **Кулиев Г.Ф., Сафарова З.Р.** Об определении начальных функций в по измеренным значениям граничных функций для гиперболического уравнения // V Международной научной конференции, -Нальчик, Кабардино-Балкарская Республика: 4-7 декабря 2018 г., -с.114.
8. **Сафарова З.Р.** Об определении коэффициента при производной в уравнении колебаний струны с разрывом // Проблемы управления и информатики, Украина: Киев, №1, -2019, с.67-71.
9. **Кулиев Г.Ф., Сафарова З.Р.** Задача быстродействия относительно одного нелинейного гиперболического уравнения второго порядка // Bakı Universitetinin xəbərləri, Fizika-riyaziyyat elmləri seriyası №1, -2020, -s.55-67.
10. **Safarova Z.R.** On a finding the coefficient of one nonlinear wave equation in the mixed problem // Archvies of Control Sciences, Poland, Volume 30 (LXVI), №2, -2020, -p.199-212. **(Scopus)**
11. **Mardanov M.J., Guliyev H.F., Safarova Z.R**. The problem of starting control with two intermediate moments of observation in the boundary value problem for the hyperbolic equation // Optimal Control Appl. and Meth., 2020, V.41, issue 5, p.1-10. **(Scopus)**
12. **Səfərova Z.R.** İki qeyri-xətti hiperbolik tənlik üçün sərhəd məsələsində əmsalın tapılmasının ədədi həll üsulu haqqında // Naxçıvan Dövlət Universiteti, Elmi əsərlər, Fizika-riyaziyyat və texniki elmlər seriyası, -2020, №7(108). -s.16-19.
13. **Səfərova Z.R.** Həllinin kəsilməsi olan membranın rəqsləri tənliyində birinci tərtib törəmələrin əmsallari ilə optimal idarəetmə məsələsi // “Naxçıvan” Universiteti, Elmi əsərlər, 2021, №3, -s. 286-295.

14. **Səfərova, Z.R.** İkitərtibli bir qeyri-xətti hiperbolik tənlik üçün ən tez təsir məsələsi // “III Fizika, Riyaziyyat və Astronomiyanın aktual problemləri” adlı Respublika elmi konfransının materialları, -12 may, -2023, s.109-111.

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