REPUBLIC OF AZERBAIJAN

On the rights of manuscript

ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

OPTIMALITY CONDITION IN THE PROBLEM OF OPTIMAL CONTROL OF POPULATION DYNAMICS

Specialty: 1214.01 – Dynamical systems and optimal control

Field of science: Mathematics

Applicant: **Aygun Isvagan Agamaliyeva**

Baku – 2024

The work was executed at department of "Mathematical cybernetics" of Baku State University.

Scientific supervisor:

doctor of physical-mathematical sciences, professor
Kamil Bayramali Mansimov

Official opponents:

doctor of physical-mathematical sciences, professor
Misreddin Allahverdi Sadiqov

doctor of physical-mathematical sciences, professor
Fikret Gulali Feyziyev

candidate of mathematical sciences, associate professor
Rashad Oktay Mastaliyev

Dissertation council ED 1.04 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at the Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan.

Chairman of the Dissertation council: corr.-member of the NASA, doctor of physical-mathematical sciences, professor

_______________ **Misir Jumail Mardanov**

Scientific secretary of the Dissertation council:
candidate of physical-mathematical sciences

_______________ **Abdurrahim Farman Guliyev**

Chairman of the scientific seminar:
doctor of physical-mathematical sciences, professor

_______________ **Hamlet Farman Guliyev**

GENERAL CHARACTERISTICS OF THE WORK

Rationale and development degree of the topic.

Mathematical models help to study the development and change of certain population. Note that under population one understands a totality of individuals of one species relatively isolated from other groups that have a common gene pool. One of the important problems of ecology and biology is the problem of rational use of biological resources to a certain extent. Problems of optimal control of certain resources reduced to various optimal control problems described by various equations.

Various problems of optimal control of population dynamics were studied in the papers of A.V. Bukina, O.I. Il'in, Yu.A. Kuznetsov, A.S. Plyatov, A. Belyakov, A.I. Abakumov, A.V. Argunchintseva and others. Note that the development of a qualitative theory of optimal control of dynamics of various classes of population problems allows working out appropriate constructive methods allowing in principle to solve the problems under consideration.

Many models that describe the dynamics of population are quite complex, but adequate to accurately describe the studied process by complex partial differential equations, two-dimensional integro-differential equations and their discrete analogs. But such optimal control problems have been still insufficiently studied.

Therefore, the topic of the dissertation work devoted to derivation of necessary and in some cases of sufficient conditions of optimality for some classes of problems of optimal control of population dynamics is urgent.

Object and subject research. The represented dissertation work is devoted to the study of a number of problems of optimal control of population dynamics described by first order two-dimensional integro-differential equations and their discrete analogues. The subject of the research is to establish various necessary and, in some cases, sufficient conditions of optimality in the considered continuous and discrete optimal control problems.

Purpose and objectives of the study. The goal of the dissertation work is to study some classes of continuous problems of optimal control of population dynamics and their discrete analogues for establishing various optimality conditions.

Research methods. The methods used in the dissertation work is based on mathematical apparatus of theory of optimal processes, the calculus variations, differential and integro-differential with partial equations and their difference analogues.

The basic theses defend. Various necessary conditions for optimality and in some cases sufficient conditions for optimality established for continuous optimal control problems described by integro-differential equations and their discrete analogues modeling population dynamics. The representation of solutions of linear differential equations and their discrete analogues obtained by introducing into consideration the Cauchy matrix and a resolvent.

Scientific novelty of the results, obtained in the dissertation work is as follows:

- For the first time, in the considered continuous and discrete optimal control problems, a necessary condition for optimality was found in the form of the maximum principle and in the linear case, both necessary and sufficient conditions were found.
- A new necessary condition for optimality in case of functional inequality is found.
- In continuous and discrete optimal control problems within additional constraints, optimality conditions were obtained in the form of a linearized maximum condition and analogue of Euler equation.

Theoretical and practical value of research. The results obtained in the dissertation work are of theoretical character. However, these results can be used both for the further development of a quality and constructive theory for the problems of optimal control of population dynamics and when solving appropriate applied problems occurring in applications.

Approbation and applications. The results obtained in the dissertation work were reported and discussed at the seminars of the chair of “Mathematical Cybernetics” of Baku State University, at the seminars of the laboratory "Control in complex dynamical systems" of the Institute of Control Systems of ANAS, at the seminars of the chair of “Differential Equations and Optimization” of Sumgait State University, at the Republican conference “Functional analysis and its applications” dedicated to 100 anniversary of prof. A.Sh.Habibzade (Baku, 2016), at the VII international conference with the joint project of Azerbaijan, Turkey and Ukraine “Mathematical analysis, differential equations and their applications” (Madea-7) (Баку, 2015), at the international conference “Dynamical systems, optimal control and mathematical modeling” (Irkutsk, 2019), at the V and VIII international conference “On control and optimization with industrial applications” (Баки, 2015, 2022).

Author’s personal contribution. All conclusions and results obtained belong to the author personally

Author's publications. The basic results of the dissertation work were published in 18 papers. 12 of them were published in the journals recommended by Higher Attestation Commission of Azerbaijan at the President of the Republic of Azerbaijan; the remaining 6 papers were published in the proceedings of international Republican conferences.

The institution where the work performed. The work was performed at the chair of “Mathematical Cybernetics” of Baku State University.

Structure and volume of the dissertation work (in signs indicating the volume of each structural subdivision separately). The total volume of the dissertation work – 234766 signs (title page – 393 signs, table of contents – 4802 signs, introduction – 41571 signs, chapter I – 24000 signs, chapter II – 114000 signs, chapter III – 50000 signs). The list of references consists of 75 names.

THE CONTENT OF THE DISSERTATION WORK

The dissertation consists of introduction, three chapters and a list of references.

In introduction the brief review of the works related to the dissertation topic is given, rationale of the topic is justified and a brief review of the obtained results is given.

Chapter I of the dissertation consists of two sections and is devoted to the representation of solutions of two classes of linear equations.

In section 1 we consider the following system of equations:

$$z_t(t,x) = A(t,x)z(t,x) + B(t,x)y(t,x) + f(t,x),$$
$$(t,x) \in D = [t_0, t_1] \times [x_0, x_1], \tag{1}$$

$$y(t,x) = \int_{x_0}^{x_1} [C(t,x,s)z(t,s) + g(t,x,s)]ds, \quad (t,x) \in D, \tag{2}$$

with the initial condition

$$z(t_0, x) = a(x), \quad x \in [x_0, x_1]. \tag{3}$$

Here $A(t,x)$, $B(t,x)$, $C(t,x,s)$ are the given $(n \times n)$ matrix functions continuous in totality of variables $f(t,x)$ and $g(t,x,s)$ are the given $n-$dimensional vector-functions continuous in totality of variables, $a(x)$ is a given $n-$dimensional continuous vector-function.

The representation of the solution to problem (1)–(3) is given studied.

In section 2 we study a difference (discrete) analogue of problem (1)–(3) in the form

$$z(t+1,x) = A(t,x)z(t,x) + B(t,x)y(t,x) + f(t,x),$$
$$t = t_0, t_0+1, \dots, t_1-1; \quad x = x_0, x_0+1, \dots, x_1, \tag{4}$$

$$y(t,x) = \sum_{s=x_0}^{x_1} [C(t,x,s)z(t,s) + g(t,x,s)], \tag{5}$$

with the initial condition (analogue of the Cauchy problem)

$$z(t_0, x) = a(x), \quad x = x_0, x_0 + 1, ..., x_1. \tag{6}$$

Here $A(t,x)$, $B(t,x)$, $C(t,x,s)$ are the given $(n \times n)$, discrete, bounded matrix functions, $f(t,x)$, $g(t,x,s)$, $a(x)$ –are the given $n-$ dimensional discrete and bounded vector-functions, $z(t,x)$ are the discrete $n-$ dimensional vector-functions.

The representation of the solution to problem (4)–(6) is found and the one special case of equation (4) is studied.

Chapter II consisting of 7 sections and is mainly devoted to finding the necessary conditions optimality. In this chapter, it is assumed that the problems of optimal control of the population dynamics are described by a system of special differential integro-differential equations of one formulation.

In section 1 we study a linear problem of optimal control with a linear quality criterion.

Let $D = [t_0, t_1] \times [x_0, x_1]-$ be a given rectangle, $a(x)$ be a given $n-$ dimensional continuous vector-function, $U \subset R^r$ be a given non-empty and bounded set $u(t,x)$ be a continuous with respect to x and piecewise-continuous with respect to $t-$with finite number of discontinuity points $r-$dimensional vector-function of control effects with the values from U, i.e.

$$u(t,x) \in U \subset R^r, \quad (t,x) \in D. \tag{7}$$

We call these control function admissible controls.

Assume that the controlled process is a system of linear integro-differential equations

$$z_t(t,x) = A(t,x)z(t,x) + \int_{x_0}^{x_1} B(t,x,s)z(t,s)ds +$$

$$+ \int_{x_0}^{x_1} C(t,x,s,u(t,s))ds + f(t,x,u(t,x)), \; (t,x) \in D, \tag{8}$$

with the initial condition

$$z(t_0, x) = a(x), \quad x \in [x_0, x_1]. \tag{9}$$

Here $A(t,x)$, $B(t,x)$ are the given $(n\times n)$ matrix functions continuous in totality of variables $C(t,x,s,u)$ and $f(t,x,u)-$ are the given $n-$ dimensional matrix functions continuous in totality. It is assumed that under the made suppositions each admissible, $u(t,x)$ corresponds to a unique piecewise smooth with respect to, $t-$and continuous with respect to $x-$ solution $z(t,x)$ of the Cauchy problem (8)–(9)

Let $T_i \in (t_0,t_1]$, $i=\overline{1,k}$ $(t_0 < T_1 < T_2 < \cdots < T_k \le t_1)$ be the given points.

Let us consider a problem on the minimum of the multipoint linear functional

$$S(u)=\int_{x_0}^{x_1}\sum_{i=1}^{k} c_i'(x) z(T_i,x)dx, \tag{10}$$

under the constraints (7)–(9) of problem (7)–(10). Here $c_i(x)$, $i=\overline{1,k}$ –are the given continuous $n-$dimensional vector-functions.

That the prime ($'$) everywhere for vectors and matrices means the transposition operation.

The admissible control affording a minimum value to the functional (10) under the constraints (7)–(9) is said to be an optimal control, the appropriate process $(u(t,x),z(t,x))-$ an optimal process.

Let $(u(t,x),z(t,x))$ be some admissible process. We introduce the denotations

$$H(t,x,u(t,x),\psi(t,x)) = \psi'(t,x) f(t,x,u(t,x)) + \int_{x_0}^{x_1} \psi'(t,s) C(t,s,x,u(t,x))ds,$$

$$\Delta_{\bar{u}(t,x)} = H(t,x,\psi) \equiv H(t,x,\bar{u}(t,x),\psi(t,x)) - H(t,x,u(t,x),\psi(t,x)),$$

where $\psi(t,x)$ is $n-$ dimensional vector-function being the solution of the Volterra equation

$$\psi(t,x) = \int_{t}^{t_1} A'(\tau,x)\psi(\tau,x)d\tau +$$

$$+\int_t^{t_1}\int_{x_0}^{x_1} B'(\tau,s,x)\psi(\tau,s)ds\,d\tau-\sum_{i=1}^{k}\alpha_i(t)c_i(x). \tag{11}$$

Here $\alpha_i(t)$ is a characteristic function of the segment $[t_0,T_i]$.

Applying one variant of the increment method we prove a necessary and sufficient optimality condition in the form of Pontryagin's maximum principle.

Theorem 1. For the admissible control $u(t,x)$ in problem (7)–(10) to be optimal, it is sufficient and necessary that the relation

$$\int_{x_0}^{x_1}\Delta_{v(x)}H[\theta,x,\psi]dx\le 0, \tag{12}$$

to be fulfilled for all $v(x)\in U,\ \ x\in[x_0,x_1]$ and $\theta\in[t_0,t_1)$. Here and in the sequel $\theta\in[t_0,t_1)$ is an arbitrary continuity point $u(t,x)$ with respect to t.

Then it is proved that in the case of nonlinear convex quality functional, the Pontryagin maximum condition is a sufficient optimality condition.

At the end of the solution we consider the case of convex control domain and prove a necessary optimality condition in the form of linearized maximum condition.

In section 2 we consider a control process described by the system of nonlinear equations

$$z_t=(t,x,z,y,u),\ (t,x)\in D=[t_0,t_1]\times[x_0,x_1], \tag{13}$$

$$y(t,x)=\int_{x_0}^{x_1} g(t,x,s,z(t,s),u(t,s))ds,\ \ (t,x)\in D \tag{14}$$

with the initial condition

$$z(t_0,x)=a(x),\ x\in[x_0,x_1]. \tag{15}$$

Here $f(t,x,z,y,u)$, $(g(t,x,s,z,u))-$ is a given $n-$ dimensional vector-function continuous in totality of variables together with partial derivatives with respect (z,y,u) $((z,u))$, $a(x)$ is a given measurable and bounded $n-$ dimensional vector-function

$u(t,x)$ $r-$dimensional and bounded control vector-function satisfying the constraint

$$u(t,x)\in U\subset R^r,\ \ (t,x)\in D,$$

U is a given non-empty, convex and bounded set. We call such functions admissible ones.

On the solutions of problem (13)–(15) generated by all possible admissible controls, we determine a nonlinear multipoint functional of the form

$$I(u)=\int_{x_0}^{x_1}\phi(x,z(T_1,x),z(T_2,x),\ldots,z(T_k,x))dx. \qquad (16)$$

Here $T_i\in(t_0,t_1]$, $i=\overline{1,k}$ $(t_0<T_1<T_2<\cdots<T_k\le t_1)$ are the given points $\phi(x,a_1,a_2,\ldots,a_k)-$is a given differentiable scalar function.

We consider a problem on finding a minimum value of the functional (16) under the constraints (13)–(15).

Let $(u(t,x),z(t,x),y(t,x))$ be some admissible process and $\psi(t,x)$ and $q(t,x)$ be $n-$ dimensional vector-functions satisfying the relation

$$\psi(t,x)=\int_t^{t_1}H_z(\tau,x,z(\tau,x),y(\tau,x),\psi(\tau,x),q(\tau,x),u(\tau,x))d\tau-$$

$$-\sum_{i=1}^{k}\alpha_i(t)\frac{\partial\phi(x,z(T_1,x),z(T_2,x),\ldots,z(T_k,x))}{\partial a_i}, \qquad (17)$$

$$q(t,x)=H_y(t,x,z(t,x),y(t,x),\psi(t,x),q(t,x),u(t,x)) \qquad (18)$$

where $\alpha_i(t)$ is a characteristic function of the segment $[t_0,T_i]$ and

$$H(t,x,z(t,x),y(t,x),\psi(t,x),q(t,x),u(t,x))=$$

$$=\int_{x_0}^{x_1}q'(t,s)g(t,s,x,z(t,x)u(t,x))ds+\psi'(t,x)f(t,x,z(t,x),y(t,x),u(t,x)).$$

Considering the introduced denotations and taking into attention the conjugated system (17)–(18), we represent the increment of the quality criterion corresponding to the admissible controls $u(t,x)$ and $\bar{u}(t,x)=u(t,x)+\Delta u(t,x)$ in the form

$$\Delta I(u)=-\int_{t_0}^{t_1}\int_{x_0}^{x_1} H'_u(t,x,z(t,x),y(t,x),\psi(t,x),q(t,x),u(t,x))\Delta u(t,x)dxdt+$$
$$+\eta(u;\Delta u). \qquad (19)$$

where $\eta(u;\Delta u)$ is a residual term of the increment formula whose explicit form is in the dissertation. By means of the increment formula we prove (19)

Theorem 2. Under the assumptions made for the optimality of the admissible control $u(t,x)$ in problem (13)–(16) it is necessary and sufficient that the inequality

$$\int_{t_0}^{t_1}\int_{x_0}^{x_1} H'_u(t,x,z(t,x),y(t,x),\psi(t,x),q(t,x),u(t,x))(v(t,x)-u(t,x))\le 0 \qquad (20)$$

be fulfilled for all $v(t,x)\in U\subset R^r$, $(t,x)\in D$. The relation (20) is an analogue of the linearized integral condition of maximum. It yields a point wise linearized maximum principle.

In section 3 we consider the following boundary control problem.

It is assumed to find the minimal value of the Bolsa type functional

$$I(v)=\phi(a(x_1))+\int_{x_0}^{x_1} G(x,z(t_1,x))dx, \qquad (21)$$

under the constraints

$$z_t=(t,x,z,y),\ \ (t,x)\in D=[t_0,t_1]\times[x_0,x_1], \qquad (22)$$

$$z(t_0,x)=a(x),\ \ x\in[x_0,x_1], \qquad (23)$$

$$y(t,x)=\int_{x_0}^{x_1} g(t,x,s,z(t,s))ds,\ \ \ (t,x)\in D, \qquad (24)$$

$$\dot{a}=F(x,a,v),\ x\in[x_0,x_1], \qquad (25)$$

$$a(x_0)=a_0, \qquad (26)$$

$$v(x)\in V\subset R^r,\quad x\in[x_0,x_1]. \qquad (27)$$

Here $f(t,x,z,y)$, $(g(t,x,s,z))-$is a give $n-$dimensional vector-function continuous in totality of variables together with partial derivatives with respect to (z,y) $((z))$, $t_0,x_0,t_1,x_1-$are given while, $F(x,a,v)-$is a given $n-$dimensional vector function continuous in totality of variables together with partial derivatives with respect to x, while $a_0-$is a given constant vector, $v(x)$ is an $r-$dimensional pieciwise continuous vector of control effects, V is a given, non-empty and bounded set, $\phi(a)$ and $G(x,z)$ are the given scalar functions continuously differentiable with respect to a and z. Such control functions are said to be admissible.

It is assumed that each admissible control $v(x)$ corresponds to a unique, continuous and piecewise smooth solution $a(x)$ of the Cauchy problem (25)–(26), and the solution of the problem (22)–(23) is understood in the classical since.

Let $(v(x),\ a(x),\ z(t,x))$ be some admissible process, while $\psi(x), p(t,x), q(t,x)$ be steel arbitrary $n-$dimensional vector-functions.

Introduce the denotations

$$H(t,x,z(t,x),y(t,x),p(t,x),q(t,x))=p'(t,x)f(t,x,z(t,x),y(t,x))+$$

$$+\int_{x_0}^{x_1}q'(t,s)g(t,s,x,z(t,x))ds,$$

$$M(x,a(x),v(x),\psi(x))=\psi'(x)F(x,a(x),v(x)).$$

We now assume that the vector-functions $\psi(x)$, $p(t,x)$ and $q(t,x)$ satisfy the relations

$$\dot\psi(x)=M_a(x,a(x),v(x),\psi(x))-p(t_0,x), \qquad (28)$$

$$\psi(x_1)=-\phi_a(a(x_1)), \qquad (29)$$

$$p_t(t,x)=H_z(t,x,z(t,x),y(t,x),p(t,x),q(t,x)), \qquad (30)$$

$$p(t_1,x)=-G_z(x,z(t_1,x)),$$

$$q(t,x) = H_y(t,x,z(t,x),y(t,x),p(t,x),q(t,x)). \qquad (31)$$

In the problem under consideration at first the formula of increment of the quality functional is built and then by means of the needle variations of control the following theorem is proved.

Theorem 3. For the admissible control $v(x)$ in (21)–(27) to be optimal, it is necessary that the relation

$$\max_{v \in V} M(\xi, a(\xi), v, \psi(\xi)) = M(\xi, a(\xi), v(\xi), \psi(\xi)) \qquad (32)$$

to be fulfilled for all $\xi \in [x_0, x_1)$. Here and in the sequel $\xi \in [x_0, x_1)$ is an arbitrary continuity point of the control $v(x)$.

Then we consider the case of convex control domain and prove the analogue of the linearized maximum condition.

In section 4 of chapter II we study the case when a control function occurs in the initial condition given in the integral form. A series of necessary optimality conditions were established by the increment's method under various assumptions.

In section 5 we consider an optimal control problem of the form

$$z_t(t,x) = \int_{x_0}^{x_1} K(t,x,s,z(t,s))ds + f(t,x,z(t,x),u(t,x)),$$

$$(t,x) \in D = [t_0,t_1] \times [x_0,x_1], \qquad (33)$$

$$z(t_0,x) = a(x), \quad x \in [x_0,x_1], \qquad (34)$$

$$u(t,x) \in U \subset R^r, \ (t,x) \in D, \qquad (35)$$

$$J(u) = \int_{x_0}^{x_1} \phi(x, z(T_1,x), z(T_2,x), \cdots, z(T_k,x))dx \to \min, \qquad (36)$$

On the assumption that $K(t,x,s,z)$, $f(t,x,z,u)$ are the given $n-$dimensional vector functions continuous in totality of variables together with partial derivatives with respect to z, while $\phi(x, a_1, a_2, \ldots, a_k)-$is a given continuously differentiable scalar function, $u(t,x)$ is a piecewise continuous with respect to t for each

$x \in [x_0, x_1]$ and continuum with respect to x for each $t \in [t_0, t_1]$, $n-$ dimensional vector-function, $U-$is a given non-empty and bounded set, $t_0, x_0, t_1, x_1 -$ are given numbers, $T_i \in (t_0, t_1]$, $i = \overline{1, k}$ $(t_0 < T_1 < T_2 < \cdots < T_k \le t_1)$ are the given points.

In the problem under consideration we construct a formula of increment of the quality functional admitting to prove Pontryagins maximum principle type necessary optimality condition.

Considering $(u(t,x), z(t,x))-$ a fixed admissible process, the analogue of the Hamilton-Pontryagin function is introduced for the considered problem in the form

$$H(t, x, z(t,x), u(t,x), \psi(t,x)) =$$

$$= \int_{x_0}^{x_1} \psi'(t,s) K(t, s, x, z(t,x)) ds + \psi'(t,x) f(t, x, z(t,x), u(t,x))$$

where $\psi(t,x)$ is $n-$dimensional vector-function of conjugated variables being the solution of the conjugated system

$$\psi(t,x) = \int_t^{t_1} H_z(\tau, x, z(\tau,x), y(\tau,x), u(\tau,x), \psi(\tau,x)) d\tau -$$

$$-\sum_{i=1}^{k} \alpha_i(t) \frac{\partial \phi(x, z(T_1,x), z(T_2,x), \ldots, z(T_k,x))}{\partial a_i} \tag{37}$$

where $\alpha_i(t)-$ is the characteristic function of the segment $[t_0, T_i]$.

Theorem 4. For the optimality of the permissible control $u(t,x)$ in problem (33)–(36) it is necessary that the inequality

$$\int_{x_0}^{x_1} [H(\theta, x, z(\theta,x), v(x), \psi(\theta,x)) - H(\theta, x, z(\theta,x), u(\theta,x), \psi(\theta,x))] dx \le 0, \tag{38}$$

was performed for all $v(x) \in U, x \in X$ и $\theta \in [t_0, t_1)$.

Then, with some additional assumptions, an analogue of the linearized maximum principle and an analogue of the Euler equation were proved.

In the sixth paragraph of the second chapter, the case of functional constraints of the inequality type is investigated.

The problem of the minimum functional is considered

$$S_0(u) = \int_{x_0}^{x_1} \phi_0\big(z(T_1, x), z(T_2, x), ..., z(T_k, x)\big)dx, \tag{39}$$

under the constraints

$$S_i(u) = \int_{x_0}^{x_1} \phi_i\big(z(T_1, x), z(T_2, x), ..., z(T_k, x)\big)dx \le 0, \ \ i = \overline{1, p}, \tag{40}$$

$$z_t(t, x) = \int_{x_0}^{x_1} K\big(t, x, s, z(t, s)\big)ds + f\big(t, x, z(t, x), u(t, x)\big), \ (t, x) \in D, \tag{41}$$

$$z(t_0, x) = a(x), \ \ x \in [x_0, x_1] = X. \tag{42}$$

Here $\phi_i(a_1, a_2, ..., a_k)$, $i = \overline{0, p}$ – are the given continuously differentiable scalar function $T_i \in (t_0, t_1]$, $i = \overline{1, k}$ $(t_0 < T_1 < T_2 < \cdots < T_k \le t_1)$ are the given points, $U \subset R^r$ – is a given non-empty and bounded set, $a(x)$ is a given n– dimensional vector-function, $K(t, x, s, z)$ and $f(t, x, z, u)$ are the given n– dimensional vector-function continuous in totality of variables together with partial derivatives with respect to z, $u(t, x)$– is an r– dimensional piecewise in t for all x and continuous with respect to x for all t.

$$u(t, x) \in U \subset R^r, (t, x) \in D = [t_0, t_1] \times [x_0, x_1]. \tag{43}$$

We call each of such a control function an accessible control.

If the solution $z(t, x)$ of the Cauchy problem (41)–(42) corresponding to the admissible control function $u(t, x)$– sufficient the constraints (40), let's call acceptable a control, and the permissible control that delivers the minimum value to the functional (39) is optimal control.

Assuming $\big(u(t, x), z(t, x)\big)$ some admissible process, introduce the denotations

$$I(u) = \{i : S_i(u) = 0, i = \overline{1, p}\},$$

$$J(u)=\{0\}\cup I(u),$$

$$H(t,x,z,u,\psi_i)=\int_{x_0}^{x_1}\psi_i'(t,s)K(t,s,x,z(t,x))ds+\psi_i'(t,x)f(t,x,z(t,x),u(t,x)),$$

where $\psi_i(t,x)$, $i=\overline{0,p}$ $n-$ dimensional vector-function being the solution of the system of control

$$\psi_i(t,x)=-\int_{x_0}^{x_1}\sum_{j=1}^{k}\frac{\partial\phi_i(z(T_1,x),z(T_2,x),\dots,z(T_k,x))}{\partial a_j}+$$

$$+\int_{t}^{t_1}\frac{\partial H(\tau,x,z(\tau,x),u(\tau,x),\psi_i(\tau,x))}{\partial z}d\tau$$

where $\alpha_j(t)$ is a characteristic function of the segment $[t_0,T_j]$, $j=\overline{1,k}.$ For complicity the assume that

$$J(u)=\{0,1,2,\cdots,m\}\qquad (m\le p).$$

The following theorem is proved under the made dissertations.

Theorem 5. For the admissible control $u(t,x)$ the optimal on problem (39)–(43) it is necessary that for any μ natural number is the inequality

$$\min_{i\in J(u)}\sum_{j=1}^{m+1}\int_{x_0}^{x_1}l_j\big[H(\theta_j,x,z(\theta_j,x),v_j(x),\psi_i(\theta_j,x))-$$

$$-H(\theta_j,x,z(\theta_j,x),u(\theta_j,x),\psi_i(\theta_j,x))\big]dx\le 0$$

be fulfilled for $l_j\ge 0,\ \theta_j\in[t_0,t_1),\ (t_0\le\theta_1\le\theta_2\le\cdots\le\theta_{m+1}<t_1),$ $v_j(x)\in U,\ x\in[x_0,x_1],\ \ j=\overline{1,m+1}.$

In the seventh paragraph of the dissertation, the problem of the minimum functional is considered

$$I(u)=\int_{x_0}^{x_1}\phi(z(T_1,x)z(T_2,x),\dots,z(T_k,x))+$$

$$+\int_{t_0}^{t_1}\int_{x_0}^{x_1}F(t,x,z(t,x),y(t,x),u(t,x))dxdt$$

under constraints (13)–(15). It is assumed that the control domain is an open set.

The first and second variations of the quality functional are calculated and with their help an analogue of the Euler equation and the Legendre-Klebsch type optimality conditions are proved.

The third chapter of the dissertation is devoted to the derivation of various optimality conditions for discrete analogues of optimal control problems of population dynamics considered in the second chapter.

It should be noted that in recent years M.J.Mardanov, T.Q. Melikov, K.T. Melikov, S.T.Melik obtained new and important necessary conditions for optimality in discrete optimal control problems described by one-dimensional, that is, ordinary difference equations.

In the first paragraph, the problem of the minimum of a linear functional is considered

$$S(u)=\sum_{x=x_0}^{x_1} c'(x)z(t_1,x), \tag{44}$$

under restrictions

$$z(t+1,x)=A(t,x)z(t,x)+B(t,x)y(t,x)+f(t,x,u(t,x)),$$
$$t=t_0,t_0+1,\dots,t_1-1;\quad x=x_0,x_0+1,\dots,x_1, \tag{45}$$

$$z(t_0,x)=a(x),\quad x=x_0,x_0+1,\dots,x_1, \tag{46}$$

$$y(t,x)=\sum_{x=x_0}^{x_1}\left[C(t,x,s)z(t,s)+D(t,x,s,u(t,s))\right], \tag{47}$$

$$u(t,x)\in U\subset R^r,\ \ t=t_0,t_0+1,\dots,t_1-1;\quad x=x_0,x_0+1,\dots,x_1, \tag{48}$$

Here $A(t,x)$, $B(t,x)$, $C(t,x,s)$ are given $(n\times n)$ discrete matrix functions, $f(t,x,u)$, $D(t,x,s,u)-$ given $n-$dimensional discrete vector-functions, $a(x)$ is a given initial discrete vector function, $U-$ is a given nonempty and bounded set, $u(t,x)$ is an $r-$dimensional discrete control vector function (permissible control), $c(x)-$ is a given $n-$dimensional discrete vector function.

Let $\left(u(t,x),z(t,x)\right)$ some valid process. Let 's introduce the notation

$$H(t,x,u,\psi)=\psi'(t,x)(t,x,u(t,x))t+\sum_{s=x_0}^{x_1}\psi'(t,s)B(t,s)D(t,s,x,u(t,x))$$

and let 's put

$$\Delta_{\bar{u}(t,x)}H[t,x]=H(t,x,z(t,x),\bar{u}(t,x),\psi(t,x))-H(t,x,z(t,x),u(t,x),\psi(t,x))$$

where $\psi=\psi(t,x)$ is an $n-$dimensional vector function that is a solution to the problem (conjugate system)

$$\psi(t-1,x)=A'(t,x)\psi(t,x)+\sum_{s=x_0}^{x_1}C'(t,s,x)B'(t,s)\psi(t,s), \tag{49}$$

$$\psi(t_1-1,x)=-c(x) \tag{50}$$

Theorem 6. For the admissible control $u(t,x)$ in the problem (44)–(48) to be optimal it is necessary and sufficient that the condition

$$\sum_{t=t_0}^{t_1-1}\sum_{x=x_0}^{x_1}\Delta_{v(t,x)}H[t,x,\psi]\le 0,$$

to be fulfilled for all $v(t,x)\in U$, $t=t_0,t_0+1,\dots,t_1-1;$

$x=x_0,x_0+1,\dots,x_1.$

Then, the case of nonlinear and convex quality functional is studied.
It is proved that in this case the discrete maximum principle is also a sufficient condition for optimality.

In section 2 of this chapter we study an optimal control problem described by the difference equation

$$z(t+1,x)=f(t,x,z(t,x),y(t,x),u(t,x)), \tag{51}$$

$$t=t_0,t_0+1,\dots,t_1-1,\quad x=x_0,x_0+1,\dots,x_1,$$

$$z(t_0,x)=a(x),\quad x=x_0,x_0+1,\dots,x_1, \tag{52}$$

$$y(t,x)=\sum_{s=x_0}^{x_1}g(t,x,s,z(t,s),u(t,s)), \tag{53}$$

$$t=t_0,t_0+1,\dots,t_1-1,\quad x=x_0,x_0+1,\dots,x_1.$$

Here $f(t,x,z,y,u)$, $(g(t,x,s,z,u))$–is a given $n-$ dimensional vector-function continuous with respect to (z,y,u) $((z,u))$, for all (t,x,s) together with partial derivatives with respect to (z,y) $((z))$, $t_0, x_0, t_1, x_1 -$ the given numbers $t_1 - t_0$, $x_1 - x_0$ are natural numbers, $a(x)-$ is a given discrete initial vector-function, $u(t,x)$ is $r-$ dimensional discrete vector of control effects with the values from the non-empty and bounded set $U \subset R^r$ i.e.

$$u(t,x) \in U \subset R^r, \quad t = t_0, t_0+1, \ldots, t_1-1, \quad x = x_0, x_0+1, \ldots, x_1 \qquad (54)$$

(admissible control).

The problem is to minimize the functional

$$I(u) = \sum_{x=x_0}^{x_1} \phi(x, z(t_1, x)), \qquad (55)$$

under the constraints (51)–(54), where $\phi(x,z)$ is a given, continuously differentiable with respect to z and discrete with respect to x scalar function.

Let $(u(t,x), z(t,x))-$ be some admissible process, the set

$$f(t,x,z(t,x),y(t,x),U) = \{\alpha : \alpha = f(t,x,z(t,x),y(t,x),v(t,x)), v(t,x) \in U,$$
$$t = t_0, t_0+1, \ldots, t_1-1; \ x = x_0, x_0+1, \ldots, x_1\},$$
$$g(t,x,s,z(t,s),U) = \{\beta : \beta = g(t,x,s,z(t,s),v(t,s)), v(t,s) \in U,$$
$$t = t_0, t_0+1, \ldots, t_1-1; \ s = x_0, x_0+1, \ldots, x_1\}, \qquad (56)$$

be convex for all (t,x) and (t,x,s) respectively. Using one variant of the increment method developed for the considered problem, we prove an analogue of the discrete maximum principle.

We introduce an analogue of the Hamilton-Pontryagin function in the form

$$H(t,x,z(t,x),y(t,x),u(t,x),p(t,x),q(t,x)) =$$
$$= p'(t,x) f(t,x,z(t,x),y(t,x),u(t,x)) + \sum_{s=x_0}^{x_1} q'(t,s) g(t,s,x,z(t,x),u(t,x))$$

where $p(t,x)$ and $q(t,x)$ are $n-$dimensional vector-functions satisfying the relation

$$p(t-1,x)=H_z(t,x,z(t,x),y(t,x),u(t,x),p(t,x),q(t,x)),$$
$$p(t_1,x)=-\phi_z(x,z(t_1,x)),$$
$$q(t,x)=H_y(t,x,z(t,x),y(t,x),u(t,x),p(t,x),q(t,x)),$$

We prove the following theorem.

Theorem 7. If the sets (56) are convex, then for the optimality of the admissible control $u(t,x)$ in the problem (51)–(55) it is necessary that the inequality

$$\sum_{t=t_0}^{t_1-1}\sum_{s=x_0}^{x_1}\Delta_{v(t,x)}H[t,x,p(t,x),q(t,x)]\le 0, \qquad (57)$$

to be fulfilled $v(t,x)\in U$, $t=t_0,t_0+1,\dots,t_1-1$; $x=x_0,x_0+1,\dots,x_1$.

Under the assumption that the control domain is open we establish the analogue of the Euler equation.

Necessary conditions for optimality of boundary control under various assumptions on the problem data (parameter) are established in the last section.

We consider a controllable discrete process described by the system of nonlinear difference equations

$$z(t+1,x)=f(t,x,z(t,x),y(t,x)),$$
$$t=t_0,t_0+1,\dots,t_1-1;\quad x=x_0,x_0+1,\dots,x_1, \qquad (58)$$
$$z(t_0,x)=a(x),\quad x=x_0,x_0+1,\dots,x_1 \qquad (59)$$

with the initial condition, where

$$y(t,x)=\sum_{s=x_0}^{x_1}g(t,x,s,z(t,s)), \qquad (60)$$

$$t=t_0,t_0+1,\dots,t_1-1;\quad x=x_0,x_0+1,\dots,x_1. \qquad (61)$$

Here $f(t,x,z,y)$, $g(t,x,s,z)$ —is a given $n-$dimensional vector-function continuous in totality of variables together with partial derivatives with respect to $(z,y)((z))$, $t_0,x_0,t_1,x_1-$are the given numbers and the differences t_1-t_0, x_1-x_0 are natural numbers $a(x)$ is $n-$dimensional vector-function being the solution of the problem

$$a(x+1)=F(x,a(x),u(x)),\ \ x=x_0,x_0+1,\dots,x_1-1, \quad (62)$$

$$a(x)=a_0 \quad (63)$$

where $F(x,a,u)$ is a given $n-$dimensional vector-function continuous in totality of variables together with $F_a(x,a,u)$, is a_0- given constant vector, $u(x)$ is $r-$dimensional discrete vector of control effects with the values from the given non-empty and bounded set $U\subset R^r$ i.e.

$$u(x)\in U\subset R^r\ ,x=x_0,x_0+1,\dots,x_1-1. \quad (64)$$

Each control function $u(x)$ with the above properties is said to be an admissible control.

An optimal control problem is to minimize the Bolsa type functional

$$S(u)=\phi(a(x_1))+\sum_{x=x_0}^{x_1}G(x,z(t_1,x)), \quad (65)$$

Determined on the solution of problem (58)–(63) generated by all possible admissible controls.

Considering $(u(x),\ a(x),z(t,x))$ as fixed admissible process, we assume that the set

$$F(x,a(x),U)=\{\alpha:\alpha=F(x,a(x),v(x)),$$
$$u(x)\in U\subset R^r,x=x_0,x_0+1,\dots,x_1-1\} \quad (66)$$

is convex for all x. We introduce the analogue of the Hamilton-Pontryagin function in the form

$$H(x,a(x),u(x)\psi(x))=\psi'(x)F(x,a(x),u(x)).$$

Here $\psi(x)$ is $n-$dimensional vector-function of conjugated variables being the solution of the problem

$$\psi(x-1)=\frac{\partial F'(x,a(x),u(x))}{\partial a}\psi(x)+p(t_0-1,x),$$

$$\psi(x_1 - 1) = -\frac{\partial \phi(a(x_1))}{\partial a},$$

where $p(t,x)$, $q(t,x)$ satisfies the relations

$$p(t-1,x) = \frac{\partial f'(t,x,z(t,x),y(t,x))}{\partial z} p(t,x) +$$

$$+ \sum_{s=x}^{x_1} \frac{\partial g'(t,s,x,z(t,x))}{\partial z} q(t,s),$$

$$q(t,x) = \frac{\partial f'(t,x,z(t,x),y(t,x))}{\partial y} p(t,x),$$

$$p(t_1 - 1, x) = -\frac{\partial G(x, z(t_1,x))}{\partial z}.$$

Theorem 8. If the set (66) is convex then for the admissible control $u(x)$ in problem (58)–(65) to be optimal, it is necessary that the inequality

$$\sum_{x=x_0}^{x_1-1} \Delta_{v(x)} H(x, a(x), u(x), \psi(x)) \le 0$$

to be fulfilled for all $v(x) \in U$, $x = x_0, x_0 + 1, \ldots, x_1 - 1$.

Then the establish the analogue of the linearized maximum principle under some additional assumptions.

Theorem 9. Let in problem (58)–(65) the set U be convex and while $F(x,a,u)$ have continuous derivative with respect to u. Then for the admissible control $u(x)$ be optimal, it is necessarily has the inequality

$$\sum_{x=x_0}^{x_1-1} H'_u(x, a(x), u(x), \psi(x))(v(x) - u(x)) \le 0$$

to be fulfilled for all $v(t,x) \in U$, $x = x_0, x_0 + 1, \ldots, x_1 - 1$.

In the case when the control domain is open, the necessary optimality condition in the form of analogue of the Euler equation is proved.

CONCLUSIONS

The dissertation work was devoted to finding representations of solutions of analogues of the Cauchy problem for linear integro-differential equations and their difference analogues in applications that model under some assumptions populations dynamics and to the study of appropriate (linear and nonlinear) problems of optimal control of population dynamics.

The work consists of three chapters.

In chapter I we consider the Cauchy problem for a class of linear two-dimensional integro-differential equations in applications modeling population dynamics.

Integral representation of the problem under consideration is obtained.

Similar representation is obtained for solving a problem being a discrete variant of the Cauchy continuous problem.

In both cases special cases are considered.

In chapter II a number of optimal control problems described by linear and nonlinear two-dimensional integro-differential equations are studied.

In the linear case, under the assumption of linearity of a multipoint aim functional, L.S.Pontryagin's type necessary and sufficient condition is proved.

In the case of nonlinear, convex quality criterion, sufficiency of the analogue of Pontryagin's maximum principle is proved.

In the case of nonlinear optimal control problem a number of L.S.Pontryagin's type necessary optimality conditions, linearized maximum principle, the analogue of the Euler equation are proved under various assumptions.

The cases of boundary controls are studied separately using the modified increment method and a number of first-order necessary optimality conditions are studied.

In the case of aviability of inequality type constraints, a necessary optimality condition of a constructive character and being equivalent to the Pontryagin classic maximum principle is proved for a number of control problems described by ordinary differential equations with inequality type functional constraints on the state of the system.

Chapter III is devoted to the study of a number of optimal control problems being a discrete analogue of optimal control problems considered in chapter II.

Discrete analogues of the Pontryagin maximum principle, linearized maximum principle, analogue of the Euler equation are proved by using the methods of implicit and explicit linearization.

The case of boundary controls is studied separately. Necessary and sufficient condition of optimality representing an analogue of the discrete maximum principle is proved in the linear case.

The basic results of the dissertation work are in the following works:

1. Агамалыева, А.И., Мансимов, К.Б. Об одной задаче управления динамикой популяции // -Баку: Вестник БГУ, сер. физ.-мат. наук, -2016, №2, -с.83-92.
2. Агамалыева, А.И. Представления решения разностного аналога одного класса уравнений, описывающих динамику популяции // -Баку: Вестник БГУ, сер. физ-мат. наук, -2016, №3, -с.64-68.
3. Агамалыева, А.И. Линеаризованные необходимые условия оптимальности в одной задаче управления динамикой популяции // -Баку: Вестник БГУ, сер. физ-мат. наук, -2016, №4, -с.46-52.
4. Агамалыева, А.И., Мансимов, К.Б. Об одной задаче управления, описываемой системой интегро-дифференциальных уравнений // -Томск: Вестник Томского Гос.Ун-та. сер. управ. выч. техники и информатика, -2017, №39, -с. 4–10.
5. Агамалыева, А.И., Мансимов, К.Б. Необходимое условие оптимальности в одной дискретной задаче оптимального управления // -Баку: Вестник БГУ, сер. физ.-мат. наук, -2018, №3, -с.20-28.
6. Агамалыева, А.И. Необходимое условие оптимальности первого и второго порядков в задаче оптимального управления динамикой популяции // -Баку: Журнал Бакинского Инжен. унив., серия матем. и комп. науки, -2020, №1, -с.40-48.
7. Агамалиева, А.И. Об одной начальной задаче управления динамикой популяции // -Баку: Вестник БГУ, сер. физ-мат. наук, -2022, №1, -с.44-52.
8. Агамалиева, А.И. Необходимые условия оптимальности в одной дискретной граничной задаче управления динамикой популяции // -Пермь: Вестник Пермского университета. Математика. Механика. Информатика, -2022, Вып. №2(57), -с. 5–13.
9. Агамалиева, А.И. Необходимое и достаточное условие оптимальности в линейной задаче управления динамики // -Баку: Вестник БГУ, сер. физ-мат. наук, -2022, №1, -с.72-77.
10. Агамалиева, А.И. Аналог формулы Коши для однолинейного интегро-дифференциального уравнения типа Фредгольма // -Улан-Удэ: Вестник БГУ. Математика, информатика, -2022, №2, -с. 11-22.

11. Агамалиева, А. И. Исследование дискретного аналога одной граничной задачи оптимального управления // -Пермь: Прикладная математика и вопросы управления, -2022, № 3. -с. 9–25.
12. Agamaliyeva, A.I. Necessary and sufficient optimality condition in one discrete optimal control problem // -Baku: Informatics and Control Problems 42 Issue, -2022, -pp. 33-38.
13. Агамалыева, А.И. Необходимое условие оптимальности в одной задаче оптимального управления, описываемой разностным аналогом интегро-дифференциального уравнения динамики популяции. // -Иркутск: Матер. междун. симпозиума «Динамические системы, оптимальное управление и математические моделирование», Иркутский гос. Ун-тет, -2019, -с. 182-186.
14. Agamaliyeva, A.I. Necessary optimality in a discrete problem of control of the initial condition // The 8th intern. conf. on Control and Optimization with Industrial Applications (COIA), -Baku 2022, Vol II, -pp 39-41.
15. Agamaliyeva, A.I., Mansimov, K.B. Linearized principle of maximum in a problem of control of population dynamics // The 5th intern. conf. on Control and Optimization with Industrial Applications (COIA), -Baku: -2015, -pp. 34-35.
16. Agamaliyeva, A.I. Representation of the solution of the Cauchy problem analogue for a class of linear difference equations // The 7th intern. conf. Azerbaijan, Turkey and Ukrainian on Mathematical analysis, differential equations and their applications, -Baku: -2015, -pp. 8-9.
17. Агамалыева, А.И., Мансимов, К.Б. Необходимое условие оптимальности в одной задаче управления динамикой популяции // “Funksional analiz və onun tətbiqləri” respublika elmi konf. mater. -Bakı: Bakı Universiteti, -22 iyun, -2016, -s. 67-69.
18. Агамалыева, А.И., Алиева, С.Т., Ахмедова, Ж.Б. Представления решения одной системы линейных неоднородных разностных уравнений типа Фредгольма. // -Иркутск: Матер. Междун. симп. «Динамические системы, оптимальное управление и математические моделирование», Иркутский гос. Ун-тет, -2019, -с. 108-110.

The defense will be held on **20 September 2024** year at **14^{00}** at the meeting of the Dissertation council ED 1.04 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan.

Address: AZ 1141, Baku city, B.Vahabzade str, 9.

Dissertation is accessible at the library of the Institute of Mathematics and Mechanics of the library.

Electronic versions of the dissertation and its abstract are available on the official website of the Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan.

Abstract was sent to the required addresses on **05 July 2024**.

Signed for print: 21.06.2024
Paper format: 60x841/16
Volume: 40000
Number of hard copies: 30