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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

EXISTENCE OF SOLUTIONS OF NON-LINEAR ELLIPTIC AND PARABOLIC TYPE EQUATIONS

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Field of science:	Mathematics
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Baku-2024

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GENERAL CHARACTERISTICS OF THE WORK

Actuality of the research theme and degree of processing. Mathematical modeling of various situations and processes of mechanics, physics, hydrodynamics, and biology is brought to the study of certain problems for special derivative differential equations. For example, quasi-linear parabolic equations describe the processes of non-stationary thermal conductivity, gas and liquid movement, and at the same time appear in the mathematical modeling of the processes of heat of chemical kinetics, population growth and stability. The wide applications of such equations are due to the fact that they are derived from fundamental conservation laws (matter, momentum, energy). For this reason, a large number of books and articles have been devoted to parabolic and elliptic equations of the second order.

Mathematical modeling of some thermal conductivity processes, for example, self-ignition of coal collected in mines, air filtration in a certain environment, prediction of the process of gas separation in a closed environment, etc. oblige us to study existence, uniqueness, and stability of global solutions of non-linear differential equations.

The posibility of predictation the state of mechanical systems described by non-linear equations at any moment in time makes the issue of studying the existence of global solutions of such equations important (solutions that can be determined at any moment in time). After the famous works of Dj. Keller and R. Osserman the interest to this problem increased even more. It can be said that the interest in this type of issues has not faded since then, and a sufficient list of literature has been formed to this day. For example, we can say that H. Berestuki, and L. Nirenberg, M.F. Bidaut-Veron, J. Serrin, H. Brezis and V.A. Strauss, H. Brezis and K. Cabre, J.L. Vazquez, L. Veron, B. Gidas, J. Spruk , V.A. Kondratyev and L. Veron, V.A. Kondratyev, E.M. Landis . A large number of works by authors such as V.V. Kurta, E. Mitidieri and S.I. Pokhojayev, K. Denk and H.A Levin, H.A Levin are devoted to this field.

The existence of a global solution for non-linear equations depends on the nature of the non-linearity. Depending on the non-linearity, the solution of the equation can either be at any instant of time or go to infinity at a finite instant of time. It turns out that there is a critical value of the degree such as if the price of the degree is less than this critical value, there is no global solution of the problem, in the other case the problem may has a global solution depending on beginning conditions. The absence of a global solution means that, no matter how small the acquisition of the system is, the solution goes to infinity at a finite value of time, that is, the studied process is not stable.

One of the main criteria for not fading interest in the existence of global solutions of non-linear elliptic, parabolic, hyperbolic equations is that these types of problems have a wide application in mechanics, physics, and at the same time in some internal problems of mathematics.

At the beginning of the 20th century, while studying the equilibrium state of a polytropic gas sphere, Emden and Fowler considered a

$$\frac{d^2\Phi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} + \alpha^2\Phi^n = 0$$

non-linear equation with a singular coefficient, where is α^2 constant and Φ is a function depending on the gas density and pressure.

In another example of the semi-linear equation

$$x^{\frac{1}{2}} y'' = y^{\frac{3}{2}}$$

the issue of the existence of solutions satisfying the

$$y(0) = 1, \quad \lim_{x \to +\infty} y(x) = 0$$

boundary conditions was considered in the works of Thomas and Fermi while studying the distribution of electrons in a heavy atom.

In 1966, Fujita1 considered such a problem in his famous work:

[1] Fujita, H. On the blowing-up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$ // J. Fac. Sci.Unix, Tokyo, Sect. I, -1966. 13, -p.109-124.

$$\frac{\partial u}{\partial t} = \Delta u + u^q, \quad (q > 1), \quad (x,t) \in R^n \times (0,+\infty)$$

$$u\big|_{t=0} = u_0(x) \geq 0 .$$

It is apparent, for any $q > 1$, there is such a $T = T(u_0)$ that this problem has a classical solution in the $R^n \times (0,T)$ cylinder. The question is, when can $T = +\infty$ be?

Fujita proved that there is such a critical point $q_{cr} = 1 + \frac{2}{n}$, if $1 < q < q_{cr}$ for any $u_0 \geq 0$, $T = +\infty$ can not be and such a $T_\infty = T_\infty(u_0) < \infty$, for $t \to T_\infty - 0$, $\int\limits_{R^n} u^q(t,x)dx \to +\infty$.

At the same time he proved that, if $q > q_{cr}$ for small u_0 $T_0 = +\infty$ can be. More specifically, he showed that if $q > 1 + \frac{2}{n}$ and $u_0(x) \leq h(0,x)$, then the solution of the considered issue is limited by the above with $h = h(t,x)$ function, where is

$$h(t,x) = \left\{ \frac{n(q-1)-2}{2(q-1)(t+a)} \right\}^{\frac{1}{q-1}} e^{-\frac{|x|^2}{4(t+a)}}, \quad a = const > 0 .$$

Thus, in this case $u(t,x)$ is valuable at all prices of t and $\lim\limits_{t\to\infty} u(t,x) = 0$. If $0 < q < 1 + \frac{2}{n}$, then $u(t,x)$ set in a limited interval for t and no matter how small $u_0(t,x) \geq 0$, $u(t,x)$ increases infinitely. Japanese scientists Hayakawa and Kobayashi showed that in the case of $q = 1 + \frac{2}{n}$, $T = +\infty$ can not be, that is, there is no global solution. This result can be interpreted as if $q > 1 + \frac{2}{n}$, the solution $u \equiv 0$ is stable, if $q \leq 1 + \frac{2}{n}$ then $u \equiv 0$ is not stable.

It should be noted that the evolution equations' solution dissipation is of mathematical interest and has wide application areas.

For example, in the theory of reactors, quantum mechanics, fluid mechanics, etc.

After Fujita's famous work, the way how the size of the non-linearity affects the presence and absence of a global solution, i.e., the process of solution decay at a finite time instant, began to be widely studied. Fujita's work was expanded in various directions. Instead of R^n various bounded and unbounded domains were considered, a system of equations was taken instead of an equation, or non-linearity was studied in another form.

The question of the existence of a global solution for stationary equations is related to both application issues and arises in the study of some internal problems of mathematics.

It is apparent that $u(x) \in W^{1,2}(R^n)$, *for* $1 \le p < n$

$$\|u\|_{\frac{np}{n-p}} \le C(n,p)\|\nabla u\|_p \cdot$$

The Sobolev inequality is true, where is, $\|u\|_p = \left(\int_{R^n} |u|^p dx\right)^{\frac{1}{p}}$. If we write this inequality like this

$$\inf\left\{\|\nabla u\|_p \quad / \quad \|u\|_{\frac{np}{n-p}} = 1, \quad u \in W^{1,p}(R^n)\right\} = C^{-1}(n,p) > 0 .$$

Then two questions arise. First, what is this infimum equal to, and second, is there a specific function that can take this infimum? Talenti answered the first question and showed that

$$C(n,p) = \frac{1}{\sqrt{\pi n}}\left(\frac{p-1}{n-p}\right)^{1-\frac{1}{p}}\left[\frac{\Gamma\left(1+\frac{n}{2}\right)\cdot\Gamma(n)}{\Gamma\left(\frac{n}{p}\right)\cdot\Gamma\left(1+n-\frac{n}{p}\right)}\right]^{\frac{1}{n}} .$$

Let's find the function that takes this infimum when $p = 2$. If we calculate the variations of the corresponding functions when p=2, we get that the function $u(x)$ which takes the minimum, is in all R^n

must have a generalized solution in the Sobolev sense of the bellow equation

$$\Delta u + \lambda u^{\frac{n+2}{n-2}} = 0 .$$

In recent years, special attention has been paid to the existence and non-existence of global solutions for different classes of differential equations and inequalities. In contrast to the previous works, the presented work deals mainly with excited equations, that is, both small-order derivatives and singular potential members take part in the equations. The effect of lower order derivatives and singular potential members on the sufficient conditions for the existence of a global solution is studied.

Object and subject of research. The research object and subject of the dissertation work is to find an exact sufficient conditions that ensures the absence of a positive global solution of the second order semi-linear elliptic and parabolic equations and systems of equations with singular potential and lower order derivatives.

Goal and tasks of the research.

1. Study of the existence of global solutions of some semi-linear elliptic, parabolic type equations and the system of equations in the infinite domain;

2. Finding sufficient conditions ensuring the absence of a global solution, checking the accuracy of these conditions;

3. Studying the effect of the coefficients of the small members on valuation which ensures the absence of a global solution, in the equations including small members with singular potential.

Research methods. Some methods of functional analysis, special derivative differential equation theory and test function method which is first used by Pokhojayev and Mitidieri were used in the study of the considered issues.

The basic aspects to be defended. The following main statements are defended:

1. Finding the exact sufficient condition that ensures the absence of a positive global solution outside the sphere of the semi-linear elliptic equation with singular potential and low-order derivatives;

2. Proof of the existence of a sufficient condition for the absence of a positive global solution of the semi-linear elliptic equation with singular potential and the divergent main part in the outer region;

3. Proof of the non-existence of a solution of the semi-linear elliptic equation in the cylindrical region that satisfies the homogeneous Neumann condition on the side surface of the cylinder;

4. Finding an exact sufficient condition for the absence of a global solution of the semi-linear parabolic equation with singular potential in the cylindrical region outside the sphere;

5. Finding an exact sufficient condition for the absence of a global solution of a semi-linear parabolic equation whose main part is a Baowendi-Grushin type operator in the cylindrical region outside the sphere;

6. Finding an exact sufficient condition ensuring the absence of a global solution for a system of weakly connected semi-linear elliptic equations with a singular potential in an infinite region;

7. Finding accurate estimates for the absence of a global solution of the system of weakly connected semi-linear parabolic equations in the infinite domain.

Scientific novelty of the research. The following results were obtained:

1. An exact sufficient condition ensuring the absence of a positive global solution of some semi-linear elliptic equations in the infinite region has been found;

2. The existence of solution of the first boundary problem for the semi-linear elliptic type equation in a limited domain under certain conditions has been proved;

3. A Fujita-type result was obtained for semi-linear parabolic equations of the second order with a singular potential and divergent main part or Baowendi-Grushin operator type;

4. Exact estimates for the absence of global solutions of the system of weakly connected second-order semi-linear elliptic and parabolic equations in external regions are found.

Theoretical and practical significance of the research. The results obtained in the dissertation work are mainly theoretical in nature. These results can be used in the theory of special differential equations, in certain problems of mechanics and physics.

Approbation and application. The results obtained in the dissertation were presented at the scientific conference dedicated to the 95th anniversary of Baku State University on "Actual problems of mathematics and mechanics" (Baku, 2014), at the scientific conference dedicated to the 93rd anniversary of the birth of Heydar Aliyev, the national leader of the Republic of Azerbaijan, "Actual problems of mathematics and mechanics" (Baku, 2016), at the International Symposium on "Spectral and Evolutionary Solutions" of the XXX Crimean School of Mathematics (КРОМШ-2019), at the scientific conference on "Actual problems of mathematics and mechanics" (Baku, 2022) dedicated to the 99th anniversary of the birth of Heydar Aliyev, the national leader of the Republic of Azerbaijan and presented at the scientific seminars of the "Differential and Integral Equations" department of Baku State University.

Personal contribution of the author. All results and suggestions obtained belong to the author.

Author's publications. According to the research, 6 articles (1 is included in Web of Science Core Collection, 1 is included in SCOPUS) and 4 theses (in total 10 works) were published in the publishing houses recommended by the HCC under the President of the Republic of Azerbaijan. The list of works is given at the end of the abstract.

Name of the organization where the dissertation work is carried out. The dissertation work was carried out at the Department of "Differential and Integral Equations" of the faculty of Mechanics and Mathematics of Baku State University.

Structure and volume of the dissertation (in signs, indicating the volume of each structural subsection separately). The dissertation consists of a list of 117 titles. The total volume of his work is 196808 characters (title page - 472 symbols, table of contents - 1916 symbols, introduction 41408 symbols, first chapter - 70000 symbols, second chapter - 38000 symbols, third chapter - 44000 symbols, conclusion - 1012 symbols).

THE CONTENT OF THE WORK

The dissertation consists of an introduction, three chapters, result and a list of references.

The **first chapter** consists of four paragraphs. In the first paragraph, the issue of non-existence of a positive global solution of the second-order semi-linear elliptic equation with singular potential and lower-order derivatives is investigated. So that, in the domain of $B_R^C = \{x; |x| > R\} \subset R^n,\ n \geq 3$

$$\Delta u + \frac{C_1}{|x|^2}(x, \nabla u) + \frac{C_0}{|x|^2} u + u^q = 0 \qquad (1)$$

equation is considered, where is, $q > 1$, $C_0, C_1 = const \geq 0$,

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right),\ \Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_1^2},\ \ (x, \nabla u) = \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i}.$$

The problem of the absence of a positive global solution of the given equation is studied in the domain B_R^C. It should be noted that the issue of non-existence positive solution of the equation in the case of $C_0 = C_1 = 0$ in whole R^n was considered in the work of B.Gidas and J. Spruck2. In the work of H. Brezis, Z. Dupaigne, A. Tesein3, the case of $C_1 = 0$ the equation was considered in the $B_R = \{x; |x| < R\}$.

In this paragraph the case of $C_1 \neq 0$ is investigated. A generalized solution is assumed as the solution of the equation. Note that the generalized solution means such a $u(x)$ function that for any

$$R_1 > R,\quad u(x) \in W_2^1(B_{R,R_1}) \cap L_\infty(B_{R,R_1})$$

[2] Gidas, B., Spruck, J. A priori bounds for positive solutions of non-linear elliptic equations// Commun. Part. Differ. Eq. -1981. 6, № 8, -p. 883–901.

[3] H.Brezis, Z.Dupaigne, A.Tesei On a semilinear elliptic equation with inverse-square potential//. Selecta math. New Ser., -2005, 11, -p. 1-7.

and for any $\psi(x)\in \overset{0}{W}{}_2^1(B_{R,R_1})$ the

$$\int\limits_{B_{R,R_1}} u^q\psi dx = \int\limits_{B_{R,R_1}} (\nabla u,\nabla\psi)dx - \int\limits_{B_{R,R_1}} \frac{C_1}{|x|^2}(x,\nabla u)\psi dx - \int\limits_{B_{R,R_1}} \frac{C_0}{|x|^2}u\psi dx$$

integral identity be satisfied.

Here $B_{R,R_1} = \{x; R<|x|<R_1\}$.

Let's point

$$D=\left(\frac{n-2+C_1}{2}\right)^2 - C_0, \qquad \mu_\pm = -\frac{n-2+C_1}{2}\pm\sqrt{D}.$$

Theorem 1. Let $n\ge 3$, $q>1$, $C_0, C_1\in R$, $D\ge 0$ and $2+(q-1)\mu_- \ge 0$.

If $u(x)\ge 0$ is solution of the equation (1) in B_R^C, then $u\equiv 0$.

Then the accuracy of the theorem is shown with an example.

In the section **1.2** the problem of non-existence of a positive global solution of the second order semi-linear elliptic equation with divergent form outside the sphere is studied. So, in the region B_R^C

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\left(\delta_{ij}+\gamma\frac{x_i x_j}{|x|^2}\right)\frac{\partial u}{\partial x_j}\right)+\frac{c}{|x|^2}u+|x|^\sigma u^q=0 \qquad (2)$$

looking at the equation (2), where

$$q>1,\ \gamma>-1,\ \sigma>-2,\ 0\le\frac{c}{\gamma+1}<\frac{(n-2)^2}{4}.$$

The aim here is to show that there exists a value such q^* that, for $1<q\le q^*$ there is no non-negative solution of equation (2) in the domain B_R^C .

The global solution of the considered equation means such a $u(x)$ function that, for any $R_1>R$,

$$u(x)\in W_2^1(B_{R,R_1})\cap L_\infty(B_{R,R_1})$$

and for any $\varphi(x)\in \overset{0}{W}{}_2^1(B_{R,R_1})$

$$\int_{B_{R,R_1}} |x|^{\sigma} u^q \varphi dx = \int_{B_{R,R_1}} \sum_{i,j=1}^{n} \left(\delta_{ij} + \gamma \frac{x_i x_j}{|x|^2} \right) \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_i} dx - \int_{B_{R,R_1}} \frac{c}{|x|^2} u \varphi dx$$

integral identity be satisfied.

$$\alpha_{\pm} = -\frac{n-2}{2} \pm \sqrt{D}, \quad D = \frac{(n-2)^2}{4} - \frac{c}{\gamma+1}.$$

Teorem 2. Let $n \geq 3,\ q > 1,\ \gamma > -1,\ \sigma > -2,$

$$0 \leq \frac{c}{\gamma+1} < \frac{(n-2)^2}{4},\ 2 + \sigma + (q-1)\alpha_{-} \geq 0.$$

If function $u(x)$ is non-negative solution of equation of (2) in the domain of B_R^C , then $u \equiv 0$.

In **1.3** the issue of the absence of positive global solutions of the semi-linear elliptic equation in cylindrical domains is discussed. Here in the domain of

$$\Omega^1 = \left\{ (\hat{x}, x_n);\ \hat{x} = (x_1, \ldots, x_{n-1}),\ \ |\hat{x}|^2 = x_1^2 + \ldots + x_{n-1}^2 < 1,\ x_n > 0 \right\},$$

$$\Omega^2 = \left\{ (\hat{x}, x_n);\ |\hat{x}| < e^{-x_n},\ x_n > 0 \right\}$$

the existence of non-negative global solutions of the

$\Delta u + u^q = 0,\ q > 1$ equation satisfying the $\frac{\partial u}{\partial n} = 0$ condition on the

$$\Gamma^1 = \left\{ (\hat{x}, x_n);\ |\hat{x}| = 1,\ x_n > 0 \right\},$$

$$\Gamma^2 = \left\{ (\hat{x}, x_n);\ |\hat{x}| = e^{-x_n},\ x_n > 0 \right\}$$

side surfaces is considered, where is the n is the unit vector of the external normal drawn to the Γ^1 and Γ^2 side surfaces.

First of all

$$\Delta u + u^q = 0,\ \ q > 1,\ \ (\hat{x}, x_n) \in \Omega^1, \tag{3}$$

$$\left. \frac{\partial u}{\partial n} \right|_{\Gamma^1} = 0 \tag{4}$$

existence of a solution of the problem (3)-(4) is investigated and the following theorem is proved.

Teorem 3. For any $q > 1$ the problem (3)-(4) does not have non-negative solution in Ω^1.

Then the existence of a solution of the equation (3) satisfying the

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma^2} = 0 . \qquad (5)$$

boundary condition in area Ω^2 is investigated .

Theorem 4. For any $q > 1$ the problem (3)-(5) does not have non-negative solution in Ω^2.

In section **1.4** the existence of a solution of the boundary problem for a high order semi-linear elliptic equation is studied using the "mountain pass" theorem. In the limited area $\Omega \subset R^n$

$$\begin{cases} (-1)^k \sum_{i,j=1}^{n} \frac{\partial^k}{\partial x_i^k}\left(a_{ij}(x) \frac{\partial^k u}{\partial x_j} \right) = f(u) \\ u = 0, \quad Du = 0, \dots, D^{\alpha} u = 0, \quad |\alpha| \le k-1, \quad x \in \partial\Omega \end{cases} \qquad (6)$$

the problem is considered. It is accepted that, $n > 2k$, $\lambda_1 |\xi|^2$ - smooth function, relations

$$|f(z)| \le C(1+|z|^p), \ |f'(z)| \le C(1+|z|^{p-1}), (z \in R). \qquad (7)$$

are satisfied for a certain number p that satisfies the inequality $1 < p < \frac{n+2k}{n-2k}$.

C=const, the $a_{ij}(x)$-functions are are bounded measurable functions in Ω and there is such a number $\lambda > 0$ that, for $\forall x \in \Omega$, $\forall \xi \in R^n$

$$\lambda^{-1} |\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \le \lambda |\xi|^2$$

condition is true. At the same time it is accepted that and, there are such $\gamma < \frac{1}{2}$, $0 < \alpha \le A$ numbers, that for

$$F(z) = \int_0^z f(s)ds, \ (z \in R)$$

$$0 \le F(z) \le \gamma f(z) \cdot z \qquad (8)$$

$$\alpha |z|^{p+1} \le |F(z)| \le A |z|^{p+1} \quad (z \in R) \qquad (9)$$

inequalities are compensated.

It is apparent that, $f(0)=0$ and $u\equiv 0$ are the solutions of the problem (6). In the current paragraph, the existence of non-trivial solutions of problem (6) is studied. The solution is understood in a weak sense, so that, the solution of the problem is understood such a $u(x)$ function from $\overset{\circ}{H}{}^{k}(\Omega)$ that, for $\forall\varphi\in \overset{\circ}{H}{}^{k}(\Omega)$ the appropriate integral identity is satisfied.

Note that the function $f(u)=|u|^{p-1}\cdot u$ meets all the requirements. The obtained result consists of the following theorem.

Theorem 5. Let, the function f meets the conductions of (7), (8), (9) and $1<p<\dfrac{n+2k}{n-2k}$.

Then the problem (6) has at least one weak solution.

Note that, the case k=1, $a_{ij}=\delta_{ij}$ (δ_{ij} - is the Kronecker symbol) was considered.

The **second chapter** is devoted to the problem of the existence of a positive global solution of the semi-linear second order parabolic equation in an infinite domain and consists of two paragraphs. In this chapter, as in the previous chapter, singular potential is included in the considered equations. Note that the existence of a positive solution of the initial-boundary problem for linear parabolic equations with singular potential has been investigated by various authors. For example the work of P. Baras, J. Goldstein4.

In section **2.1** the issue of the existence of global solutions of semi-linear second-order parabolic type equations where main part is divergent type is investigated. (10)-(11) problem is considered in area Q_R' :

[4] Baras, P., Goldstein, J.A. The heat equation with a singular potential // Trans. Amer. Math. Soc. -1984. 284 , № 1, -p. 121 – 139.

$$\frac{\partial u}{\partial t}=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)+\frac{C_0}{|x|^2}u+|x|^{\sigma}|u|^{q}, \qquad (10)$$

$$u\big|_{t=0}=u_0(x)\geq 0, \qquad (11)$$

where, $\quad a_{ij}=\delta_{ij}+\gamma\frac{x_i x_j}{|x|^2}, \quad \delta_{ij}=\begin{cases}1, & i=j,\\ 0, & i\neq j,\end{cases} \quad \gamma>-1,$

$q>1, \quad \sigma>-2, \quad 0\leq C_0<(\gamma+1)\left(\frac{n-2}{2}\right)^2, \quad u_0(x)\in L_{1,loc}(B'_R).$

Note that, here

$$x\in R^n,\ n\geq 3,\ \ r=|x|=\sqrt{x_1^2+\ldots+x_n^2},$$

$$B_R=\{x;\ |x|<R\},\ B'_R=\{x;\ |x|>R\},\ Q_R=B_R\times(0,+\infty),$$

$$Q'_R=B'_R\times(0,+\infty).$$

In this paragraph, the question of the existence of a non-negative global solution of the problem is investigated. The global solution of the problem is understood as a weak solution. $u(x,t)$ function is called a weak solution of the considered problem, for any

$$u(x,t)\in W_{2,loc}^{1,0}(Q'_R)\cap L_{\infty,loc}(Q'_R)$$

and for any

$$\psi(x,t)\in C_0^{\infty}\left(B'_R\times[0,+\infty)\right)$$

$$\iint_{Q'_R}u\frac{\partial\psi}{\partial t}dxdt+\iint_{Q'_R}\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial\psi}{\partial x_i}dxdt-\iint_{Q'_R}\frac{C_0}{|x|^2}u\psi dxdt-$$

$$-\int_{B'_R}u_0(x)\psi(x,0)dx=\iint_{Q''_R}|x|^{\sigma}|u|^{q}\psi dxdt$$

the integral identity be satisfied.

$$D=\left(\frac{n-2}{2}\right)^2-\frac{C_0}{\gamma+1}\ .$$

Teorem 6. Let $n\geq 3,\ \ q>1,\ \sigma>-2,\ \gamma>-1,$

$$0 \le C_0 < (\gamma+1)\left(\frac{n-2}{2}\right)^2 \text{ and } 1 < q \le 1 + \frac{\sigma+2}{\frac{n+2}{2}+\sqrt{D}}.$$

If the function $u(x,t)$ is non-negative solution of the given problem (10)-(11), then $u(x,t) \equiv 0$.

In section **2.2** talks about the absence of a global solution of the semi-linear parabolic equation with singular potential where the main part is a Baowendi-Grushin type operator.

Make markings as follows:

$$x = (x_1, x_{2,} \ldots, x_m), \; y = (y_1, y_{2,} \ldots, y_n), \; |x| = \sqrt{x_1^2 + \cdots + x_m^2},$$

$$|y| = \sqrt{y_1^2 + \cdots + y_n^2}, \; B_x(r) = \{x; |x| < r\}, \; B_y(r) = \{y; |y| < r\},$$

$$B_x(r_1, r_2) = \{x; r_1 < |x| < r_2\}, \; B_y(r_1, r_2) = \{y; r_1 < |y| < r_2\},$$

$$B'_x(R) = R^m \setminus B_x(R), \; B'_y(R) = R^n \setminus B_y(R), \; B'(R) = B'_x(R) \times B'_y(R),$$

$$Q'(R) = B'(R) \times (0; +\infty), \; B'_x(1) = B'_x, B'_y(1) = B'_y,$$

$$B'(1) = B', \; Q'(1) = Q'.$$

In $Q'(R)$

$$\frac{\partial u}{\partial t} = div_x\left(|x|^\alpha \nabla_x u\right) + \Delta_y u + \frac{c_1}{|x|^{2-\alpha}} u + \frac{c_2}{|y|^2} u + |x|^{\sigma_1} |y|^{\sigma_2} |u|^q \qquad (12)$$

considered solutions of equation (12) satisfying the initial condition

$$u\big|_{t=0} = u_0(x,y) \ge 0, \qquad (13)$$

where,

$$0 \le c_1 < \left(\frac{\alpha+m-2}{2}\right)^2, \qquad 0 \le c_2 < \left(\frac{n-2}{2}\right)^2,$$

$$q > 1, \quad \sigma_1, \sigma_2 \in R, \quad \alpha < 2,$$

$$\nabla_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_m}\right), \qquad \nabla_y = \left(\frac{\partial}{\partial y_1}, \cdots, \frac{\partial}{\partial y_n}\right),$$

$$div_x(f_1, \cdots, f_m) = \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_m}{\partial x_m}, \qquad \Delta_y = \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_n^2}.$$

The non-existence of a global solution to the given problem is investigated and the solution of the problem is understood in the classical mean.

The similar questions examined in this paragraph were previously answered in the works of E.Mitidieri and S.I. Pokhojayev5 , Azman, I. Jleli and M. Samet6, P.Baras and J.A. Goldstein, K. Deng and H.A. Levine7.

$$D_1 = \left(\frac{\alpha + m - 2}{2} \right)^2 - c_1, \quad D_2 = \left(\frac{n-2}{2} \right)^2 - c_2,$$

$$\lambda_+ = -\frac{\alpha + m - 2}{2} + \sqrt{D_1}, \quad \lambda_- = -\frac{\alpha + m - 2}{2} - \sqrt{D_1},$$

$$\mu_+ = -\frac{n-2}{2} + \sqrt{D_2}, \qquad \mu_- = -\frac{n-2}{2} - \sqrt{D_2}$$

are indicated.

The main result here is the next theorem.

Theorem 7. Let, $m > 2 - \alpha,\ n > 2,\ \alpha < 2,\ q > 1,$

$$0 \le c_1 < \left(\frac{\alpha + m - 2}{2} \right)^2, \quad 0 \le c_2 < \left(\frac{n-2}{2} \right)^2,$$

$$2 - \alpha + \sigma_1 + \frac{2-\alpha}{2}\sigma_2 > 0 \text{ and } q \le 1 + \frac{2 - \alpha + \sigma_1 + \frac{2-\alpha}{2}\sigma_2}{\lambda_+ + m + \frac{2-\alpha}{2}(\mu_+ + n)}.$$

If the function $u(x, y, t)$ is the solution of the problem (12)-(13), then $u(x, y, t) \equiv 0$.

The **third chapter** of the work is devoted to the existence of global solutions of the system of weakly connected semi-linear

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[5] Похожаев, С.И., Митидиери, Э. Априорные оценки и отсутствие решений нелинейных уравнений и неравенств в частных производных. //-Москва: Наука, Труды МИАН, -2001, 234, -с.1-383.

[6] Azman, I, Jleli, M, Samet, B. Blow-up of solutions to parabolic inequalities in the Heisenberg group// Electron. J. Differ. Equ. -2015 , 167, p.1-9.

[7] Deng, K., Levine, H.A. The role of critical exponent in blow-up theorem: the sequel // J. Math. Anal. and Appl. -2000. v. 243, -p. 85–126.

elliptic and parabolic equations of the second order in external domains and is divided into two paragraphs. In the first paragraph, the existence of global solutions of the system of weakly connected semi-linear elliptic equations of the second order in external domains is investigated. An exact sufficient condition for the non-existence of a positive solution of the system considered in this paragraph has been found.

Markings are carried out in the following way:

$$x=(x_1,\ldots,x_n)\in R^n,\ n\geq 3,\ r=|x|=\sqrt{x_1^2+\cdots+x_n^2},$$

$$B_R=\{x;|x|<R\},\ B'_R=\{x;|x|>R\},$$

$$B_{R_1,R_2}=\{x;R_1<|x|<R_2\},\ Q_R=B_R\times(0,+\infty),$$

$$Q'_R=B'_R\times(0,+\infty),\ Q_R(\rho)=Q'_R\cap\{(x,t);t+|x|^2<2\rho^2\}.$$

With $C_{x,t}^{k,l}(Q'_R)$ denotes the set of continuously differentiable functions with respect to the variable x from k order, and with respect to the variable t from l order.

In section **3.1** the absence of a non-trivial global solution of the following system of semi-linear elliptic equations in domain B'_R is investigated.

$$\begin{cases} div(A\nabla u)+\dfrac{C_1}{|x|^2}u+|x|^{\sigma_1}|\vartheta|^{q_1}=0 \\ div(A\nabla\vartheta)+\dfrac{C_2}{|x|^2}\vartheta+|x|^{\sigma_2}|u|^{q_2}=0, \end{cases} \tag{14}$$

where

$$q_1,q_2>1,\ A=(a_{ij}(x))_{i,j=1}^n,\ a_{ij}(x)=\delta_{ij}+\gamma\frac{x_ix_j}{|x|^2},\ \gamma>-1,$$

δ_{ij} - Kronecker symbol, $0\leq\dfrac{C_1}{\gamma+1}<\left(\dfrac{n-2}{2}\right)^2$ $\quad 0\leq\dfrac{C_2}{\gamma+1}<\left(\dfrac{n-2}{2}\right)^2$

$$A\nabla u=\left(\sum_{j=1}^n a_{ij}\frac{\partial u}{\partial x_j}\right)_{i=1}^n,\quad div(A\nabla u)=\sum_{i,j=1}^n\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right).$$

The global solution of the system means the generalized solution. If the integral identities are satisfied for

$$u(x),\vartheta(x)\in W^{1}_{2,loc}(B'_R)\cap L_{\infty,loc}(B'_R)$$

and for any $R_1 > R,\ \varphi(x)\in \overset{o}{W}{}^{1}_{2}\left(B_{R,R_1}\right)$

$$\int_{B'_R}|x|^{\sigma_1}|\vartheta|^{q_1}\varphi dx=\int_{B'_R}(A\nabla u,\nabla\varphi)dx-\int_{B'_R}\frac{C_1}{|x|^2}u\varphi dx,$$

$$\int_{B'_R}|x|^{\sigma_2}|u|^{q_2}\varphi dx=\int_{B'_R}(A\nabla \vartheta,\nabla\varphi)dx-\int_{B'_R}\frac{C_2}{|x|^2}\vartheta\varphi dx.$$

Then, the $(u(x),\vartheta(x))$ a pair of functions is called a solution of the considered system.

Such markings are carried out:

$$D_k=\left(\frac{n-2}{2}\right)^2-\frac{C_k}{\gamma+1},\quad \alpha_k^{\pm}=-\frac{n-2}{2}\pm\sqrt{D_k},$$

$$\theta_1=\frac{\sigma_1+2+q_1(\sigma_2+2)}{q_1q_2-1},\quad \theta_2=\frac{\sigma_2+2+q_2(\sigma_1+2)}{q_1q_2-1},\ k=1,2.$$

Theorem 8. Let, $n\geq 3$, $q_k>1$, $\gamma>-1$,

$$0\leq C_k<(\gamma+1)\left(\frac{n-2}{2}\right)^2,\ \max\left\{\theta_1+\alpha_1^{-},\theta_2+\alpha_2^{-}\right\}\geq 0,\quad k=1,2.$$

If the (u,ϑ) a pair of functions is the solution of the problem (15), then $u\equiv 0,\ \vartheta\equiv 0$.

After the proof of the theorem is finished, the accuracy of its ruling is shown on the example. The solution of the system is searched in the form as $u=A_1|x|^{\mu_1}$ and $v=A_2|x|^{\mu_2}$. After these solutions are written in the system, we obtain that the functions $u=A_1|x|^{-\frac{\sigma_1+2+q_1(\sigma_2+2)}{q_1q_2-1}}$ and $v=A_2|x|^{-\frac{\sigma_2+2+q_2(\sigma_1+2)}{q_1q_2-1}}$ are the global solutions, where,

$$A_2^{q_1}=A_1\left(-(1+\gamma)(\theta_1+\alpha_1^{+})(\theta_1+\alpha_1^{-})\right),$$

$$A_1^{q_2} = A_2\left(-\left(1+\gamma\right)\left(\theta_2+\alpha_2^+\right)\left(\theta_2+\alpha_2^-\right)\right).$$

In section **3.2** the issue of the existence of global solutions of the system of weakly connected second order semi-linear parabolic equations in the external domain is studied. In this paragraph, a system of semi-linear parabolic equations is considered, but instead of the Laplace operator, an operator of the form $div\left(A(x)\nabla u\right)$ is taken, where has a special form as mentioned above. In addition, small terms with singular potential are also included in our equations. It should be noted that the question of the existence of global solutions of semi-linear parabolic equations with singular potential was discussed by L.A. Dupaigne, Daomin, Cao, Pigong and Han, M. Fall, Huilai and Li, V.A. Kondratyev, V. Liskevich and Z. Sobol, V.A., Kondratyev, D. Smets and Tesei.

As shown in the work of V.A. Kondratyev, V. Liskevich, and Z. Sobol8, if we replace the singular potential Laplace operator in the equation with a more general $div\left(A(x)\nabla u\right)$ operator, then it will be impossible to find an exact critical value that ensures the absence of a non-negative global solution. In the current paragraph, the matrix $A(x)$ is chosen in a special way and the effect of constant γ on the critical value is shown. Here, also, a critical value for the absence of a global solution is found. Thus, the next weakly coupled system is considered in the domain Q'_R.

$$\begin{cases} \dfrac{\partial u}{\partial t} = div\left(A\nabla u\right) + \dfrac{C_1}{|x|^2}u + |x|^{\sigma_1}|v|^{q_1} \\ \dfrac{\partial v}{\partial t} = div\left(A\nabla v\right) + \dfrac{C_2}{|x|^2}v + |x|^{\sigma_2}|u|^{q_2}. \end{cases} \qquad (15)$$

Here solutions of the system (16) that satisfy the initial conditions

[8] Kondratiev, V., Liskevich, V., Sobol, Z. Second–order semilinear elliptic inequalities in exterior domains// J. Differential Equations, -2003. 187, -p. 429-455.

$$u\big|_{t=0} = u_0(x), \quad v\big|_{t=0} = v_0(x), \tag{16}$$

are considered, where, $0 \le u_0(x),\ v_0(x) \in C\left(\overline{B'_R}\right)$,

$$\sigma_k > -2,\ q_k > 1,\ 0 \le C_k < (\gamma+1)\frac{(n-2)^2}{4},\ k=1,2,\ \ \gamma > -1,$$

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}\right),\ \ A = \left(a_{ij}(x)\right)_{i,j=1}^{n},\ \ a_{ij}(x) = \delta_{ij} + \gamma\frac{x_i x_j}{|x|^2},$$

here δ_{ij} - are the Kronecker symbols and

$$A\nabla u = \left(\sum_{j=1}^{n} a_{1j}\frac{\partial u}{\partial x_j}, \dots, \sum_{j=1}^{n} a_{nj}\frac{\partial u}{\partial x_j}\right),$$

$$div(A\nabla u) = \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right).$$

The existence of a global solution of the given problem is investigated. The global solution means such a $u(x,t), v(x,t) \in C_{x,t}^{2,1}\left(\overline{Q'_R}\right) \cap C_{x,t}^{1,0}\left(\overline{Q'_R}\right)$ pair of functions that is satisfy system (15) at each point of domain Q'_R, for $t=0$ (16) initial conditions.

Markings are made as follows:

$$D_k = \sqrt{\left(\frac{n-2}{2}\right)^2 - \frac{C_k}{\gamma+1}},\ \ \lambda_k^+ = -\frac{n-2}{2} + D_k,\ \lambda_k^- = -\frac{n-2}{2} - D_k,$$

$$\theta_1 = \frac{\sigma_1 + 2 + q_1(\sigma_2+2)}{q_1 q_2 - 1} - \lambda_1^+ - n,$$

$$\theta_2 = \frac{\sigma_2 + 2 + q_2(\sigma_1+2)}{q_1 q_2 - 1} - \lambda_2^+ - n,\ k=1,2.$$

Teorem 9. Let, $n \ge 3$, $q_k > 1$, $\gamma > -1$,

$$0 \le C_k < (\gamma+1)\left(\frac{n-2}{2}\right)^2,\ \max\{\theta_1, \theta_2\} \ge 0,\ \ k=1,2.$$

If the $(u(x,t),v(x,t))$ pair of functions are non-negative solution of the problem (15)-(16), then $u \equiv 0, v \equiv 0$.

The author expresses respect and gratitude to her scientific supervisor, doctor of physicsal and mathematical sciences, professor Nizameddin Shirin Iskenderov and scientific consultant, candidate of physical and mathematical sciences, associate professor Bagirov Shirmayil Hasan, for setting the issues, discussing the obtained results, and for their attention to the work.

CONCLUSION

In the dissertation work, the question of the presence and absence of global solutions of elliptic and parabolic type equations of the second order and the system of equations was studied. Unlike the previous works, the presented work deals with excited equations, that is, the equations include both small order derivatives and singular potential terms. It is investigated how lower order derivatives and singular potential bounds affect the sufficiency estimates found for the existence of a global solution.

The following results were obtained.

1. An exact estimate was found that ensures the absence of a positive global solution of the second order semi-linear elliptic equation with a singular potential in different regions [2,3,5].

2. The sufficiency theorem for the existence of global solutions of semi-linear parabolic equations of the second order with the main part divergent type is proved [1].

3. A Fujita-type result is proved for a semi-linear parabolic equation with a singular potential, where the main part is a Bauendi-Grushin type operator [6].

4. An exact sufficient condition for the absence of global solutions of the system of weakly connected semi-linear second order elliptic and parabolic type equations in external domains has been found [4].

The main results of the dissertation were published in the following works:

1. Quliyeva, K.A. Sinqulyar potensiallı ikinci tərtib yarımxətti parabolik tənliyin mənfi olmayan qlobal həllərinin yoxluğu.//-Bakı: Azərbaycan Dövlət Pedaqoji Universitetinin Xəbərləri, riyaz. və təbiət elmləri ser. – 2018. №1, – s. 59-70.
2. Багыров, Ш.Г., Гулуева, К.А. Об отсутствии неотрицательных решений полулинейного эллиптического уравнения второго порядка во внешности шара //-Баку: Вестник Бакинского Университета. – 2016, №3, – с. 89-94.
3. Багыров, Ш.Г., Гулуева, К.А. Отсутствие положительных решений полулинейного эллиптического уравнения второго порядка с младшими производными и с сингулярным потенциалом // – Москва: Математические заметки. – 2017, т.101, вып.2, – с. 313-317.
4. Багыров, Ш.Г., Гулуева, К.А., Отсутствие глобальных решений слабо связанной системы параболических уравнений второго порядка //-Польша: Wschodnioeuropejskie Czasopismo Naukowe (East European Scientific Journal) 9(49), – 2019. – с. 79-85.
5. Гулуева, К.А. Об отсутствии положительных глобальных решений полулинейных эллиптических уравнений в неограниченных областях // – Баку: Вестник Бакинского Университета. – 2015. №3, – с. 108-116.
6. Bagyrov, Sh.G., Guliyeva K.A. Non existence of global solution of a semi-linear parabolic equation with a singular potential//Transactions of ANAS, ser. phys.- tech. mat. sc., - 2020. v.40(1), – pp. 79-87.
7. Quluyeva, K.Ə. İkinci tərtib yarımxətti elliptik tənliyin silindrik tip oblastda müsbət həllinin varlığı // Bakı Dövlət Universitetinin 95-ci ildönümünə həsr olunmuş "Riyaziyyat və mexanikanın aktual problemləri" adlı Respublika elmi konfransının materialları, Bakı Dövlət Universiteti, Mexanka-riyaziyyat fakultəsi, – Bakı, – 2014, 25-26 dekabr – s. 90-91.
8. Bağırov, Ş.H. Quluyeva, K.Ə. Aşağı tərtib törəmələr daxil olan sinqulyar potensiallı yarımxətti ikinci tərtib elliptik tənliyin

müsbət qlobal həllinin varlığı // Azərbaycan xalqının ümummilli lideri Heydər Əliyevin anadan olmasının 93-cü ildönümünə həsr olunmuş “Riyaziyyat və mexanikanın aktual problemləri” adlı Respublika elmi konfransının materialları, Bakı Dövlət Universiteti, Mexanka-riyaziyyat fakultəsi, – Bakı, 2016, – s. 6-7.

9. Багыров, Ш.Г., Гулуева, К.А. Отсутствие глобальных решений слабо связанной системы полулинейных параболических уравнений второго порядка // Сборник материалов международной конферецuи КРОМШ, – 2019, –с. 153-154.
10. Bağırov, Ş.H., Quluyeva, K.Ə. Yüksək tərtib yarımxətti elliptik tənlik üçün sərhəd məsələsinin həllinin varlığı // Azərbaycan xalqının ümummilli lideri Heydər Əliyevin anadan olmasının 99-cu ildönümünə həsr olunmuş “Riyaziyyat və mexanikanın aktual problemləri” adlı Respublika elmi konfransının materialları, Bakı Dövlət Universiteti, Mexanka-riyaziyyat fakültəsi, – Bakı, – 2022, – s. 27-28.

The defense will be held on **11 june 2024** at **14^{00}** at the meeting of the Dissertation council ED 2.17 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at Baku State University.

Address: AZ 1148, Baku city, Acad. Z. Khalilov street, 23.

Dissertation is accessible at the Baku State University Library

Electronic versions of dissertation and its abstract are available on the official website of the Baku State University

Abstract was sent to the required addresses on **07 may 2024**.

Signed for print: 01.05.2024
Paper format: 60 × 84 1/16
Volume: 37170
Number of hard copies: 20