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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

SOLVABILITY OF THE RIEMAN BOUNDARY VALUE PROBLEMS IN NON-STANDARD HARDY SPACES AND APPLICATIONS TO THE BASICITY PROBLEMS

Specialty: 1202.01– Analysis and functional analysis
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GENERAL CHARACTERISTICS OF THIS WORK

Relevance and development of the topic.

The dissertation work is devoted to the solvability of the Riemann boundary value problem in non-standard Hardy spaces, namely in weighted Hardy spaces with a variable summability index, as well as its application to study the basis properties of perturbed trigonometric systems in non-standard Banach spaces. It should be noted that application of the Fourier method to many equations in mathematic physic, mechanic and in general in theory of the particle differential equations in respect of an ordinary differential operator is bought to the spectral problem. In many cases differential operators appear, eigenfunctions of which are perturbed trigonometric functions. Differential equations of mixed type (or elliptic) given in special domain belong to similar problems. On related of these issues can be dating in the series of the works of S.M. Ponomaryev, E.I. Moiseyev, S.A.Qabova, P.A.Krutisko and others. In connection with those listed above, it is believed that subject of the dissertation work is actual and from the point of view the theory of approximation, basis and Riemann boundary value problem is interesting. The spectral problems taken in most of special cases have eigenfunctions of which are consisting of perturbed system of sines

$$\left\{\sin(nt+\alpha(t))\right\}_{n\in N} , \qquad (1)$$

where $\alpha:[0,\pi]-R$, in general, is a piece-wise linear function. If seek a solution of the differential equation in non-standard Sobolev spaces of the functions, then basis properties (completeness, minimality, basicity) of the system (1) is studied in corresponding non-standard Banach function spaces. Research of basis properties of the perturbed trigonometric system has a deep history. It assumed that, it got its start from the famous work of Palley-Winner[1] and N. Levinson[2]. It should be noted that the basis properties of system (1) is closely related to the analogical properties of system of exponent

$$\left\{e^{i(nt-\alpha(t)signn)}\right\},\ n\in Z, \qquad (2)$$

with the appropriate function $\alpha:[-\pi,\pi]-R$. Criterion for the basicity with regard to the parameter $a\in R$ is found by A.M. Sedleski[3] in 1982, when $\alpha(\cdot)$ is linear function $\alpha(t)=at$. Regarding to the systems of sines and cosines (also in relation to the system of exponent) analogical results are found by E.M. Moiseyev[4] in 1984, and when $a\in C$ is a complex parameter, in these results is bought by Q.Q. Devdariani[5]. Note that, the works in this direction is divided into two groups. The works, where it is applied the methods of the theory of integer functions can be attributed to the first group (for example the works of Palley-Winer, N. Levinson, A.M. Sedleski, B.Y. Levin and others). The works, where it is applied the methods of the theory of boundary value problems for analytical functions can be attributed to the second group. The idea, where it is applied the methods of the theory of boundary value problems to study basis properties of the perturbed trigonometric system is belonged to Bisadze[6] (in 1950 year). Later this method was applied to define basis properties of the system of sines, with concrete linear phase by

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[1] Palley R. Wiener N. Fourier transforms in complex domain.// Amer. Math. Soc. Colloq. Publ. 19 (Amer. Math. Soc. RI. 1934).

[2] Levinson N. Gap and density theorems // New-York: Publ. Amer. Math. Soc. 1940, 204 p.

[3] Седлецкий А.М. Классы аналитических преобразований Фурье и экспоненциальные аппроксимации // Москва, Физматлит, 2005, 504 с.

[4] Моисеев Е.И. О базисности систем синусов и косинусов// ДАН СССР, 1984, т. 274, №4, с. 794-798.

[5] Девдариани Г.Г. О базисности одной системы функций // Дифференциальные уравнения, т. 22, №1, с. 170-171, 1986.

[6] Бицадзе А.В. Об одной системе функции // УМН, т. 5, в. 4(38), с.150-151

Ponomarev[7] and systems of sines, cosines and exponents an arbitrary linear phase by Moiseyev in Lebesgue spaces L_p, $1 < p < +\infty$. Further developing of this method and application to definition basis properties of double system of exponents, system of sines whose a piece-wise continuous phase, unitary and double systems of power belongs to B.T. Bilalov. The works of A.Y. Kazmin, A.N. Barmenkov, O.I. Lyubarskiy, V.A. Tkachenko, A.A. Shkalikov and other mathematics can be attributed to this group, too.

The subject and object of the study.

The subject and object of the study in dissertation work are Hardy classes with variable summability index, Hardy classes generated by the norm of Morrey space, homogenous and non-homogenous Riemann boundary value problems in these classes, exponential systems with linear and piecewise-linear phases, systems of sines and cosines with linear phase.

The goal and objectives of the study.

The goal of this work is to study the solvability of the Riemann boundary value problem in weighted Hardy classes with variable summability index in which the weight has general form, as well Morrey-Hardy classes and application of the obtained results to study the basis properties of the system exponent, with a piece-wise linear phase.

General technique of the studies.

In achieving main results was applied Riemann boundary value problem method of the theory analitic functions, methods of the theory of approximation and basises, methods of the functional and complex analysis.

Main provisions of dissertation. The following main provisions will be removed for defense:

- establishment of solvability conditions and finding general solution of the homogenous Riemann boundary value problem in

[7] Пономарев С.М. Об одной задаче на собственные значение // ДАН СССР, 1979, т. 243, № 5, 1068-1070.

weighted Hardy classes with variable summability index, in which weight has general form;

- establishment the solvability condition and finding the general solution, if resolved of the non-homogenous Riemann boundary value problem in these classes;
- to study these problems in relation to Hardy-Morrey classes;
- to study basis properties of a system of exponent (with unit) with linear phase (and with piece-wise linear phase) in Morrey-type linear space;
- to consider a relationship between the basis properties of double and unitary systems in Morrey-type spaces;

Scientific novelty. The following main results was obtained in the work:

- a homogeneous Riemann boundary value problem with piecewise continuous coefficients in weighted Hardy classes with variable summability index is considered, a sufficient condition for a weight function of general form to solve this problem is found, and a general solution is constructed;
- the condition of the solvability of the non-homogenous Riemann boundary value problem in the generalized weighted Hardy classes with variable summability index is found;
- the solvability of homogenous Riemann boundary value problem in Hardy-Morrey classes is investigated and the general solution is obtained;
- a sufficient condition for the solvability of the non-homogeneous Riemann boundary value problem is found in Hardy-Morrey classes;
- system of exponent with a linear phase is considered and basis properties of this system is established in a subspace of Morrey space where continuous functions are dense;
- analogical problems for system of exponent with a piece-wise linear phase is investigated;
- a relationship between the basis properties of the double and unitary systems in Morrey-type space is established;
- criterion for the basicity of system of sines and cosines with a linear phase in Morrey-type spaces are found.

Theoretical and practical value of study.

The work is theoretical. Its results can be used in the theory of boundary value problems, the theory of basis, to justify of the Fourier method for many particle differential equations and others.

Approbation and application.

Main results of the dissertation was repeatedly reported at the International conference "Modern methods of the boundary-value problems theory" dedicated to the 90^{th} anniversary of V.A.Ilin (Moscow 2018), International conference on Mathematical Advances and Applications (ICOMAA 2018, Istanbul), International Workshop "Spectral Theory and its applications" dedicated to 80^{th} anniversary of outstanding mathematician, academician Mirabbas Gasimov (Baku 2019), "Operators, functions and system of mathematical physics" Intern. conference (Baku 2019), 4-th International E-Conference on Mathematical Advances and Applications (ICOMAA 2021).

Personal contribution of the author. All the results obtained in the work are the personal contribution of author.

Publications of the author.

The main results of the work-7 of them were published in the recommended journals of the EAC under the President of the Republic of Azerbaijan, including 6 in periodicals published in the international summarization and indexing systems, 2 of them without co-author. Published on the results of national and international scientific events are 7 (3 of them were published abroad).

The name of the institution where the dissertation was completed. The dissertation work was executed at the department of "Mathematical analysis" of Ganja State University.

Structure and volume of the dissertation (in signs, indicating the volume of each structural unit)**.** The total volume of the dissertation work -195498 signs (title page -365 signs, content -1797 signs, introduction -46050 signs, I chapter-60000 signs, II chapter -86000 signs, conclusion -1286 signs), The list of used literature consists of 94 titles. The total volume of work is 107 pages.

THE MAIN CONTENT OF THE DISSERTATION

Dissertation consists of introduction, two chapters, conclusions and the list of used literature.

In the introduction, the relevance of the topic is substantiated, a brief summary of the results on the topic of the dissertation is given, and the main results of the dissertation are presented.

The first chapter of the dissertation dedicated to the solvability of the Riemann boundary value problem in Hardy classes which is generated by the norm of the considered function spaces.

In paragraph 1.1 the main concepts used in this chapter are accepted.

Thus firstly accept some notions. Let C – be the complex plane, $\omega \equiv \{z \in C : |z| < 1\}$ – be the unit circle, and $\partial\omega \equiv \{z \in C : |z| = 1\}$ – be the unit circumference. We will need some notion from the theory of generalized Lebesgue spaces.

Let $p : [-\pi, \pi] \to [1, +\infty)$ – be some Lebesgue-measureable function. Denote

$$I_p(f) \stackrel{def}{\equiv} \int_{-\pi}^{\pi} |f(t)|^{p(t)} dt.$$

With respect to the norm

$$\|f\|_{p(\cdot)} \stackrel{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\}$$

$L_{p(\cdot)}$ is Banach space of measurable functions on $[-\pi, \pi]$. Let

$$WL \stackrel{def}{\equiv} \{p : p(-\pi) = p(\pi); \exists C > 0, \quad \forall t_1, t_2 \in [-\pi, \pi] : |t_1 - t_2| \le \frac{1}{2} \Rightarrow$$

$$\Rightarrow |p(t_1) - p(t_2)| \le \frac{C}{-\ln|t_1 - t_2|}\}.$$

We will also use the following variable Muckhenhoupt condition.

Definition 1. Given $p(\cdot):[-\pi,\pi]\to[1,+\infty]$, and weight function $\omega(\cdot)$. We say that $\omega\in A_{p(\cdot)}$ if

$$\sup_{I\in[-\pi,\pi]}|I|^{-1}\|\omega(\cdot)\chi_I(\cdot)\|_{p(\cdot)}\|\omega^{-1}(\cdot)\chi_I(\cdot)\|_{q(\cdot)}<+\infty.$$

We will also need the weighted Hardy classes. By H_p^+ we denote the usual Hardy class of analytical functions in ω. Let $\mathscr{A}$ be the $\sigma-$algebra of Borel sets in $[-\pi,\pi]$ and $\rho-$be a $\sigma-$finite measure on $\mathscr{A}$. $L_{p(\cdot);d\rho}\equiv L_{p(\cdot);d\rho}(-\pi,\pi)-$will denote the Banach space $\mathscr{A}$ of measurable functions on $[-\pi,\pi]$ with norm

$$\|f\|_{p(\cdot)}\overset{def}{\equiv}\inf\left\{\lambda>0:I_{p(\cdot);d\rho}\left(\frac{f}{\lambda}\right)\le 1\right\},$$

where

$$I_{p(\cdot);d\rho}(f)=\int_{-\pi}^{\pi}|f(t)|^{p(t)}\,d\rho(t).$$

Assume

$$\tilde{H}\equiv\left\{f\in H_1^+:f^+\in L_{p(\cdot),d\rho}\right\},$$

where $f^-(e^{it})=f/_{\partial\omega}-$ are non-tangential boundary values of function f on $\partial\omega$. The norm in space $\tilde{H}$ is defined by

$$\|f\|_{\tilde{H}}=\|f^+(e^{it})\|_{p(\cdot);d\rho}.$$

Let ${}_mH_{p(\cdot)}^-$ be usual Hardy classes of analytical functions outside ω, which at the infinity has a pole of order not great from m. Denote

$$\tilde{H}^-\equiv\left\{f\in{}_mH_1^-:f^-(e^{it})\in L_{p(\cdot),d\rho}\right\},$$

where $f^-(e^{it})=f/_{\partial\omega}-f-$ are non-tangential boundary values of on $\partial\omega$ from the outside of ω. The norm in $\tilde{H}^-$ is defined as follows

$$\|f\|_{\tilde{H}^-}\equiv\|f^-(e^{it})\|_{p(\cdot);d\rho},\ \forall f\in\tilde{H}^-.$$

In paragraph 1.2 Riemann boundary value problem of theory of analytical functions in weighted Hardy classes with variable summability index is considered. Consider the following Riemann boundary value problem

$$F^{+}(\tau)-G(\tau)F^{-}(\tau)=g(\tau),\ \tau\in\partial\omega, \qquad (3)$$

where $G:\partial\omega\to C$ is coefficient of the problem, $g(\tau)$ is a given function.

We will assume that $g\in L_{p(\cdot);\vartheta}$ and G satisfies the following conditions:

$\alpha)$ $G^{\pm1}\in L_{\infty}(\partial\omega)$;

$\beta)$ $\theta(t)\equiv\arg G(e^{it})$ is a piecewise Hölder function on $[-\pi,\pi]$ and $-\pi<s_1<...<s_r<\pi-$ are the corresponding point of discontinuity.

Let $h_k=\theta(s_k+0)-\theta(s_k-0)$, $k=\overline{1,r}$, $h_0=\theta(-\pi+0)-\theta(\pi-0)$, are jumps of the function $\theta(\cdot)$ at these points.

The following theorem is proved.

Theorem 1. *Let the coefficient $G(\tau)$ of the problem (3) satisfy the conditions $\alpha);\beta)$. Assume that the jumps of the function $\theta(t)\equiv\arg G(e^{it})$ satisfy the relations*

$$\int_{-\pi}^{\pi}\left|w(t)\vartheta^{-1}(t)\right|^{q(t)}dt<+\infty\ , \qquad (4)$$

$$h_k<2\pi,\ k=\overline{0,r}, \qquad (5)$$

$$\int_{-\pi}^{\pi}\left|w^{-1}(t)\ \ \vartheta(t)\right|^{p(t)}dt<+\infty. \qquad (6)$$

where $w(t)$ is defined by

$$w(t)=\left|\sin\frac{t-\pi}{2}\right|^{\frac{h_0}{2\pi}}\prod_{k=1}^{r}\left|\sin\frac{t-s_k}{2}\right|^{\frac{h_k}{2\pi}}. \qquad (7)$$

Then the general solution of the homogenous problem

$$F^{+}(\tau)-G(\tau)F^{-}(\tau)=0,\ \tau\in\partial\omega,$$

in classes $H^{+}_{p(\cdot);\vartheta} \times H^{-}_{p(\cdot);\vartheta}$ *can be represented in the form*

$$F(z) \equiv Z(z) P_m(z).$$

where $Z(z)-$ *is a canonical solution, and* $P_m(z)-$*is an arbitary polinomial of order* $\le m$.

In paragraph 1.3 the solvability of the non-homogenous Riemann problem with a piece-wise continuous coefficient in weighted Hardy classes with variable summability index is studied. The following theorem is proved.

Theorem 2. *Let the coefficient* $G(\tau)$ *of the problem (3) satisfy the conditions* $\alpha)\beta)$. *Assume that the jumps of the function* $\theta(t) \equiv \arg G(e^{it})$ *satisfy the relations (4), (5), (6), where the weight function* $w(t)$ *is defined by the expression (7). Let* $\nu \in A_{p(\cdot)}$, $1 < p < +\infty$ *,where* $\nu(t) = w^{-1}(t)\vartheta(t)$ *and* $\upsilon^{-1} \in L_{q(\cdot)}$ *. Then the non-homogenous Riemann problem is solvable in classes* $H_{p(\cdot),\vartheta} \times_m H^{-}_{p(\cdot),\vartheta}$, *if the ortogonality conditions*

$$\int_{-\pi}^{\pi} \frac{g(e^{i\sigma})}{Z^{+}(e^{i\sigma})} e^{in\sigma} d\sigma = 0 \, , n = \overline{1,-m},$$

hold. For $m \ge 0$ *the general solution of (3) can be represented as*

$$F(z) = Z(z) P_m(z) + F_1(z),$$

where $Z(z)-$ *is a canonical solution of homogeneous problem,* $P_m(z)$ *is an arbitary polynomial of order* $\le m$, *and* $F_1(z)$ *is defined by the expression*

$$F_1(z) = \frac{Z(z)}{2\pi} \int_{-\pi}^{\pi} \frac{g(e^{i\sigma})}{Z^{+}(e^{i\sigma})} \frac{d\sigma}{1 - ze^{-i\sigma}}.$$

Moreover, for $m \le -1$ *the problerm (3) uniquely solvable, and for* $m = -1$ *it has a solution for* $\forall g \in L_{p(\cdot);\vartheta}$.

In paragraph 1.4 homogeneous Riemann boundary value problerm of theory of analitical functions in Morrey-Hardy classes is considered. For this purpose, some properties of functions of this class and the possibility of describing these functions using a Cauchy type integral are studied.

First we define the Morrey space on the unit circle $\gamma = \{z \in C : |z| = 1\}$ on the complex plane C. By $L_0(-\pi, \pi)$ denote the linear space of all Lebesgue measurable functions on $(-\pi, \pi)$.

Denote by $L^{p,\alpha}(\gamma)$, $1 \le p < +\infty$, $0 \le \alpha \le 1$, the normed space of measurable functions $f(\cdot)$ on γ with a finite norm

$$\|f\|_{L^{p,\alpha}(\gamma)} = \sup_B \left(|B \cap \gamma|_\gamma^{\alpha - 1} \int_{B \cap \gamma} |f(\xi)|^p |d\xi| \right)^{1/p} < +\infty,$$

($|B \cap \gamma|_\gamma$ – is linear measure of $B \cap \gamma$), where $\sup$ is taken over all balls centered on γ and with an arbitrary positive radius. With respect to this norm $L^{p,\alpha}(\gamma)$ is a Banach space.

Define the class Morrey-Hardy $H_+^{p,\alpha}$, $1 \le p < +\infty$, $0 \le \alpha \le 1$, of analytic functions $f(\cdot)$ inside the unit ball ω with a norm

$$\|f\|_{H_+^{p,\alpha}} = \sup_{0<r<1} \|f_r(\cdot)\|_{p,\alpha},$$

where $f_r(t) = f(re^{it})$. It is easy to see that the inclusion $H_+^{p,\alpha} \subset H_1^+$, $1 \le p < +\infty$, holds true, where H_1^+ is an ordinary Hardy class.

In a very similar way, we can construct the Morrey-Hardy class outside the circle ω. Let, $\omega^- = C \setminus \overline{\omega} \left(\overline{\omega} = \omega \cup \gamma \right)$.

If the expansion of the function f into the Laurent series

$$f(z) = \sum_{k=-\infty}^{m} a_k z^k, a_m \ne 0, \qquad (8)$$

in a neighborhood of the infinitely point then the function f which is analytic in ω^-, has a finite order m at infinity.

So that, in the case of $m>0$ it has a pole of order m at the point $z=\infty$; in the case of $m=0$ the function $f(z)$ is bonded and different from zero at the point $z=\infty$; in the case of $m<0$ it has a zero order $(-m)$ at the point $z=\infty$. Let us $f(z)=f_0(z)+f_1(z)$, where $f_0(z)-$ is a regular part, and $f_1(z)-$ is a main part of the expansion (8) (that is $f_0(z)=\sum_{k=0}^{m} a_k z^k$). So, if $m<0$, then $f_0(z)\equiv 0$, if $m\geq 0$ then f_0 is m order polynomial. If $\deg f_0 \leq m$ and $F\in H_{+}^{p,\alpha}$, then f belongs to class ${}_m H_{-}^{p,\alpha}$, where $F(z)=\overline{f_1\left(\frac{1}{\bar{z}}\right)}, z\in\omega$.

The following theorem from B.T. Bilalov's works was used to obtain the main results of this chapter.

Theorem 3. *Let the coefficient* $G(\cdot)$ *of the problem*

$$\left.\begin{array}{l} F^{+}(\tau)-G(\tau)F^{-}(\tau)=0, \tau\in\gamma, \\ F^{+}(\cdot)\in H_{+}^{p,\alpha}; F^{-}(\cdot)\in {}_m H_{-}^{p,\alpha}, \end{array}\right\} \tag{9}$$

satisfy the conditions $\alpha),\beta)$ *and the jumps* $\{h_k\}_0^r$ *of the function* $\theta(t)=\arg G(e^{it})$ *on* $[-\pi,\pi]$ *satisfy the inequality*

$$1+\frac{\alpha}{p}<\frac{h_k}{2\pi}\leq\frac{\alpha}{p}, \; k=\overline{0,r}, \tag{10}$$

where $h_0=\theta(-\pi)-\theta(\pi)$.

Then:

$\alpha)$ *if* $m\geq 0$ *general solution of the problem (9) can be represented in the form*

$$F(z)\equiv Z(z)P_k(z),$$

where $Z(z)-$ *is a canonical solution, and* $P_k(z)-$ *is an arbitrary polynomial of order* $k\leq m$;

$\beta)$ *if* $m<0$ *the problem (9) has only trivial solution.*

Let's look the following non-homogeneous Riemann boundary value problem

$$F^{+}(\tau)-G(\tau)F^{-}(\tau)=f(\arg\tau), \qquad \tau\in\gamma, \qquad (11)$$

in Morrey-Hardy classes $H_{+}^{p,\alpha}\times_m H_{-}^{p,\alpha}, 1<p<+\infty, 0<\alpha<1$, where $f\in L^{p,\alpha}-$ is arbitrary function.

We use some corollaries that immediately follow from theorem 3.

Corollary 1. *Let coefficient $G(\cdot)$ of the problem (11) satisfy the conditions $\alpha),\beta)$ and $\{h_k\}_0^r$ be the jumps of the functions $\theta(t)=\arg(G(e^{it}))$ at the discontinuity points $\{s_k\}_1^r\subset(-\pi,\pi)$: $h_0=\theta(-\pi)-\theta(\pi)$ for $\forall f\in L^{p,\alpha}, 1<p<+\infty, 0<\alpha<1$ the problem (11) has a unique solution in Morrey-Hardy classes $H_{+}^{p,\alpha}\times_{-1}H_{-}^{p,\alpha}$, which can be represented in terms of Cauchy-type integral of the form*

$$F_1(z)=\frac{Z(z)}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)}{Z^{+}(e^{it})}K_z(t)dt \ . \qquad (12)$$

From the same theorem it also follows

Corollary 2. *Let all the conditions of corollary 1 hold. Then, for $\forall f\in M^{p,\alpha}, 1<p<+\infty, 0<\alpha<1$, the problem (11) has a unique solution in Morrey-Hardy classes $MH_{+}^{p,\alpha}\times_{-1}MH_{-}^{p,\alpha}$, defined by Cauchy-type integral (12).*

The second chapter of the dissertation is dedicated to investigation basis properties of the exponential system

$$E_\beta\equiv\left\{e^{i(nt+\beta|t|\,sign\ n)}\right\}_{n\in Z}, \qquad (13)$$

where $\beta\in C-$ is a complex parameter, and $Z-$ is an integer.

This system is another modification of the exponential system

$$e_\beta\equiv\left\{e^{i(n+\beta sign\, n)t}\right\}_{n\in Z},$$

basis properties (completeness, minimality and basicity) of this system is investigated in the subspace of the Morrey space in which continuous functions are dense. Criterion of the completeness of this system in this subspace is found. Basis properties of the system (13) is differs from corresponding basis properties of the system e_β.

The basis properties of the system e_β in Morrey-type spaces were published in new papers by B. T. Bilalov.

In paragraph 2.1 the main concepts are accepted used in this chapter.

In paragraph 2.2 applying the boundary value problem method basicity of the system of exponents E_β in $M^{p,\alpha}$ Morrey type spaces is determined.

This method requires the determination of basicity of parts of system of exponents for corresponding $MH_+^{p,\alpha}$ and ${}_{-1}MH_-^{p,\alpha}$ Morrey-Hardy spaces.

In paragraph 2.3 it is considered the perturbed system of exponents with a unit, whose phase is a linear function depending on a real parameter. The following theorem is proved.

Theorem 4. *Let* $2\operatorname{Re}\beta+\frac{\alpha}{p}\notin Z$. *Then the system of exponents*

$$1\cup\left\{e^{i(n-\beta\, sign\, n)t}\right\}_{n\neq 0},$$

forms a basis for $M^{p,\alpha}, 0<\alpha<1, 1<p<+\infty$, *if and only if* $d(\beta)=\left[2\operatorname{Re}\beta+\frac{\alpha}{p}\right]=0$. *For* $d(\beta)<0$ *it is not complete, but it is minimal in* $M^{p,\alpha}$; *for* $d(\beta)>0$ *it is complete, but not minimal* $M^{p,\alpha}$.

In paragraph 2.4 a perturbed system of exponents with a piece-wise linear phase depending on two real parameters is considered.

Let us consider the exponential system of

$$E_{\beta;\gamma}\equiv\left\{e^{i(nt-\lambda_n(t))}\right\}_{n\in Z},$$

where $\lambda_n(t) = -(\beta t + \gamma\, sign t)\, sign\, n$; $\beta, \gamma \in R -$ are real parameters, and $Z -$ is an integer.

In this paragraph the following theorem is proved

Theorem 5. *Let the real parameters* $\beta; \gamma \in R$ *satisfy the following inequalities*

$$-1 + \frac{\alpha}{p} < -\frac{2\gamma}{\pi} < \frac{\alpha}{p}; -1 + \frac{\alpha}{p} < -2\beta - \frac{2\gamma}{\pi} < \frac{\alpha}{p}.$$

Then the system of exponents $E_{\beta;\gamma}$ *forms a basis for* $M^{p,\alpha}, 0 < \alpha < 1, 1 < p < +\infty$.

Let us consider the most general case, namely, consider the system of exponents

$$\tilde{E}_{\beta,\gamma} \equiv \left\{ e^{i(nt + \tilde{\lambda}(t))} \right\}_{n \in Z},$$

where

$$\tilde{\lambda}_n(t) = -\frac{1}{2}\tilde{\theta}(t) signn, \; n \in Z,$$

$$\tilde{\theta}(t) = \begin{cases} -2\beta t + 2\gamma + 2m_1\pi, t \in [-\pi, 0], \\ -2\beta t + 2\gamma + 2m_2\pi, t \in (0, \pi], \end{cases}$$

$m_1, m_2 \in Z -$ are arbitrary integers. Function $\tilde{\theta}(\cdot)$ has a discontinuity point $t = 0$ and its jump at this point is

$$\tilde{h}_1 = \tilde{\theta}(+0) - \theta(-0) = -2\gamma + 2m_2\pi - (2\gamma + 2m_1\pi) = -4\gamma + 2(m_2 - m_1)\pi.$$

We also get that

$$\tilde{h}_0 = \tilde{\theta}(\pi) - \tilde{\theta}(-\pi) = -2\beta\pi - 2\gamma - 2m_1\pi + (-2\beta\pi - 2\gamma + 2m_2\pi) = -4\beta\pi - 4\gamma - 2(m_1 - m_2)\pi.$$

Let's choose integers $m_1; m_2$ from the following conditions

$$-1 + \frac{\alpha}{p} < \frac{\tilde{h}_k}{2\pi} < \frac{\alpha}{p}, \; k = 0,1.$$

We have

$$\left.\begin{aligned} 1+\frac{\alpha}{p} < -\frac{2\gamma}{\pi} + m_2 - m_1 < \frac{\alpha}{p} \\ 1+\frac{\alpha}{p} < -2\beta - \frac{2\gamma}{\pi} - m_1 + m_2 < \frac{\alpha}{p} \end{aligned}\right\} . \qquad (14)$$

Assuming that $m = m_2 - m_1$ we obtain the validity of the following theorem.

Theorem 6. *Let there be an integer* m *such that the inequalities*

$$\left.\begin{aligned} -1+\frac{\alpha}{p} < -\frac{2\gamma}{\pi} + m < \frac{\alpha}{p}, \\ -1+\frac{\alpha}{p} < -2\beta - \frac{2\gamma}{\pi} + m < \frac{\alpha}{p} \end{aligned}\right\},$$

are satisfied. Then the system of exponents $\tilde{E}_{\beta,\gamma}$ *forms a basis for* $M^{p,\alpha}, 0 < \alpha < 1, 1 < p < +\infty$.

In paragraph 2.5 are considered double and unitary systems of functions in Morrey-type spaces $M^{p,\alpha}(-a,a)$ and $M^{p,\alpha}(0,a)$. A relationship between the completeness and minimality properties of these systems in these spaces are established.

Consider the following unitary system of functions of the form

$$v_n^{\pm} \equiv a(t)\omega_n^{+}(t) \pm b(t)\omega_n^{-}(t),\ n \in N,$$

and its associated double system

$$\{A(t)W_n(t); A(-t)W_n(-t)\}_{n\in N}, \qquad (15)$$

where

$$A(t) = \begin{cases} a(t), t \in [0,a], \\ b(-t), t \in [-a,0), \end{cases}$$

$$W_n(t) = \begin{cases} \omega_n^{+}(t), t \in [0,a], \\ \omega_n^{-}(-t), t \in [-a,0) \end{cases}$$

So, the following theorem is proved.

Theorem 7. *The double system*

$$V_{n,m} \equiv \left(A(t)W_n(t); A(-t)W_m(-t)\right),\ n, m \in N,$$

is complete in $M^{p,\alpha}(-a,a)$, $1 \le p < +\infty$, $0 < \alpha \le 1$, *if and only if the unitary systems* $\{v_n^+\}_{n\in N}$ *and* $\{v_n^-\}_{n\in N}$ *are complete in* $M^{p,\alpha}(0,a)$.

The following theorem is also true.

Theorem 8. *The double system* $1 \cup \{V_{n,n}\}_{n\in N}$ *is complete in* $M^{p,\alpha}(-a,a)$, $1 \le p < +\infty$, $0 < \alpha \le 1$, *if and only if the unitary systems* $1 \cup \{v_n^+\}_{n\in N}$ *and* $\{v_n^-\}_{n\in N}$ *are complete in* $M^{p,\alpha}(0,a)$.

The following corollary follows directly from these theorems

Corollary 3. *The double system of exponents*

$$\left\{e^{i(nt+\beta(t)\operatorname{sign} n)}\right\}_{n\neq 0},$$

is complete in $M^{p,\alpha}(-\pi,\pi)$, $1 \le p < +\infty, 0 < \alpha \le 1$, *if and only if the systems of sines* $\{\sin(nt+\beta(t))\}_{n\in N}$ *and cosines* $\{\cos(nt+\beta(t))\}_{n\in N}$ *are complete in* $M^{p,\alpha}(0,\pi)$.\

We also have the following

Corollary 4. *The double system of exponents*

$$1 \cup \left\{e^{i(nt+\beta(t)\operatorname{sign} n)}\right\}_{n\neq 0},$$

is complete in $M^{p,\alpha}(-\pi,\pi)$, $1 \le p < +\infty, 0 < \alpha \le 1$, *if and only if the systems of sines* $\{\sin(nt+\beta(t))\}_{n\in N}$ *and cosines* $1 \cup \{\cos(nt+\beta(t))\}_{n\in N}$ *are complete in* $M^{p,\alpha}(0,\pi)$

We define the space $\left(M^{p,\alpha}(a,b)\right)'$ associated with $M^{p,\alpha}(a,b)$ and for brevity, denote it by M'. Let S be the unit ball in $M^{p,\alpha}(a,b)$, i.e

$$S = \left\{f \in M^{p,\alpha}(a,b) : \|f\|_{L^{p,\alpha}(a,b)} \le 1\right\}.$$

M' is the Banach space of all measurable functions on (a,b) for which the norm

$$\|g\|_{M'} = \sup_{f \in S} \left| \int_a^b fgdt \right| < +\infty,$$

is finite.

Let the system $\{V_{n,n}\}_{n\in N}$ be minimal in $M^{p,\alpha}(-a,a)$ and $\{h_n^+; h_n^-\}_{n\in N} \subset M'(-a,a)$ be a biortogonal system to it. Define

$$\vartheta_k^+ = h_k^+(t) + h_k^+(-t), \forall k \in N,$$

$$\vartheta_k^+ = h_k^+(t) + h_k^+(-t), \forall k \in N.$$

The following theorem is true.

Theorem 9. *The system* $\{V_{n,n}\}_{n\in N}$ *is minimal in* $M^{p,\alpha}(-a,a), 1 \le p < +\infty, 0 < \alpha \le 1,$ *if and only if the systems* $\{\vartheta_n^+\}_{n\in N}$ *and* $\{\vartheta_n^-\}_{n\in N}$ *are minimal in* $M^{p,\alpha}(0,a)$.

Quite similarly the following statement is proved.

Theorem 10. *The system* $1 \cup \{V_{n,n}\}_{n\in N}$ *is minimal in* $M^{p,\alpha}(-a,a), 1 \le p < +\infty, 0 < \alpha \le 1,$ *if and only if the systems* $1 \cup \{\vartheta_n^+\}_{n\in N}$ *and* $\{\vartheta_n^-\}_{n\in N}$ *are minimal in* $M^{p,\alpha}(0,a)$.

In paragraph 2.6 the system of sines $\sin(n+\beta)t \ n = 1,2,\ldots$ and cosines $\cos(n-\beta)t \ n = 0,1,2,\ldots$ are considered, where $\beta \in R$ is a real parameter. Criterion for the completeness, minimality and basicity of these systems with respect to the parameter β in the subspace $M^{p,\alpha}(0,\pi), \ 1 < p < +\infty$ are found.

Theorem 11. *Let* $2\operatorname{Re}\beta + \frac{\alpha}{p} \notin Z$, $1 < p < +\infty, 0 < \alpha < 1$.*Then the system of sines* $\{\sin(n-\beta)t\}_{n\ge 0}$ *forms a basis for* $M^{p,\alpha}(0,\pi)$ *if and only if* $\left[\operatorname{Re}\beta + \frac{\alpha}{2p}\right] = 0$. *Moreover, for* $\left[\operatorname{Re}\beta + \frac{\alpha}{2p}\right] < 0$ *it is* *not complete, but is minimal; and for* $\left[\operatorname{Re}\beta + \frac{\alpha}{2p}\right] > 0$ *it is complete, but it is not minimal in* $M^{p,\alpha}(0,\pi)$.

Analogical result is true for the system of cosines, namely, the following theorem is true

Theorem 12. *Let* $2\operatorname{Re}\beta + \frac{\alpha}{p} \notin Z, 1 < p < +\infty, 0 < \alpha < 1.$ *System of cosines* $\{\cos(n-\beta)t\}_{n\geq 0}$ *forms a basis in* $M^{p,\alpha}(0,\pi)$ *if and only if* $\left[\operatorname{Re}\beta + \frac{\alpha}{2p} - \frac{1}{2}\right] = 0$. *For* $\left[\operatorname{Re}\beta + \frac{\alpha}{2p} - \frac{1}{2}\right] < 0$ *it is not complete, but minimal; for* $\left[\operatorname{Re}\beta + \frac{\alpha}{2p} - \frac{1}{2}\right] > 0$ *it is complete, but it is not minimal in* $M^{p,\alpha}(0,\pi)$.

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CONCLUSIONS

The dissertation work is dedicated to investigation of Riemann boundary value problem and its applications to basicity problems in Morrey-type spaces. The following main results was obtained in the work:

- a homogenous Riemann boundary value problem with piece-wise continuous coefficients in weighted Hardy classes with variable summability index is considered, a sufficient condition for a weight function of general form to solve this problem is found, and a general solution is constructed;
- the condition of the solvability of the non-homogenous Riemann boundary value problem in the generalized weighted Hardy classes with variable summability index is found;
- the solvability of homogenous Riemann boundary value problem in Hardy-Morrey classes is investigated and the general solution is obtained;
- a sufficient condition for the solvability of the non-homogeneous Riemann boundary value problem is found in Hardy-Morrey classes;
- system of exponent with a linear phase is considered and basis properties of this system is established in a subspace of Morrey space where continuous functions are dense;
- analogical problems for system of exponent with a piece-wise linear phase is investigated;
- a relationship between the basis properties of the double and unitary systems in Morrey-type space is considered;
- criterion for the basicity of system of sines and cosines with a linear phase in Morrey-type spaces are found.

The main clauses of the dissertation work were published in the following scientific works:

1. Bilalov, B.T., Huseynli, A.A., Seyidova, F.S. Riemann boundary value problems in generalized weighted Hardy spaces // - Baku: Proceedings of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan. -2017. v. 43, №2, p. 240–251.
2. Huseynli, A.A., Seyidova, F.S. Basis properties of the system of exponents in weighted Morrey spaces // International conference "Modern methods of the boundary-value problems theory" dedicated to the 90th anniversary of V.A.Ilyn, -Moscow: - 2-6 May, -2018, -p. 253-254.
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4. Nasibova, N.P., Seyidova F.S. On the basicity of a perturbed system of exponents with a unit in Morrey-type spaces // - Baku: Transaction of National Academy of Sciences of Azerbaijan, Series of phys.-tech. and math. sciences, -2019. v. 39 (1), -p. 151-161.
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functions and systems of mathematical physics" Intern. conference, -Baku: Khazar University, -10-14 June, -2019, -p. 91.
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13. Seyidova, F.S. On the completenes and minimality of double and unitary systems in Morrey-type spaces // -Nur-Sultan: Eurasian Mathematical Journal, -2021. v.12, №2, -p.74-81.
14. Muradov, T.R., Seyidova, F.S. Basis properties of perturbed system of exponents with a piecewise linear phase in Morrey-type spaces // 4-th International E-Conference on Mathematical Advences and Applications, -Istanbul, Turkey: -26-29 May, -2021, -p. 197.

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