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ABSTRACT

of the dissertation for the degree of Doctor of Science

DIRECT AND INVERSE PROBLEMS FOR NON-SELF-ADJOINT HILL OPERATOR

Speciality: 1211.01 –Differential equation

Field of science: Mathematics

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GENERAL CHARACTERISTICS OF THE WORK

Rationale of the topic and the degree of development. This dissertation is devoted to solve direct and inverse problems for Schrödinger operators with complex-valued, periodic potentials in various settings.

In various branches of mathematical physics, in particular, in non-Hermitian quantum mechanics and crystal theory, it becomes necessary to study differential operators with complex-valued and periodic coefficients.

The main idea of non-Hermitian quantum mechanics is that quantities measured in an experiment are always described by real numbers, not complex numbers, therefore, each observable quantity is associated with an operator acting in the space of state vectors, the eigenvalues of which are the measurement results. Our requirement in constructing operators is that their eigenvalues must be real. The implementation of non-Hermitian quantum mechanics consists of replacing the Hermitian conjugation by a PT- symmetry transformation. The P-symmetry transformation (reflection of spatial coordinates) consists, for example, in changing the sign in front of the coordinate operator, and the T-symmetry transformation (time reversal) consists in changing the sign of the impulse (but not the coordinate), as well as replacing i by $-i$. The potentials considered in the thesis are of the form

$$q(x)=\sum_{n=1}^{\infty} q_n e^{inx},$$

where certain conditions are fulfilled for numbers q_n and are PT-symmetric for $q_n=\overline{q}_n$, i.e. ... $q(x)=\overline{q(-x)}$. These potentials were first considered in M.G.Gasymov in various formulations for the Schrödinger operator and was investigated by I.M.Guseinov, A.Orudzheva, E.Orudzheva,V. Guillemin and A.Uribe, R.Carlson, K.Shin, L.Pastur, V.Tkachenko and Fegan.

In the dissertation work, three main areas are considered.

- We study inverse problems for a pencil of differential operators, as well as for higher-order operators with a complex-valued periodic potential. These cases are very complex. It turned out that the Goursat-type problem for transformation operators is ill-posed and has a solution in the general case only for equations with analytic coefficients. In this regard, it is of great interest to choose some classes of ordinary differential operators for which transformation operators exist.
- We study inverse problems for various types of discontinuous differential operators with complex-valued periodic potential. In connection with important applications in quantum physics, it is of interest to study the spectral characteristics of discontinuous differential operators. As a rule, the problems under consideration are related to the discontinuous properties of the physical characteristics of the medium.
- Scattering problems on quantum graphs are studied. Hill's problem with a complex-valued potential on quantum graphs arises in the most diverse problems of natural science - from waveguide systems and neural networks to discrete-continuous approximations of the Laplacian on a Riemannian manifold.

Goal and tasks of the research.

- Solution of a characterization problem, i.e., a complete inverse problem for a pencil of differential operators.
- Investigation of spectral characteristics, as well as solution of the inverse problem for high-order differential equations with a spectral parameter polynomially included in the equations.
- Investigation of the indefinite spectral problem for the Hill operator from two spectra.
- Solution of the inverse problem for the wave equation with discontinuous velocity.
- Study of wave propagation in a non-conservative medium by studying the spectral characteristics of a beam of differential operators with a delta-shaped potential.

- Study of wave propagation on branching strings by studying the spectral characteristics of the Schrödinger operator on quantum graphs.

Investigation methods. The dissertation work uses the methods of the spectral theory of operators, the theory of functions of a complex variable, the theory of differential equations and equations of mathematical physics.

Key points of the dissertation which will be defended:

- complete inverse problems for a pencil of differential operators
- inverse problems for high-order differential equations with spectral parameter polynomially included in the equations
- inverse problems for different types of discontinuous differential operators with complex-valued periodic potential
- scattering problems on quantum graphs

Scientific novelty of the research. The following main results were obtained.

- Necessary and sufficient conditions are obtained for a given sequence of complex numbers to be a set of spectral data for a second-order operator pencil and higher-order differential operators.
- The direct and inverse problems of spectral analysis for $2n$ order ordinary differential equations with polynomially depending on the spectral parameter are solved. It is shown that the continuous spectrum of the operator pencil fills out the rays $\{k\omega_j : 0 \le k \le \infty, j = \overline{0.2m-1}\}$, where $\omega_j = \exp(\frac{ij\pi}{n})$, and on the continuous spectrum there are spectral singularities that coincide with numbers of the form $\frac{n\omega_j}{2}, j = \overline{0.2m-1}, n = 1,2,3,\ldots$ By generalized normalization numbers, the inverse problem of recovering the coefficients is solved.
- The inverse problem for the Schrödinger operator with complex-valued periodic potentials and a discontinuous coefficient on the whole real axis is solved. The main characteristics of fundamental solutions are investigated, the spectrum of the operator is

studied. The inverse problem is formulated, the uniqueness theorem is proved, and a constructive procedure for solving the inverse problem is proposed

- The classical Hill problem with complex potential is extended to star and loop graphs. The definition of the Hill operator on such graphs is given. The operator is defined by complex, periodic potentials and special boundary conditions are used to connect the values of the functions at the vertices. An effective description of the form of the resolvent is given, and the spectrum is accurately described, and the inverse problem for reflection coefficients is solved.

Theoretical and practical value of the study.

Mathematical methods are given for the study of various direct and inverse problems for the Hill operator with a complex, periodic potential. For the first time, the complete solution of the inverse problem for a pencil of differential operators with complex periodic potentials is investigated, the inverse problem for normalization numbers for a high-order operator with a polynomially depending spectral parameter is solved. Various discontinuous inverse problems are investigated. The results obtained for the classical Hill problem are generalized to quantum graphs. The obtained results of the dissertation can be applied in the theory of direct and inverse problems of mathematical and quantum physics.

Approbation and application. The results of the dissertation were presented at the following seminars: at the seminar of the Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan (acad. F.G.Maksudov); at the seminat of the "Non-harmonic analysis" (corr-member of ANAS, prof.B.T.Bilalov), "Functional analysis" (prof. H.I.Aslanov) və "Differential equation" (prof. A.B.Aliyev), at the department of Applied Mathematics of Baku State University (acad. M.G.Gasimov); the Institute of Applied Mathematic Baku State University (acad. F.A.Aliyev); at the University of Nantes (prof. A.Nachaoui, France, 2011); at the Polytechnic University of Valencia (prof. Luis García Raffi, Spain, 2016); at a seminar on differential equations of the Masaryk

University, Brno, (the Czech Republic, 2017); at Keel University, (prof. Y.Kaplunova, England, 2017, 2019); at the international conference on ill-posed and inverse problems dedicated to the 70th anniversary of academician M.M. Lavrentieva (August 5-9, 2002 Novosibirsk, Russia); at the 38-th annual Iranian Mathematical conference (03-10 September 2007, Zanjan, Iran); Third conference of the World Mathematical Society of Turkic countries (June 30 - July 4, 2009, Baku, Azerbaijan); Bulgarian-Turkish-Ukrainian scientific conference "Mathematical analysis, Differential Equations and their applications" (September 15-20, 2010, Sunny Beach, Bulgaria); International conference on functional analysis dedicated to tthe 90th anniversary of V.E.Lyantse (3-10 November 2010, Lviv, Ukraine); VI congress of the Turkic World Mathematical Society (July 11-13 -2018, Baku, Azerbaijan); VI Congress of the Turkic World Mathematical Society (July 11-13 -2018, Baku, Azerbaijan).

Personal contribution of the author. All the results obtained in the dissertation belong to the author.

- The complete inverse problem is solved for operator pencils of the second-order and operators of higher-order
- Spectral problems for discontinuous spectral problems are solved
- Classical results obtained for operators with complex periodic potentials extended to star and loop-shaped graphs

Publications of the author. Publications in editions recommended by HAC under the President of the Republic of Azerbaijan -24, conference materials - 11.

The name of the institution where the thesis is performed Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan.

Structure and volume of the dissertation (in signs, indicating the volume of each structural subsection separately). The total volume of the dissertation –401001 signs (the title page-645 and content 3524 signs, introduction – 75000 signs, chapter I – 122000 signs, chapter II – 38000 signs, chapter III – 76000 signs chapter IV-84000 signs, conclusion – 1832 signs).

THE MAİN CONTENT OF THE DISSERTATION

The thesis consists of an introduction, four chapters and a list of used literature.

The introduction justifies the relevance of the research topic and shows the degree of its elaboration, formulates the purpose and task of the research, provides scientific novelty, notes the theoretical and practical value of the research, and also provides information on the approbation of the work.

In the **first chapter** of the dissertation, the problem of characterization is solved, which consists of determining the necessary and sufficient conditions for the scattering data so that the reconstructed potentials belong to a certain class.

Introduced the class Q^2 of all complex-valued, 2π periodic functions on the real axis R, belonging to space $L_2[0,2\pi]$ and its subclass Q_+^2, consisting of functions of the form

$$q(x)=\sum_{n=1}^{\infty} q_n e^{inx}, \qquad (1)$$

The object of research is a non-self-adjoint operator pencil L generated by the differential expression

$$l\left(\frac{d}{dx},\lambda\right)\equiv -\frac{d^2}{dx^2}+2\lambda p(x)+q(x)-\lambda^2 \qquad (2)$$

in a space $L_2(-\infty,+\infty)$ with potentials $p(x),q(x)\in Q_+^2$, for which the conditions

$$\sum_{n=1}^{\infty} n\cdot|p_n| = p<\infty;\ \sum_{n=1}^{\infty}|q_n| = q<\infty \qquad (3)$$

λ - spectral parameter.

In 1.1.2, the analytic properties of solutions to the equation $Ly=0$ are investigated, for which the fundamental system of solutions of the equation

$$-y''(x)+2\lambda p(x)y(x)+q(x)y(x)=\lambda^2 y(x) \qquad (4)$$

is explicitly constructed

Theorem 1. Let the potentials $p(x), q(x)$ belong to space Q_+^2, for which conditions (3) are satisfied. Then the differential equation (4) has particular solutions that can be represented in the form of

$$f^{\pm}(x,\lambda) = e^{\pm i\lambda x}\left(1 + \sum_{n=1}^{\infty} V_n^{\pm} e^{inx} + \sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty} \frac{V_{n\alpha}^{\pm}}{n \pm 2\lambda} e^{i\alpha x}\right) \tag{5}$$

Here $V_n^{\pm}, V_{n\alpha}^{\pm}$, are the numbers for which the series converge

$$\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\alpha=n}^{\infty} \alpha \left|V_{n\alpha}^{\pm}\right|; \quad \sum_{n=1}^{\infty} n^2 \left|V_n^{\pm}\right| \tag{6}$$

It follows from representation (5) that $f^{\pm}(x,\lambda)$ is a holomorphic function for λ and can have first-order poles at points $\lambda = \pm\frac{n}{2}, n = 1,2,3\ldots$

Lemma 1. For the $\lambda \neq 0, \lambda \neq \pm n/2,$ functions form a fundamental system of the solution to the equation (4).
It is shown that with the notation

$$f_n^{\pm}(x) = \lim_{\lambda \to \pm n/2} (n \pm 2\lambda) f^{\pm}(x,\lambda) = \sum_{\alpha=n}^{\infty} V_{n\alpha}^{\pm} e^{i\alpha x} e^{-i(n/2)x} \tag{7}$$

the relation holds

$$f_n^{\pm}(x) = S_n^{\pm} f^{\mp}(x, \mp n/2). \tag{8}$$

Comparison of the formulas for these expressions shows

$$S_n^{\pm} = V_{nn}^{\pm} \tag{9}$$

Definitions 1. A sequence $\left\{S_n^{\pm}\right\}_1^{\infty}$ constructed using formulas (9) is called a set of spectral data of an operator L with potentials $p(x), q(x) \in Q_+^2$.

Theorem 2. (the main result of this chapter). To a given sequence of complex numbers to $\left\{S_n^{\pm}\right\}_1^{\infty}$ be a set of spectral data of an operator L with potentials, $p(x), q(x) \in Q_{+}^{2}$ it is necessary and sufficient, the simultaneous fulfilment of the condition

1. $$\sum_{n=1}^{\infty} n\left|S_{n}^{\pm}\right| < \infty \tag{10}$$

2. Infinite determinant

$$D(z) = \det\left\|\delta_{nm} - \sum_{k=1}^{\infty} \frac{4s_{m}^{-}s_{k}^{+}}{(m+k)(n+k)} e^{i\frac{m+k}{2}z} e^{i\frac{n+k}{2}z}\right\|_{n,m=1}^{\infty} \tag{11}$$

exists, is continuous, does not vanish in a closed half-plane $\overline{C_{+}} = \{z : \operatorname{Im} z \geq 0\}$, and is analytic inside an open half-plane $C_{+} = \{z : \operatorname{Im} z > 0\}$.

In 1.1.3 the inverse problem of scattering theory is considered. First, the spectral properties of the operator L are investigated, which are based on the study of the Floquet solutions to the equation

$$-y''(x) + 2\lambda p(x)y(x) + q(x)y(x) = \lambda^{2} y(x).$$

Since $q, p \in Q_{2\pi}^{+}$ analytically continued into the upper half-plane $\Pi^{+} = \{z : \operatorname{Im} z > 0\}$, then, of course, it is convenient to consider (1) for all $x \in \Pi^{+}$. Assuming $x = it,\ \lambda = -i\mu,\ Y(t) = y(x)$, $t \in R^{+}$, we obtain the equation

$$-Y''(t) + 2\mu\,\overline{p}(it)Y(t) + \overline{q}(it)Y(t) = \mu^{2} Y(t) \tag{12}$$

in which

$$\overline{p}(t) = ip(it) = i\sum_{n=1}^{\infty} p_{n}e^{-nt} \tag{13}$$

$$\overline{p}(t) = -q(it) = -\sum_{n=1}^{\infty} q_{n}e^{-nt}. \tag{14}$$

As a result, obtained the equation (12) where the potentials exponentially decrease at $t \to \infty$. Whence it follows that it is possible to use the method of transformation operators, well-known in the theory of inverse problems.

According to the definition of the transformation operator to infinity, the solution to equation (12) can be represented in the form of

$$f_{\pm}(t,\mu)=\Psi^{\pm}(t)e^{\pm i\mu t}+\int_{t}^{\infty}K^{\pm}(t,u)e^{\pm i\mu u}du. \tag{15}$$

Comparing the formulas (15) and (12), we have

$$K^{\pm}(t,u)=\frac{1}{2i}\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}V_{n\alpha}^{\pm}e^{-\alpha t}e^{-(n-t)n/2} \tag{16}$$

$$\Psi^{\pm}(t)=1+\sum_{n=1}^{\infty}V_{n}^{\pm}e^{-nt} \tag{17}$$

The kernel of the transformation operator $K^{\pm}(t,u)$ and the $\Psi^{\pm}(t)$ in our case are constructed constructively. Thus, we have proved the following theorem.

Theorem 3. The function $\Psi^{\pm}(t)$ and the kernel $K^{\pm}(t,u)$, $u\geq t$, transformation operator to infinity to the equation (12) allows the representation (16) and (17) respectively, where the series

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\alpha=n}^{\infty}\alpha\left|V_{n\alpha}^{\pm}\right| \ :\sum_{n=1}^{\infty}n^{2}\left|V_{n}^{\pm}\right|$$

converge.

Further, the questions of the solvability of the main equation and the uniqueness of the solution of the inverse problem are studied.

Section 1.2 solves the complete inverse problem for the high-order operator L_1 generated by the differential expression

$$l(y)=(-1)^{m}y^{(2m)}(x)+\sum_{\gamma=0}^{2m-2}P_{\gamma}(x)y^{(\gamma)}(x) \tag{18}$$

in space $L_2(-\infty,+\infty)$ with potentials $P_{\gamma}(x)\in Q_{+}^{2}$, where Q_{+}^{2} consists of functions of

$$P_\gamma(x)=\sum_{n=1}^{\infty}P_{\gamma n}e^{inx};\quad \sum_{\gamma=0}^{2n-2}\sum_{n=1}^{\infty}n^\gamma\left|P_{\gamma n}\right|<\infty \tag{19}$$

and is a subclass of the class of Q^2 all $2\pi-$ periodic complex-valued functions on the real axis R belonging to space $L_2[0,2\pi]$.

In this section, we introduce some well-known facts and concepts that are used in solving the problem and also formulate the main result.

Theorem 4. For a given sequence of complex numbers to $\left\{\widetilde{S}_{nj}\right\}_{n=1,\,j=1}^{\infty,\,2m-1}$ to be a set of spectral data of an operator L_1 generated by differential expression (18), it is necessary and sufficient to simultaneously satisfy the condition

(1) $\left\{n\ \widetilde{S}_{nj}\right\}_{n=1,\,j=1}^{\infty,\,2m-1}\in l_1$ (20)

2) infinite determinant

$D(z)\equiv$

$$\equiv\det\left\|\delta_{rn}E_{2m-1}-\left\|\frac{i\left(1-\omega_j\right)\widetilde{S}_{nj}}{r\omega_l\left(1-\omega_j\right)-n\left(1-\omega_j\right)}e^{i\frac{n}{1-\omega_j}z}e^{-i\frac{r\omega}{1-\omega_l}z}\right\|_{j,l=1}^{2m-1}\right\|_{r,n=1}^{\infty} \tag{21}$$

exists, is continuous, does not vanish in a closed half-plane, $\overline{C_+}=\{z:\operatorname{Im}z\geq 0\}$ and is analytic inside an open half-plane $C_+=\{z:\operatorname{Im}z>0\}$.

The direct and inverse problem of spectral analysis for one class of differential equations of even order with a polynomially entering spectral parameter is solved in section 1.3.1.

A differential operator pencil $L_2(k)$ generated by an expression of the form

$$l(y)=(-1)^m y^{(2m)}(x)+\sum_{\gamma=0}^{2m-2} P_\gamma(x,k) y^{(\gamma)}(x)-k^{2m} y(x), \quad (22)$$

where

$$P_\gamma(x,k)=\sum_{s=0}^{2m-\gamma-1}\sum_{n=1}^{\infty} p_{\gamma sn} k^s e^{inx} \quad (23)$$

and a number

$$\sum_{\gamma=0}^{2m-2}\sum_{s=0}^{2m-\gamma-1}\sum_{n=1}^{\infty} n^{\gamma+s}\left|p_{\gamma sn}\right| \quad (24)$$

converges, but k is a spectral parameter.

A resolvent is constructed and the spectrum of an operator pencil $L_2(k)$ is studied using the theorem

Theorem 5. The operator pencil $L_2(k)$ has a purely continuous spectrum that fills out the rays

$$\left\{k\omega_j : 0 \le k < \infty; j=\overline{0.2m-1}, \omega_j=\exp\left(\frac{ij\pi}{m}\right)\right\},$$

and on the continuous spectrum, there may be spectral singularities that coincide with numbers of the form $\frac{n\omega_j}{2}, n=1,2,\ldots$

Let be

$$a_m=\max_{\substack{1\le j\le l\le 2m-1\\ 1\le n,\ r<\infty}} \frac{\left|(1-\omega_j)(n+r)\right|}{\left|r(1-\omega_j)-n(1-\omega_l)\omega_j\right|},$$

$$S_n=\sum_{\nu=0}^{2m-1}\sum_{j=1}^{2m-1} n^{2m-2}\left|S_{j\nu n}\right|.$$

Expansion in eigenfunctions is obtained, and the inverse problem is solved using the following theorem.

Theorem 6. Let for the given numbers $S_{j\nu n}, j=\overline{0.2m-1}$, $\nu=0.2m-1, n\in N$ the following conditions be satisfied

$$\sum_{n=1n}^{\infty} n\left|S_n\right| < \infty \text{ ; } 4^{m-1} a_m \sum_{n=1}^{\infty} \frac{\left|S_n\right|}{n+1} = \rho < 1$$

Then there are functions $p_\gamma(x,k), \gamma = \overline{0.2m-2}$, of the form (23) for which condition (24) is satisfied and the numbers $S_{j\pi n}$ are "generalized normalization numbers" of the operator with recovered coefficients.

Section 1.4 solves the problem of determining the Hill operator from two spectra.

Let us consider the differential equation

$$-y'' + q(x)y = \lambda^2 y, \; 0 \le x \le 2\pi, \qquad (25)$$

where $q(x)$ is a complex periodic function of the form (1), (3) with the boundary conditions

$$y'(0) = y(2\pi) = 0, \qquad (26)$$

$$y'(0) = y'(2\pi) = 0. \qquad (27)$$

The eigenvalues $\{\lambda_n\}$ and $\{\mu_n\}$ of problems (25), (26) and (25), (27) are also introduced . It turns out that, if $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ solutions of equation (25), satisfies the initial conditions

$$\phi(0,\lambda) = 1, \;\; \phi'(0,\lambda) = 0,$$

$$\psi(0,\lambda) = 0, \;\; \psi'(0,\lambda) = 1,$$

then the eigenvalues of boundary value problems (25), (26) and (25), (27) coincide, respectively, with the zeros of the functions

$$\Phi_1(\lambda) = \varphi(2\pi,\lambda) \text{ and } \Phi_2(\lambda) = \varphi'(2\pi,\lambda).$$

Theorem 7. The potential $q(x)$ is uniquely determined by the spectra $\{\lambda_n\}$ and $\{\mu_n\}$ of the problems (25), (26) and (25), (27), respectively.

Chapter II is devoted to the study of indefinite spectral problems for potentials linearly depending on the λ so-called energy-dependent potentials, that is, the pencil of L_3 non-self-adjoint differential operators generated by the formal differential expression

$$l\left(\frac{d}{dx},\lambda\right)\equiv\frac{1}{\rho(x)}\left\{-\frac{d^2}{dx^2}+2\lambda p(x)+q(x)\right\}$$

in space $L_2(-\infty,+\infty)$. There λ is a complex number, and the coefficients $p(x),q(x)$ are defined as (1), (3) and for the $\rho(x)$ fulfilled

$$\rho(x)=\begin{cases}1 & x\geq 0,\\ -1 & x<0.\end{cases} \tag{28}$$

In 2.2, special solutions of the equation $l\left(\frac{d}{dx},\lambda\right)=\lambda^2 y(x)$ are studied using the following theorem

Theorem 8. Let it $p(x),q(x)$ have the form (1)-(3), respectively, and for $\rho(x)$ satisfies condition (28).
Then the equation

$$-y''(x)+2\lambda p(x)y(x)+q(x)y(x)=\lambda^2\rho(x)y(x) \tag{29}$$

has a solution of the form

$$f_1^{\pm}(x,\lambda)=e^{\pm\lambda x}\left(1+\sum_{n=1}^{\infty}V_n^{\pm}e^{inx}+\sum_{n=1}^{\infty}\frac{1}{n\mp 2i\lambda}\sum_{\alpha=n}^{\infty}V_{n\alpha}^{\pm}e^{i\alpha x}\right)$$

for $x\geq 0$

$$f_2^{\pm}(x,\lambda)=e^{\pm\lambda x}\left(1+\sum_{n=1}^{\infty}V_n^{\pm}e^{inx}+\sum_{m=1}^{\infty}\frac{1}{n\mp 2i\lambda}\sum_{\alpha=n}^{\infty}V_{n\alpha}^{\pm}e^{i\alpha x}\right)$$

for $x<0$

where the numbers $V_n^{\pm}$ and $V_{n\alpha}^{\pm}$ are determined from the recurrence relations and the series

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\alpha=n}^{\infty}\alpha\left|V_{n\alpha}^{\pm}\right| : \sum_{n=1}^{\infty}n^2\left|V_n^{\pm}\right|$$

converge.

It will be easy to notice that the functions $f_1^+(x,\lambda)$, $f_1^-(x,\lambda)$, $\left(f_2^+(x,\lambda),\ f_2^-(x,\lambda)\right)$ form a fundamental system of solutions to equation. (29) for $\lambda \neq 0, \lambda \neq \pm\frac{n}{2}$ and $\lambda \neq \pm\frac{in}{2}$.

Then, using the conjugation conditions

$$y(0+) = y(0-)$$
$$y'(0+) = y'(0-)$$

we find that each solution to equation. (29) can be represented as linear combinations of these solutions

$$f_2^+(x,\lambda) = C_{11}(\lambda) f_1^+(x,\lambda) + C_{12}(\lambda) f_1^-(x,\lambda) \quad x \geq 0 \tag{30}$$

$$f_1^+(x,\lambda) = C_{22}(\lambda) f_{21}^+(x,\lambda) + C_{21}(\lambda) f_{21}^-(x,\lambda) \quad x < 0 \tag{31}$$

Using (30), (31), we obtain the following relations for the coefficients $C_{21}(\lambda)$, $C_{22}(\lambda)$,

$$C_{22}(\lambda) = iC_{11}(-\lambda); \quad C_{21}(\lambda) = iC_{12}(\lambda)$$

Let us denote

$$f_n^\pm(x) = \lim_{\lambda \to \mp\frac{n}{2}} (n \pm 2\lambda) f_1^\pm(x,\lambda) = \sum\nolimits_{\alpha=n}^{\infty} V_{n\alpha}^\pm e^{i\alpha x} e^{ii\frac{n}{2}x}$$

It was found that the functions $f_1^\pm\left(x, \mp\frac{n}{2}\right)$ and $f_n^\pm(x)$ are linearly dependent. Hence,

$$f_1^\pm(x) = S_n^\pm f_1^\pm\left(x, \mp\frac{n}{2}\right) \tag{32}$$

Analysis of formula (32) shows that $S_n^\pm = V_{nn}^\pm$.

The 2.3 studied the spectrum of L_3, in space $L_2(-\infty,\infty)$.

Let us divide the λ plane into sectors

$$S_k=\left\{\frac{k\pi}{2}<\arg\lambda<\frac{(k+1)\pi}{2},k=\overline{0,3}\right\}$$

The following representation holds for the kernel of the resolvent of the operator L_3,

$$R_{11}(x,t,\lambda)=\frac{1}{W[f_1^+,f_2^+]}\begin{cases}f_1^+(x,\lambda)f_2^+(t,\lambda) & t<x\\ f_1^+(t,\lambda)f_2^+(x,\lambda) & t>x\end{cases}\qquad \lambda\in S_0$$

$$R_{12}(x,t,\lambda)=\frac{1}{W[f_1^+,f_2^-]}\begin{cases}f_1^+(x,\lambda)f_2^-(t,\lambda) & t<x\\ f_1^+(t,\lambda)f_2^-(x,\lambda) & t>x\end{cases}\qquad \lambda\in S_1$$

$$R_{21}(x,t,\lambda)=\frac{1}{W[f_1^-,f_2^-]}\begin{cases}f_1^-(x,\lambda)f_2^-(t,\lambda) & t<x\\ f_1^-(t,\lambda)f_2^-(x,\lambda) & t>x\end{cases}\qquad \lambda\in S_2$$

$$R_{22}(x,t,\lambda)=\frac{1}{W[f_1^-,f_2^+]}\begin{cases}f_1^-(x,\lambda)f_2^+(t,\lambda) & t<x\\ f_1^-(t,\lambda)f_2^+(x,\lambda) & t>x\end{cases}\qquad \lambda\in S_3$$

Directly from the form of the resolvent, one can obtain that for all λ outside of $\operatorname{Re}\lambda=0$ $\operatorname{Im}\lambda=0$ and $C_{11}(\pm\lambda)\neq 0$, $C_{12}(\pm\lambda)\neq 0$, the resolvent of the operator exists and is bounded.

Theorem 9. The operator L_3 has no real and purely imaginary eigenvalues. The continuous spectrum fills out the axes $\operatorname{Re}\lambda=0$ and $\operatorname{Im}\lambda=0$ on which there may be spectral singularity that coincides with numbers of the form $\frac{in}{2},\frac{n}{2},n=\pm1,\pm2,\ldots$ The eigenvalues of the operator L_3 are bounded and coincide with the zeros of the functions $C_{11}(-\lambda),C_{11}(\lambda),C_{12}(\lambda),C_{12}(-\lambda)$ on the sectors

$$S_k=\left\{\frac{k\pi}{2}<\arg\lambda<\frac{(k+1)\pi}{2},k=\overline{0,3}\right\}$$

respectively.

Definition 2. The set of quantities $\{C_{11}(\lambda), C_{12}(\lambda)\}$ is called the spectral data of the operator L_3.

Theorem 10. All the numbers $V_{n\alpha}^{\pm}, n < \alpha$ and $V_{\alpha}^{(\mp)}$ can be uniquely determined by using the numbers $V_{nn}^{\pm}$.

The **third chapter** is devoted to the spectral analysis of differential operators with truncated coefficients.

Section 3.1 consists of four parts, where discontinuous inverse problems of spectral analysis are solved.

The inverse problem of spectral analysis is solved for the wave equation with discontinuous wave propagation.

For this, the differential equation is considered

$$-y''(x) + q(x)y(x) = \lambda^2 \rho(x) y(x) \tag{33}$$

in space $L_2(-\infty, +\infty, \rho(x))$ under the assumption that the potential $q(x)$ is determined using (1), (3), and

$$\rho(x) = \begin{cases} 1 & for \;\; x \geq 0, \\ \beta^2 & for \;\; x < 0, \beta \neq 1, \beta > 0 \end{cases}$$

a λ is a complex number.

In 3.1.2, the main properties of particular solutions of equation (33) are studied.

It is shown that the equation

$$-y''(x) + q(x)y(x) = \lambda^2 \rho(x) y(x)$$

has solutions of the form

$$f_1^{\pm}(x,\lambda) = e^{\pm i\lambda x}\left(1 + \sum_{n=1}^{\infty} \frac{1}{n \pm 2\lambda} \sum_{\alpha=n}^{\infty} V_{n\alpha} e^{i\alpha x}\right) \quad for \quad x \geq 0$$

$$f_2^{\pm}(x,\lambda) = e^{\mp ii\lambda\beta x}\left(1 + \sum_{n=1}^{\infty} \frac{1}{n \mp 2\lambda} \sum_{\alpha=n}^{\infty} V_{n\alpha} e^{i\alpha x}\right) \quad for \quad x < 0$$

where the numbers are $V_{n\alpha}$ determined from the recurrence relations

$$\alpha(\alpha-n)V_{n\alpha}+\sum_{s=n}^{\alpha-1}q_{\alpha-s}V_{ns}=0,\ \ 1\le n\le\alpha$$

$$\alpha\sum_{s=n}^{\alpha-1}V_{n\alpha}+q_{\alpha}=0$$

for what converged the series

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\alpha=n}^{\infty}\alpha\left|V_{n\alpha}\right|.$$

Note that the functions $f_1^+(x,\lambda)$ and $f_1^-(x,\lambda)$ $f_1^+(x,\lambda)$ $\left(f_2^+(x,\lambda)\right.$ and $\left.f_2^-(x,\lambda)\right)$are linearly independent and their Wronskian is equal to $2i\lambda(-2i\lambda\beta)$.

Then each solution to equation (33) can be represented as a linear combination of these solutions.

$$f_2^+(x,\lambda)=C_{11}(\lambda)f_1^+(x,\lambda)+C_{12}(\lambda)f_1^-(x,\lambda)\quad for\quad x\ge 0$$

$$f_1^+(x,\lambda)=C_{22}(\lambda)f_{21}^+(x,\lambda)+C_{21}(\lambda)f_{21}^-(x,\lambda)\quad for\quad x<0$$

The coefficients $C_{ij}(\lambda), i,j=1,2$ are expressed in terms of the Wronskian of solutions $f_1^{\pm}(x,\lambda)$ and $f_2^{\pm}(x,\lambda)$.

Indeed

$$C_{11}(\lambda)=\frac{1}{2i\lambda}W\left[f_2^+(0,\lambda),f_1^-(0,\lambda)\right],$$

$$C_{12}(\lambda)=\frac{1}{2i\lambda}W\left[f_1^+(0,\lambda),f_2^+(0,\lambda)\right],$$

$$C_{22}(\lambda)=-\frac{1}{\beta}C_{11}(-\lambda),\ C_{21}(\lambda)=\frac{1}{\beta}C_{12}(\lambda).$$

Following the physical meaning of the solutions $f_1^{\pm}(x,\lambda)$ and $f_2^{\pm}(x,\lambda)$, we will call the coefficient $\frac{C_{11}(\lambda)}{C_{12}(\lambda)}-$ the amplitude of the reflection coefficient to the right, $\frac{1}{C_{12}(\lambda)}$ and $\frac{1}{C_{21}(\lambda)}$ the amplitudes of the transmission coefficients .

Lemma 2. The eigenvalues of the operator L_4 are finite and coincide with the squares of the zeros of functions $C_{12}(\lambda)$, $C_{12}(-\lambda)$, from the sectors $S_k, k=0,1,$ respectively.

It is shown that the operator L_4, generated by the differential expression $\frac{1}{\rho(x)}\left(-\frac{d^2}{dx^2}+q(x)\right)$ in space $L_2(-\infty,\infty,\rho(x))$, the kernel of the resolvent has the form

$$R_{11}(x,t,\lambda)=\frac{1}{C_{12}(\lambda)}\begin{cases} f_1^+(x,\lambda)f_2^+(t,\lambda) & for \quad t\le x \\ f_1^+(t,\lambda)f_2^+(x,\lambda) & for \quad t\ge x \end{cases}$$

when $\operatorname{Im}\lambda>0$ and

$$R_{12}(x,t,\lambda)=\frac{1}{C_{12}(-\lambda)}\begin{cases} f_1^-(x,\lambda)f_2^-(t,\lambda) & for \quad t\le x \\ f_1^-(t,\lambda)f_2^-(x,\lambda) & for \quad t\ge x \end{cases}$$

for $\operatorname{Im}\lambda<0$.

Thus, there exists $R_{\lambda^2}(x,t,\lambda)=\left(L_4-\lambda^2 I\right)^{-1}$ and is bounded for all λ^2 outside the positive semiaxis and $C_{12}(\pm\lambda)\neq 0$. Moreover, it was proved that the coefficient $C_{12}(\lambda)$, is an analytic function for $\operatorname{Im}\lambda>0$ and has a limited number of zeros, moreover, if $C_{12}(\lambda_n)=0$, then

$$\frac{d}{d\lambda}\left[C_{12}(\lambda)\right]_{\lambda=\lambda_n}=-i\int_{-\infty}^{+\infty}\rho(x)f_1^+(x,\lambda_n)f_2^+(x,\lambda_n)dx.$$

We get that the eigenvalues of the operator L_4 are finite and coincide with the squares of the zeros of the functions $C_{12}(\lambda), C_{12}(-\lambda)$ from the sectors S_k, $k=0,1$, respectively. Note that we can obtain a further useful relation

$$\beta=2\lim_{IM\lambda\to\infty}C_{12}(\lambda)-1$$

Theorem 11. The spectrum of an operator L_4 is real, purely continuous and fills out the semiaxis $[0,\infty)$, while the continuous spectrum may have spectral singularities of the first order, which coincide with numbers of the form $(n/2)^2, (n/2\beta)^2, n = 1,2,\dots$

The problem of eigenfunction expansion discussed in paragraph 3.1.3, where it is proved that for an arbitrary function $\Psi(x)$ belonging to space $L_2(-\infty,+\infty,\rho(x))$, we have the following expansion in eigenfunctions

$$\Psi(x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \rho(t)\Psi(t)\left[\int_{\Gamma_0^-} \frac{f_1^+(x,\lambda)f_1^+(t,\lambda)}{2i\lambda C_{12}(\lambda)C_{22}(\lambda)} d\lambda + G_{11}(\lambda_n, x, t) + \right.$$

$$\left. + \frac{2}{in} V_{nn} f_1^+(x, n/2) f_1^+(t, n/2) + F\left(x, t, \frac{n}{2\beta}\right)\right] dt$$

In 3.1.4, the inverse problem for an operator L_4 based on spectral data $\{C_{12}(\lambda), V_{nn}\}$ is solved. It is proved that the spectral data $\{C_{12}(\lambda), V_{nn}\}$ uniquely determines the function $q(x)$, and the number β.

Section 3.2 considers the problem of the spectrum of a non-self-adjoint operator pencil with a discontinuous coefficient.

A spectral problem for an operator pencil L_5 with complex periodic potentials in the space $L_2(-\infty,+\infty,\rho(x))$ generated by the differential expression

$$l\left(\frac{d}{dx},\lambda\right) \equiv \frac{1}{\rho(x)}\left\{-\frac{d}{dx^2} + 2\lambda p(x) + q(x)\right\}, \tag{34}$$

where λ - is a complex number, and for the coefficients $p(x), q(x)$ conditions (1), (3) are satisfied and $\rho(x)$ has the form

$$\rho(x)=\begin{cases}1 & for \quad x\in(-\infty,b_1)\\ \gamma_1^2 & for \quad x\in(b_1,b_2)\\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \gamma_n^2 & for \quad x\in(b_n,\infty)\end{cases} \qquad (35)$$

where $\gamma_j \neq 0(j=1,2,\ldots,n)$.

It is known that the study of the spectral properties of an operator pencil L_5 is based on the analysis of solutions to the equation

$$-y''+[2\lambda p(x)+q(x)]y=\lambda^2\rho(x)y, x\in R \qquad (36)$$

Theorem 12. Let the potentials $p(x),q(x)$ have the form (1), (3) and condition (35) is satisfied for $\rho(x)$. Then the equation $L_5(y)=\lambda^2\rho(x)y$ has solutions of the form

$$f_j^{\pm}(x,\lambda)=e^{\pm i\lambda\gamma_j x}\left(1+\sum_{n=1}^{\infty}\upsilon_n^{\pm}e^{inx}+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{\upsilon_{n\alpha}^{\pm}}{n\pm 2\lambda\gamma_j}e^{i\alpha x}\right)$$

for $x\in(b_j,b_{j+1})$ $j=0,1,2,\ldots m,$ $\gamma_0=1,$ $b_0=-\infty,$ $b_{m+1}=\infty$ where the numbers $\upsilon_n^{\pm}$ and $\upsilon_{n\alpha}^{\pm}$ are determined from recurrence relations and $f_j^{\pm}(x,\lambda)$ admits double term differentiation.

Theorem 13. The spectrum of the operator pencil L_5 consists of a continuous spectrum filling the axis $(-\infty,+\infty)$ on which there can be spectral singularities that coincide with numbers of the form $\left(\pm\frac{n}{2\gamma_j}\right)$, $n\in N,$ $j=0,1,2,3,\ldots,m,$ and a finite number of eigenvalues.

In 3.3 we study the spectral analysis of one class of non-self-adjoint differential operator pencils with a generalized function.

An inverse problem is considered for a pencil L_6 of non-self-adjoint differential operators generated by a formal differential expression

$$l\left(\frac{d}{dx},\lambda\right) \equiv -\frac{d}{dx^2} + 2\lambda p(x) + q(x) + \beta\delta(x) - \lambda^2,$$

with a generalized function in space $L_2(-\infty,+\infty)$. Here $\delta(x)$ is the Dirac delta function, $\beta < 0$ is a real number, λ is a complex number, and the coefficients $p(x), q(x)$ satisfy the conditions (1), (3).

It is proved that the spectrum of the operator pencil L_6 consists of a continuous spectrum filling the axis $\{-\infty < \lambda < +\infty\}$ on which there can be spectral singularities that coincide with numbers of the form $\pm n/20,\ n \in N$ and at most one eigenvalue.

The inverse problem is solved, where the problem of determining the functions, $p(x), q(x)$ and β is posed, by numbers $\{S_n^{\pm}\}$. In the beginning, we found clear links between the sequences $\{S_n^{\pm}\}$ and $\{V_{n\alpha}^{\pm}\}$, $\{V_n^{\pm}\}$.

$$V_{mm}^{\pm} = S_m^{\pm} V_{m,\alpha+m}^{\pm} = S_m^{\pm}\left(V_{\alpha}^{\mp} + \sum_{n=1}^{\alpha} \frac{V_{n\alpha}^{\mp}}{n+m}\right),$$

a β is defined using the equality

$$\beta = -\frac{y'(+0) - y'(-0)}{y(0)}$$

These relations are the basic equations for determining $\{q_\alpha\}$, $\{p_\alpha\}$ and β by the numbers $\{S_n^{\pm}\}$.

Theorem 14. For the numbers $\{S_n^{\pm}\}$ to be the “normalization” numbers of an operator pencil of type L_6 with a potential of the form (1), (3), it is sufficient that the conditions

$$\sum_{m=1}^{\infty} m \cdot \left|S_m^*\right| = \delta < \infty; \quad \sum_{m=1}^{\infty} \frac{\left|S_m^*\right|}{1+m} = p < 1,$$

where $\left|S_m^*\right| = \max\left\{\left|S_m^+\right|, \left|S_m^-\right|\right\}$.

Section 3.4 investigates the spectral analysis of a non-self-adjoint Hill operator with a step potential.

Considered the differential equation

$$-y''(x) + q(x)y(x) = \lambda^2 y(x) \qquad (37)$$

in space $L_2(-\infty,+\infty)$ where the prime denotes the derivative for the spatial coordinate under the assumption that the potential

$$q(x) = \begin{cases} \sum_{n=1}^{\infty} q_n^+ e^{inx} & for \quad x \geq 0 \\ \sum_{n=1}^{\infty} q_n^- e^{inx} & for \quad x < 0 \end{cases};$$

$\sum_{n=1}^{\infty} \left|q_n^{\pm}\right| = q < \infty, \quad q_n^+ \neq q_n^-$ and λ is a complex number.

Then the equation

$$-y''(x) + q(x)y(x) = \lambda^2 y(x)$$

has a solution of the form

$$f(x,\lambda) = \begin{cases} f^+(x,\lambda) \ for \quad x \geq 0 \\ f^-(x,\lambda) \ for \quad x < 0 \end{cases}$$

where

$$f^{\pm}(x,\lambda) = e^{i\lambda x}\left(1 + \sum_{n=1} \frac{1}{n \pm 2\lambda} \sum_{\alpha=n} V_{n,\alpha}^{\pm} e^{i\alpha x}\right)$$

and the numbers $V_{n,\alpha}^{\pm}$ are determined from recurrence relations and the series $\sum_{n=1} \frac{1}{n} \sum_{\alpha=n} \alpha \left|V_{n\alpha}^{\pm}\right|$ converges.

Since the functions $f^+(x,\pm\lambda)$ and $f^-(x,\pm\lambda)$ are linearly independent solutions of equaiton (37), respectively, for $x \geq 0$ and $x < 0$, then continuing as a solution to the equation using the conjugation conditions

$$\begin{pmatrix} y(+0) \\ y'(+0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y(-0) \\ y'(-0) \end{pmatrix}$$

it is easy to see that the following equalities hold

$$f^{+}(x,\lambda) = C_{11}(\lambda) f^{-}(x,\lambda) + C_{12}(\lambda) f^{-}(x,-\lambda), \ \ for \ \ x < 0$$

$$f^{-}(x,\lambda) = C_{22}(\lambda) f^{+}(x,\lambda) + C_{21}(\lambda) f^{+}(x,-\lambda), \ \ for \ \ x \geq 0 .$$

Let

$$f_n^{\pm}(x) = \lim_{\lambda \to \mp \frac{n}{2}} (n \pm 2\lambda) f^{+}(x, \pm\lambda) = \sum_{\alpha=n}^{\infty} V_{n\alpha}^{+} e^{i\alpha x} e^{-i\frac{n}{2}x},$$

$$\varphi_n^{\pm}(x) = \lim_{\lambda \to \mp \frac{n}{2}} (n \pm 2\lambda) f^{-}(x, \pm\lambda) = \sum_{\alpha=n}^{\infty} V_{n\alpha}^{-} e^{i\alpha x} e^{-i\frac{n}{2}x},$$

then

$$f_n^{\pm}(x) = V_{nn}^{+} f^{+}\left(x, \frac{n}{2} \right) \text{ and } \varphi_n^{\pm}(x) = V_{nn}^{-} f^{-}\left(x, \frac{n}{2} \right).$$

In section 3.4.2 proved that the spectrum of the operator $L = \left(-\frac{d^2}{dx^2} + q(x) \right)$ consists of a continuous spectrum that fill out the axis $[0, \infty)$ which may be the spectral features that coincide with numbers of the form $\left(\frac{n}{2} \right)^2, n = 1,2,\ldots$ and has a finite number of eigenvalues, defined as the root of the equations $C_{12}(\lambda) = 0$.

In 3.4.3, the inverse problem is studied.

Inverse problem: Reconstruct a potential $q(x)$ based on the given spectral data $\{C_{11}(\lambda), C_{12}(\lambda)\}$.

In this section, we give a constructive procedure for solving the inverse problem from given spectral data. Thus, from the spectral data $\{C_{11}(\lambda), C_{12}(\lambda)\}$, the potential $q(x)$ can be recovered unambiguously and efficiently.

The purpose of the fourth chapter is the spectral analysis of wave propagation in a layered, inhomogeneous medium, such as a branching tube or a system of connected strings.

Section 4.1 gives the formulation of the direct problem.

To study the propagation of waves on branching strings, we must consider the system of equations

$$\begin{cases} -y_1''(x_1,\lambda)+[2\lambda p_1(x_1)+q_1(x_1)]y_1(x_1,\lambda)=\lambda^2 y_1(x_1,\lambda) \\ -y_2''(x_2,\lambda)+[2\lambda p_2(x_2)+q_2(x_2)]y_2(x_2,\lambda)=\lambda^2 y_2(x_2,\lambda) \\ -y_3''(x_3,\lambda)+[2\lambda p_3(x_3)+q_3(x_3)]y_3(x_3,\lambda)=\lambda^2 y_3(x_3,\lambda) \\ \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \quad \ldots\ldots \\ -y_n''(x_n,\lambda)+[2\lambda p_n(x_n)+q_n(x_n)]y_n(x_n,\lambda)=\lambda^2 y_n(x_n,\lambda) \end{cases}$$

where the potentials $p_k(x_k)$ and $q_k(x_k)$, $k=1,2,3,$ are of the form

$$p_k(x_k)=\sum_{n=1}^{\infty} p_{kn}e^{inx_k}, \sum_{n=1}^{\infty} n|p_{kn}|<\infty;$$

$$q_k(x_k)=\sum_{n=1}^{\infty} q_{kn}e^{inx_k}, \sum_{n=1}^{\infty} |q_{kn}|<\infty;$$

with the following system of boundary conditions at the initial points of the positive semiaxis

$$\begin{cases} y_1(0,\lambda)=y_2(0,\lambda)=y_3(0,\lambda)=\cdots=y_n(0,\lambda) \\ y_1'(0,\lambda)+y_2'(0,\lambda)+y_3'(0,\lambda)+\cdots+y_n'(0,\lambda)=0 \end{cases} \tag{38}$$

in the space $L_2(G)=\bigoplus_{k=1}^{n} L_2[0_k,\infty)$ where the notation with the index o_k is used to denote the starting point of the k th positive semiaxis, and the direct sum of the spaces is denoted by $\oplus$. The prime means the derivative for the spatial coordinate and λ is a complex number.

For simplicity of obtaining results without loss of generality in the future, we will consider the case $n=3$.

On the outside space $L_2(G)$

$$L_2(G)=\overset{3}{\underset{k=1}{\oplus}} L_2(N_k),$$

where $N_k[o_k,\infty)$ with dot product $(f,g)_{L_2(G)}=\sum_{k=1}^{3}(f_k,g_k)_{L_2(N_k)}$,

and consider the operator $L_G=\overset{3}{\underset{k=1}{\oplus}} L_k$.

Where $L_k=-d^2/dx_k^2+2\lambda p_k(x_k)+q_k(x_k)$ with domain

$$D(L_G)=\{y(x)|y_k(x),y'_k(x)\in AC[0,R]$$

for all $R>0, y_1(0)=y_2(0)=y_3(0)$

$$y'_1(0)+y'_2(0)+y'_3(0)=0,\ y_k(x),y''_k(x)\in L_2(R^+),k=1,2,3\}$$

Then the problem can be interpreted as a study of an operator $L_G=\overset{3}{\underset{k=1}{\oplus}} L_k$, on the above non-compact graph. The spectral problem can be described as follows:

Find a function $y_k(x_k,\lambda)=(y_{k1}(x_1,\lambda),y_{k2}(x_2,\lambda),y_{k3}(x_3,\lambda))$ satisfying the Sturm-Liouville equation

$$\begin{aligned}&-y''_k(x_k,\lambda)+2\lambda p_k(x_k)y_k(x_k,\lambda)+\\&+q_k(x_k)y_k(x_k,\lambda)=\lambda^2 y_k(x_k,\lambda),\end{aligned}\tag{39}$$

on $N_k, k=1,2,3$ connected at zero with usual Kirchhoff condition and by the initial conditions for the functions $y_k(x_k,\lambda)$, $k=1,2,3$.

a) y_k is continuous at the nodes of the graph, in particular, for our graph

$$y_{k1}(0,\lambda)=y_{k2}(0,\lambda)=y_{k3}(0,\lambda);\tag{40}$$

b) the sum of derivatives over all branches outgoing from a node, calculated for each node, is equal to zero

$$y'_{k1}(0,\lambda)=y'_{k2}(0,\lambda)=y'_{k3}(0,\lambda)=0.\tag{41}$$

For each fixed $k=1,2,3$ on the edge N_k, there is a fundamental system of solutions $f_k^{\pm}(x_k,\lambda)$ to equation (39), which for $f_k^{\pm}(x_k,\lambda)$ $\lambda\neq\pm n/2,\ n\in N$ and $\lambda\neq 0$ has the form:

$$f_k^{\pm}(x_k,\lambda)=e^{\pm i\lambda x_k}\left(1+\sum_{n=1}^{\infty}V_n^{(\pm k)}e^{inx_k}+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{V_{n\alpha}^{(\pm k)}}{n\pm 2\lambda}e^{i\alpha x_k}\right)$$

where $V_n^{(\pm k)}\,V_{n\alpha}^{(\pm k)}$ are numbers and for them the following series converge:

$$\sum_{n=1}^{\infty}n^2\left|V_n^{\pm}\right|;\quad \sum_{n=1}^{\infty}\frac{1}{n}\sum_{\alpha=n+1}^{\infty}\alpha(\alpha-n)\left|V_{n\alpha}^{\pm}\right|;\quad \sum_{n=1}^{\infty}n\cdot\left|V_{nn}^{\pm}\right|$$

As a solution to the problem, we will understand the matrix $Y(x,\lambda)=\left\lfloor y_{jk}(x_k,\lambda)\right\rfloor_{k,j=1,2,3}$ on a non-compact graph based on the following conditions:

1. $L_G Y=\lambda^2 Y$;
2. $y_{jk}(x_k,\lambda)$ a solution on the edge $N_k=[0,\infty),k=1,2,3$
3. $y_{jk}(x_k,\lambda)=T_{jk}(\lambda)f_k^{+}(x_k,\lambda),k\neq j$

and

$$y_{kk}(x_k,\lambda)=f_k^{-}(x_k,\lambda)+R_{kk}(\lambda)f_k^{+}(x_k,\lambda),k=1,2,3,$$

The coefficients $T_{kj}(\lambda)$ and $R_{kk}(\lambda)$ can be found by writing down the boundary conditions (40), (41) for the solution $y_{jk}(x_k,\lambda)$. To be specific, suppose $k=1$, then

$$f_1^{-}(0,\lambda)+R_{11}(\lambda)f_1^{+}(0,\lambda)=T_{12}(\lambda)f_2^{+}(0,\lambda)=T_{13}(\lambda)f_3^{+}(0,\lambda),$$

$$f_1^{-'}(0,\lambda)+R_{11}(\lambda)f_1^{+'}(0,\lambda)+T_{12}(\lambda)f_2^{+'}(0,\lambda)+T_{13}(\lambda)f_3^{+'}(0,\lambda)=0.$$

We solve these equations for $R_{11}(\lambda)$, $T_{12}(\lambda)$ and $T_{13}(\lambda)$. Noting that for the Wronskian solutions $W[f_1^{+}(0,\lambda),f_1^{-}(0,\lambda)]=2i\lambda,$ we get

$$R_{11}(\lambda) = -\frac{f_1^-(0,\lambda)}{f_1^+(0,\lambda)} + \frac{2i\lambda}{f_1^+(0,\lambda) f_1^+(0,\lambda) G(\lambda)}$$

$$T_{12}(\lambda) = \frac{2i\lambda}{f_1^+(0,\lambda) f_2^+(0,\lambda) G(\lambda)}$$

$$T_{13}(\lambda) = \frac{2i\lambda}{f_1^+(0,\lambda) f_3^+(0,\lambda) G(\lambda)}$$

where

$$G(\lambda) = \frac{f_1^{+'}(0,\lambda)}{f_1^+(0,\lambda)} + \frac{f_2^{+'}(0,\lambda)}{f_2^+(0,\lambda)} + \frac{f_3^{+'}(0,\lambda)}{f_3^+(0,\lambda)}.$$

The coefficients $T_{jk}(\lambda)$ and $R_{kk}(\lambda)$ can be found by writing down the boundary conditions (61), (62) for the solution $y_{jk}(x_k, \lambda)$.

In 4.2 we study the properties of the spectrum of an operator L_G. It is proved that the operator L_G has no real eigenvalue.

Theorem 15. The eigenvalues of the operator are finite and coincide with the zeros L_G of the function

$$\Delta(\lambda) = \det F(\lambda) = \left[f_1^+ \cdot f_2^+ \cdot f_3^+\right]$$

where

$$F(\lambda) = \begin{pmatrix} f_1^+ & -f_2^+ & 0 \\ f_1^+ & 0 & -f_3^+ \\ -f_1'^+ & f_2'^+ & f_3'^+ \end{pmatrix}$$

Theorem 16. The spectrum of the operator L_G consists of a continuous spectrum filling out the axis $-\infty < \lambda < +\infty$, on which there may be spectral singularities that coincide with the numbers of the form $\pm n/2,\ n \in N$.

In 4.1.2, we solve the inverse spectral problem on a star graph.

Inverse problem: Taking into account the spectral data, the reflection coefficients $R_{kk}(\lambda)$ on each N_k edge, construct and the potentials $p_k(x_k)$ and $q_k(x_k)$ where $k=1,2,3$.

Theorem 17. On each fixed edge , N_k, $k=1,2,3\ldots$ $n \in N$, the following relation is fulfilled

$$\lim_{\lambda \to n/2}(n-2\lambda)R_{kk}(\lambda)=V_{nn}^{(-k)},$$

$$\lim_{\lambda \to -\frac{n}{2}}(n+2\lambda)\frac{1}{R_{kk}(\lambda)}=V_{nn}^{(-k)}.$$

Shown, that

$$V_{m\alpha+m}^{(\pm k)}=V_{mm}^{(\pm k)}\left(V_{\alpha}^{(\mp k)}+\sum_{n=1}^{\alpha}\frac{V_{n\alpha}^{(\mp k)}}{n+m}\right), m,\alpha=1,2,3,\ldots$$

These relations are fundamental equations for the recovery $p_{nk}(x_k)$ and $q_{nk}(x_k)$ using predetermined integers $V_{n\alpha}^{\pm k}$, $V_{n}^{\pm k}$.

Theorem 18. All the numbers $V_{n\alpha}^{\pm k}, n>\alpha$ and $V_{n\alpha}^{(\mp k)}$ can be uniquely determined from the known numbers $V_{nn}^{\pm k}$.

Theorem 19. Specification of the spectral data uniquely determines the potentials $p_k(x_k)$, $q_k(x_k)$ on each edge N_k, $k=1,2,3$.

In 4.2 considered the inverse spectral problem for the Hill operator on a loop graph.

Considered a non-compact graph G where the half-line is attached to a loop. The graph consists of a non-compact part, which is the axis $\gamma_0=\{x|0<x<\infty\}$, the compact part of the loop, the $\gamma_1=\{z|0<x<2\pi\}$ length of which, for the sake of definiteness, is taken equal to 2π and $\gamma_2 = \{\{x=0\}=\{z=0\}=\{z=2\pi\}\}$ which correspond to the attachment point.

We will investigate a spectral problem describing the one-dimensional scattering of a quantum particle on a graph G.

$$-Y'' + \{q(X) - \lambda^2\}Y = 0, \quad X \in G \setminus \{\gamma_2\} \tag{42}$$
$$Y(x=0) = Y(z=0) = Y(z=2\pi),$$
$$Y'(x=0+0) + Y'(z=0+0) - Y'(z=2\pi-0) = 0.$$

In (42), differentiability X is understood as differentiability for x if $X \in \gamma_0$ and differentiability to z, when $X \in \gamma_1$. At the vertex of the graph, differentiability is undefined. We assume that the potential is defined as $q(X)$ where

$$q(X) = \begin{cases} q_1(x) = \sum\limits_{n=1}^{\infty} q_{1n} e^{inx}, & X \in \gamma_0 \\ q_2(x) = \sum\limits_{n=1}^{\infty} q_{2n} e^{inx}, & X \in \gamma_1 \end{cases}$$

with the condition $\sum_{n=1}^{\infty} |q_{kn}| < \infty$, $k = 1,2$; λ is a spectral parameter.

The solution to the spectral problem (42) will be found as

$$Y(X,\lambda) = \begin{cases} y(x,\lambda), & X \in \gamma_0 \\ u(z,\lambda), & X \in \gamma_1 \end{cases}$$

on a non-compact graph G provided that the following conditions are satisfied.

1) $LY = \lambda^2 Y$;

2) $y(x,\lambda) = f(x,-\lambda) + R_{11}(\lambda) f(x,\lambda)$

where $R_{11}(\lambda)$ is the reflection coefficient and

$$f(x,\lambda) = e^{i\lambda x}\left(1 + \sum_{n=1}^{\infty} \frac{1}{n+2\lambda} \sum_{\alpha=n}^{\infty} V_{n\alpha}^{\gamma_0} e^{i\alpha x}\right)$$

in this case, the numbers are $V_{n\alpha}^{\gamma_0}$ determined from the following recurrence relations

$$\begin{cases} \alpha(\alpha-n)V_{n\alpha}^{\gamma_0}+\sum\limits_{s=n}^{\alpha-1}q_{1\alpha-s}V_{ns}^{\gamma_0}=0, \ 1\le n\le\alpha \\ \alpha\sum\limits_{n=1}^{\alpha}V_{n\alpha}^{\gamma_0}+q_{1\alpha}=0 \end{cases}$$

for which the series $\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\alpha=n}^{\infty}\alpha(\alpha-n)\left|V_{n\alpha}^{\gamma_0}\right|;\ \sum_{n=1}^{\alpha}n\left|V_{n\alpha}^{\gamma_0}\right|$ converges. Note that in our case the following relation is valid

$$\lim_{\lambda\to\mp\frac{n}{2}}(n\pm2\lambda)f(x,\pm\lambda)=V_{nn}^{\gamma_0}f\left(x,\mp\frac{n}{2}\right)$$

To construct a solution $u(z,\lambda)$ on the compact part of the graph, first, we construct the Green's function of the operator on the compact part of the graph on the loop γ_1. Green's function on a loop can be constructed using the fundamental solutions $\varphi(z,\lambda),\theta(z,\lambda)$ satisfying the following conditions

$$\varphi(0,\lambda)=\theta'(0,\lambda)=1$$
$$\varphi'(0,\lambda)=\theta(0,\lambda)=0.$$

In this case, the boundary conditions for the Green's function on the loop are the conditions

$$\begin{cases} G(0,y,\lambda)=G(2\pi,y,\lambda) \\ G'(0,y,\lambda)=G'(2\pi,y,\lambda) \\ \lim_{z\to y+0}G(z,y,\lambda)=\lim_{z\to y-0}G(z,y,\lambda) \end{cases} \tag{43}$$

The function required to construct a solution to the spectral problem on a loop has the form

$$G(z,0,\lambda)=G(z,\pi,\lambda)=\frac{\theta(\pi,\lambda)}{\varphi(\pi,\lambda)+\theta'(\pi,\lambda)-2}\varphi(z,\lambda)+$$
$$+\frac{1-\varphi(\pi,\lambda)}{\varphi(\pi,\lambda)+\theta'(\pi,\lambda)-2}\theta(z,\lambda).$$

Thus, we can look for a solution to the spectral problem as follows

$$Y(X,\lambda)=\begin{cases} f(x-\lambda)+R_{11}(\lambda)f(x,\lambda), & X\in\gamma_0 \\ \alpha\ G(z,0,\lambda), & X\in\gamma_1 \end{cases}$$

where α is constant.

Then using boundary conditions (42) we have

$$f(0,-\lambda)+R_{11}(\lambda)f(0,-\lambda)=\alpha G(0,0,\lambda)=\alpha G(2\pi,0,\lambda)$$

$$f'(0,-\lambda)+R_{11}(\lambda)f'(0,-\lambda)+\alpha[G'(0+0,0,\lambda)-G(0+0,0,\lambda)]=0.$$

Taking into account conditions (43) for the Green's function on the loop, we have

$$f(0,-\lambda)+R_{11}(\lambda)f(0,-\lambda)=\alpha G(0,0,\lambda)$$

$$f'(0,-\lambda)+R_{11}(\lambda)f'(0,\lambda)=\alpha.$$

So

$$f(0,-\lambda)+R_{11}(\lambda)f(0,-\lambda)=[f'(0,-\lambda)+R_{11}(\lambda)f'(0,-\lambda)]G(0,0,\lambda)$$

or

$$G(0,0,\lambda)=\frac{f(0,-\lambda)+R_{11}(\lambda)f(0,-\lambda)}{f'(0,-\lambda)+R_{11}(\lambda)f'(0,-\lambda)},$$

and for the reflection coefficient, we have

$$R_{11}(\lambda)=\frac{\alpha-f'(0,-\lambda)}{f'(0,\lambda)}.$$

The main idea of the solution of the inverse problem for the considered system is its reduction to two independent problems of reconstruction of the potentials $q(x)=[q_1(x),q_2(z)]$ at the edges γ_0 and γ_1 respectively.

Since the coefficients $R_{11}(\lambda)$ can be found using the matching conditions

$$y(0)=u(0)=u(2\pi)$$

$$y'(0+0)+u'(0+0)-u'(2\pi-0)=0$$

at the central vertex, it is natural to formulate the inverse problem – recovering of the potential $q(x)$ on a non-compact graph G from the reflection coefficients, and the set of eigenvalues of the Dirichlet problems

$$\begin{cases} -u''(z,\lambda)+q_2(z)u(z,\lambda)=\lambda^2 u(z,\lambda), \ z\in[0,2\pi] \\ u(0,\lambda)=u(2\pi,\lambda)=0 \end{cases}$$

and Neumann problems

$$\begin{cases} -u''(x,\lambda)+q_2(z)u(z,\lambda)=\lambda^2 u(z,\lambda), \ z\in[0,2\pi] \\ u(0,\lambda)=u'(\pi,\lambda)=0 \end{cases}$$

Inverse problem: Given the spectral data: the set of eigenvalues of the Dirichlet and Neumann problems, as well as the reflection coefficient $R_{11}(\lambda)$, reconstruct the potential $q(X)=[q_1(x),q_2(z)]$ on the loop graph.

Theorem 20. Specification of spectral data uniquely determines the potential $q(X)$ on the loop graph.

In 4.3, we consider the inverse spectral problem for the Dirac operator on a star graph.

Consider a non-compact graph G with a single vertex, in which a finite number of edge $N_j=[o_j,\infty)$; $j=1,2,3,\ldots,n,$ are connected where the notation o_j is used, to denote the initial point of the j -th positive semiaxis.

We define space $L_2(G)=\bigoplus_{j=1}^{n} L_2(N_j)$, with a dot product

$$(f,g)_{L_2(G)}=\sum_{j=1}^{n}(f,g)_{L_2(G)}$$

and consider the operator

$$L_2(G)=\bigoplus_{j=1}^{n} L_j,$$

here

$$L_j \equiv B\frac{d}{dx_j} + \Omega_j(x_j),$$

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\Omega_j(x_j) = \begin{Vmatrix} p_j(x_j) & q_j(x_j) \\ q_j(x_j) & -p_j(x_j) \end{Vmatrix} \qquad (44)$$

in which the potentials have the form

$$p_j(x_j) = \sum_{n=1}^{\infty} p_{jn} e^{inx_j};\ q_j(x_j) = \sum_{n=1}^{\infty} q_{jn} e^{inx_j},\ \sum_{n=1}^{\infty} \{|p_{jn}| + |q_{jn}|\} < \infty$$

with domain

$$D(L_G) = \left\{ \begin{array}{l} y(x) | y_j(x), y_j'(x) \in AC[0,R] \quad R > 0, \\ \quad y_1(0) = y_2(0) = y_3(0) = \cdots = y_n(0), \\ \quad y_1'(0) + y_2'(0) + y_3'(0) + \cdots + y_n'(0) = 0, \\ \quad y_j(x), y_j''(x) \in L_2(R^+), j = 1,2,\ldots,n. \end{array} \right\}$$

Then the problem can be interpreted as a study of the operator $L_G = \bigoplus_{j-1}^{n} L_j$, on a non-compact graph. The spectral problem can be described as follows.

Find a vector $y_j(x_j,\lambda) = (y_{j1}(x_1,\lambda), y_{j2}(x_2,\lambda), y_{j3}(x_3,\lambda))$ where $y_{jk}(x_k,\lambda)$ is a solution to the Dirac equation

$$By_j' + \Omega_j(x_j)y_j = \lambda^2 y_j \qquad (45)$$

on an edge $N_j = [o_j, \infty),\ j = 1,2,3,\ldots n,$ for which the following conditions are satisfied:

1. y_j is continuous at the nodes of the graph, in particular, in our case

$$y_{j1}(0) = y_{j2}(0) = y_{j3}(0) = \cdots = y_{jn}(0);$$

2. The sum of derivatives over all branches outgoing from a node, calculated for each node is equal to zero

$$y'_{j1}(0)+y'_{j2}(0)+y'_{j3}(0)+\cdots+y'_{jn}(0)=0;$$

As a solution to the problem, we will understand the matrix

$$Y(X,\lambda)=\left\|y_{jk}(x_k,\lambda)\right\|_{j,k=1,2,3}$$

on a non-compact graph based on the following conditions.

1. $BY'+\Omega(X)Y=\lambda Y$
2. $y_{jk}(x_k,\lambda)=T_{jk}(\lambda)f_k(x_k,\lambda)$
3. $y_{jj}(x_j,\lambda)=\varphi_j(x_j,\lambda)+R_{jj}(\lambda)f_j(x_j,\lambda)$

where $\varphi_j(x_j,\lambda), f_j(x_j,\lambda)$ are the Jost solutions on the infinite edge of the graph and are determined by using the following theorem.

Theorem 21. Let $\Omega_j(x_j)$ be defined according to (44). Then equation (44) has particular solutions that can be represented in the form of $\varphi_j(x_j,\lambda), f_j(x_j,\lambda)$

$$f_j(x_j,\lambda)=\left(I+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{\vartheta^j_{n\alpha}}{\frac{n}{2}+\lambda}e^{i\alpha x_j}\right)\begin{pmatrix}1\\-i\end{pmatrix}e^{i\lambda x_j}$$

$$\varphi_j(x_j,\lambda)=\left(I+\sum_{n=1}^{\infty}\sum_{\alpha=n}^{\infty}\frac{\vartheta^j_{n\alpha}}{\frac{n}{2}-\lambda}e^{i\alpha x_j}\right)\begin{pmatrix}1\\i\end{pmatrix}e^{-i\lambda x_j}$$

here $I=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and $\upsilon^j_{n\alpha}=\begin{pmatrix}\upsilon^{11j}_{n\alpha}&\upsilon^{12j}_{n\alpha}\\\upsilon^{21j}_{n\alpha}&\upsilon^{22j}_{n\alpha}\end{pmatrix}$ where the numbers $\upsilon^{ikj}_{n\alpha}, i,k=1,2,\quad j=1,2,\ldots$ are determined using recurrence relations for which the series converges

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\alpha=n}^{\infty}\left|\vartheta^{ikj}_{n\alpha}\right|<\infty.\ .$$

The inverse problem of recovering a differential operator on each edge is considered. Namely,

Inverse problem. Construct the potentials $p_j(x_j)$ and $q_j(x_j)$ given spectral data, the reflection coefficients $R_{jj}(\lambda)$ on each edge $N_j, j = 1,2,\ldots$.

Theorem 22. Specification of spectral data uniquely determines the potentials $p_j(x_j)$ and $q_j(x_j)$ on each edge $N_j, j = 1,2,\ldots$.

On each edge $N_j, j = 1,2,\ldots$ the following relations are fulfilled

$$\lim_{\lambda \to \frac{n}{2}} (n - 2\lambda) R_{jj}(\lambda) = S_{nj}^{+},$$

$$\lim_{\lambda \to \frac{n}{2}} (n + 2\lambda) \frac{1}{R_{jj}(\lambda)} = S_{nj}^{-}.$$

Finally using the formulas

$$\upsilon_{nn}^{11j} = -\upsilon_{nn}^{22j} = \frac{S_{nj}^{+} + S_{nj}^{-}}{2},$$

$$\upsilon_{nn}^{12j} = \upsilon_{nn}^{21j} = \frac{S_{nj}^{+} - S_{nj}^{-}}{2i}$$

we reconstruct the potentials $p_j(x_j)$ and $q_j(x_j)$ from the given spectral data, the reflection coefficients $R_{jj}(\lambda)$ on each edge $N_j, j = 1,2,\ldots$.

CONCLUSIONS

The dissertation work is devoted to the study of spectral problems for differential operators with complex-valued periodic potential. Inverse problems for various types of discontinuous differential operators with complex-valued periodic potential are studied. Scattering problems on quantum graphs are investigated.

The following main results were obtained in the dissertation.

- Necessary and sufficient conditions are obtained for a given sequence of complex numbers to be a set of spectral data for a second-order operator pencil and higher-order differential operators.
- The direct and inverse problem of spectral analysis for $2n$ order ordinary differential equations with a polynomially depending spectral parameter has been solved. It is shown that the spectrum of the operator pencil is continuous and fills the rays $\{k\omega_j : 0 \le k \le \infty, j = \overline{0.2m-1}\}$, where $\omega_j = \exp\left(\frac{ij\pi}{n}\right)$, and on the continuous spectrum there are spectral features that coincide with numbers of the form $\frac{n\omega_j}{2}, j = \overline{0.2m-1}, n = 1,2,3,\dots$. The inverse problem of recovering the coefficients is solved by the generalized normalization numbers.
- Solved the inverse problem for the Schrödinger operator with complex-valued periodic potentials and a discontinuous coefficient on the whole real axis. The main characteristics of fundamental solutions are investigated, the spectrum of the operator is studied. The inverse problem is formulated, the uniqueness theorem is proved, and a constructive procedure for solving the inverse problem is proposed
- The classical Hill problem with complex potential is extended to star and loop graphs. The definition of the Hill operator on such graphs is given. The operator is defined by complex, periodic potentials and special boundary conditions are used to connect the values of the functions at the vertices. An explicit description of the

form of the resolvent is given, and the spectrum is described, and the inverse problem for reflection coefficients is solved.

The main results of the dissertation were published in the following works:

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