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ABSTRACT

of the dissertation for the scientific degree of Doctor of Philosophy

BOUNDEDNESS OF THE WEIGHTED HARDY OPERATOR AND ITS COMMUTATOR IN ORLICZ-MORREY SPACE

Specialty: 1202.01- Analysis and functional analysis
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GENERAL CHARACTERISTICS OF THE WORK

Rationale of the work. The dissertation work is devoted to the boundedness of many-variable weighted Hardy adjoint operators and their commutators in variable exponent generalized weighted Morrey, generalized "complementary" weighted Morrey spaces, boundedness of many variable weighted Hardy-Cesaro, weighted adjoint Hardy-Cesaro operators and their commutators in Morrey spaces, boundedness of weighted adjoint Hardy-Cesaro operators and their commutators in Morrey space, boundedness of many variable weighted fractional Hardy, weighetd adjoint fractional Hardy operators and their commutators in the generalized Orlicz-Morrey, generalized "complementary Orlicz-Morrey spaces. Furthermore, some applications of the obtained results have been studied. Since the beginning of the last century, D. Hilbert proved the inequality known in literature as a Hilbert inequality. After this work many mathematicians worked on various inequalities.

In 1920 Hilbert proved an inequality in integral form which is called the Hardy inequality which has a different and simpler proof that the Hardy inequality.

In 1925 G. Hardy in his paper has proved this inequality for sequences. After Hardy's works many mathematicians could show the Hardy inequality in a different way. Since the Hardy inequality is widely used in many fields of mathematics, mathematicians have conducted extensive and varied studies. The Hardy inequality is widely applied in theory of ordinary differential equations, in properties of the solutions of differential equations, spectral theory of a differential operator, in interpolation theorems for operators, in embedding theorems in weighted Sobolev spaces, in Fourier series, in Laplace transforms. Furthermore, the Hardy inequality is used in mathematical expressions of electromagnetic waves of physics.

At the late 1950 and early 1960, P.R. Beesach succeeded a systematic research of the Hardy inequalities. In this works, P.R.Beesach alongside with the case $p=q$ has studied the case $p<0$ and even the case $0<p<1$ and obtained two-weighted inverse inequalities. P.R. Beesach extended the weighetd Hardy inequalities to the class of inequalities not including $\varepsilon = p - 1$. In 1963, in their papers obtained a new result taking the weight function $x^{\varepsilon} |\log x|^{\eta}$ instead of the power function x^{ε}. For $p = q$ on a semi-axis G.Tanelli in 1966, 1967 and 1969, G.Tomaselli in 1969 have succeed to obtain necessary and sufficient conditions in weighetd Hardy inequalities.

Weighted Morrey space has a great role when studying local properties of the solutions of partial differential equations along with the Lebesgue space. Morrey spaces were formed in 1938 due to some problems of elliptic equations and variationals calculus and was introduced by Ch.B. Morrey. The finite norm of the classic Morrey space is determined as

$$\| f \|_{L^{p,\lambda}} := \sup_{x, r > 0} r^{-\frac{\lambda}{p}} \| f \|_{L^p(B(x,r))}.$$

Here $0 \le \lambda \le n,\ 1 \le p \prec \infty$.

In 1969, J.Petre noted the boundedness of the Riess potential in Morrey space. In 1975, D.Adams changing the conditions of parameters has proved the boundedness of the Riess potential in Morrey space. Furthermore, in 1975, V.P. Ilin has introduced modified Morrey space and proved embedding theorems in this work.

Important results in Morrey and modified Morrey spaces have been obtained in the works of D.Adams, S.Kampanato, C.Petre, M.Taylor, S.Lu, J.Xiao, D.Yang, Y.Qiga, T.Miyakama, A.J.Mazzukato, X.Zhon, G.D.Fazio, V.P.Ilin, V.S.Guliyev, V.I.Burenkov, R.V.Huseynov, A.M.Najafov, M.Ragusa, L.Softova, A.Gogatashvili, D.Palagachev, Y.Sawano, D.Fan, L.Tang, Jingshi Xu, J.J.Hasanov, R.Ch.Mustafayev and others. Afterwords important studies have been conducted in Navier-Stokes, Schrödinger equations, in discontinuous coefficient elliptic equations and potentials theory in Morrey space.

Boundedness problem of classic operators of harmonic analysis as maximum operator, fractional-maximum operator, Riesz potential, singular integral operator have been widely studied. These results can be successfully applied in theory of partial differential equations. It should be noted that in last years Morrey-type spaces, play an important role in theory of partial differential equations.

In the ninetieth of the XX century, an extensive investigation of generalized Morrey-type spaces has began. In 1991, T. Mizuhara introduced generalized Morrey space and proved the boundedness of some classic operators in this space. In 1994, E.Nakai proved the boundedness of the Riesz potential and singular integral operators in the generalized Morrey space. In his doctor's dissertation, V.S.Guliyev introduced local and complementary local Morrey type spaces and studied fractional integral operators in functional spaces determined in Lie groups and boundedness of singular integral operators. Classic results related to the boundedness of classic operators in generalized Morrey type spaces have been obtained by several authors. But in all these results only sufficient conditions characterizing Morrey type spaces, on parameters providing the boundedness have been obtained.

At the beginning of the XXI century a new, extensive development has taken in this field of investigation. Especially, V.S.Guliyev and V.I.Burenkov developed a new perspective in harmonic analysis related to the study of classical operators in generalized Morrey-type spaces. The importance of the worked out methods is that necessary and sufficient conditions for the boundedness of a class of singular type operators have been obtained. And this was applied to regularity problems for solving elliptic and parabolic partial differential equations. As a result, some necessary and sufficient conditions on numerical parameters have been found.

Conditions on functional parameters providing the boundedness of classic operators of harmonic analysis from generalized local Morrey type spaces to another one have been obtained. Such type results are very important for the development of modern harmonic analysis and in the first turn for partial differential equations.

It is known that commutators of integral operators play a special role in harmonic analysis. It is known that a commutator is an important integral operator and plays a great role in harmonic analysis. In 1965, Kalderon studied some type of commutator of the Lipschitz curve arising in Cauchy integral problems. Let K be Kalderon-Zigmund singular integral operator and $b \in BMO(R^n)$ Koifman, Rochberq and Weiss proved the famous result, the boundedness of the commutator operator $[b,K] = K(bf) - Kf$ in space $L^p(R^n)$ for $1 < p < \infty$.The commutator of the Kalderon-Ziqmund operators plays a great role in studying regularity of the solution of second degree elliptic partial differential equation.

For $1 < p < \infty$, $r > 0$ for the function $f \in L^p\left(R^n \setminus B(x_0,r)\right)$ the finite norm of local “complementary” generalized Morrey space ${}^C M^{p,\omega}_{\{x_0\}}(R^n)$ is in the form

$$\|f\|_{{}^c M^{p,\omega}_{\{x_0\}}(\mathrm{R}^n)} = \sup_{r>0} \frac{r^{\frac{n}{p'}}}{\omega(r)} \|f\|_{L^p(\mathrm{R}^n \setminus B(x_0,r))}.$$

The finite norm of local “complementary” classic Morrey space is in the form

$$\|f\|_{{}^c L^{p,\lambda}_{\{x_0\}}(R^n)} = \sup_{r>0} r^{\frac{\lambda}{p'}} \|f\|_{L^p(R^n \setminus B(x_0,r))} < \infty, \quad x_0 \in R^n$$

where $1 < p < \infty$, $f \in L^p(R^n \setminus B(x_0,r))$ and $0 \le \lambda < n$. For $\lambda = 0$ ${}^c L^{p,0}_{\{x_0\}}(R^n) = L^p(R^n)$. Taking $\omega(r) = r^{\frac{n-\lambda}{p'}}$ in local “complementary” generalized Morrey space, we obtain local “complementary” classic Morrey space, i.e.

$${}^c L^{p,\lambda}_{\{x_0\}}(R^n) = {}^c M^{p,\omega}_{\{x_0\}}(R^n)\Big|_{\omega(r)=r^{\frac{n-\lambda}{p'}}}.$$

In 2008, A.Almeida, J.J.Hasanov, S.G.Samko introduced variable exponent Morrey space in a finite-dimensional set, studying the properties of this space they studied the boundedness of a maximum operator and variable parameter potential-type operator in this space as well. Afterwards, in this space, V.Kokilashvili and A.Meskhi proved the boundedness of maximum and singular integral operators in homogeneous groups, P.Hästö proved the boundedness of the maximum operator in these spaces in an infinite-dimensional set.

The subject and object of the study. Boundedness theorems of the weighted Hardy operator and their commutator in variable exponents generalized weighted Morrey space, variable exponent generalized complementary weighted Morrey space, generalized Orlich-Morrey space and generalized complementary Orlich-Morrey spaces have been studied.

Goals and objectives of the study. The goal of the dissertation work is to study weighted Hardy operator and its commutator in variable exponent generalized weighted Morrey space, in variable exponent generalized complementary weighted Morrey space, in generalized Orlicz-Morrey space and in generalized complementary Orlicz-Morrey space.

Research methods. In the work, the methods of real variable functions theory, function space theory, harmonic analysis, operator theory, ebbedding theorems and method of functional analysis were used.

The main thesis to be definded.

1. In the work the boundedness of weighted Hardy, weighted adjoint Hardy operators and their commutators in variable exponent weighted generalized and generalized complementary Morrey spaces was studied.
2. The boundedness of weighted Hardy-Cesaro, weighted adjoint Hardy-Cesaro operators and their commutators in Morrey spaces is studied.
3. The boundedness of weighted fractional Hardy operators and their commutators in generalized Orlicz-Morrey and generalized complementary Orlicz-Morrey space was proved.

Scientific novelty of the work.

1. The boundedness of the weighted Hardy operator and its commutator in variable exponent weighted generalized Morrey and generalized complementary Morrey spaces has been proved.
2. The boundedness of the weighted adjoint Hardy operator and its commutator in variable exponent weighted generalized

Morrey and generalized complementary Morrey spaces has been proved.

3. Necessary and sufficient conditions for the boundedness of the weighted Hardy-Cesaro operator in Morrey space have been found.

4. Sufficient conditions for the boundedness of the commutator of the weighted Hardy-Cesaro in Morrey space have been found.

5. Necessary and sufficient conditions for the boundedness of the adjoint weighted Hardy-Cesaro operator in Morrey space have been found.

6. Sufficient conditions for the boundedness of the commutator of the weighted adjoint Hardy-Cesaro operator have been found.

7. The boundedness of the weighted Hardy operator and its commutator in generalized Orlicz-Morrey and generalized complementary Orlicz-Morrey space has been proved.

8. The boundedness of the weighted adjoint fractional Hardy operator and its commutator in generalized Orlicz-Morrey and generalized complementary Orlicz-Morrey spaces has been proved.

Theoretical and practical importance of the study. The results obtained in the dissertation work are of theoretical character. The obtained new results are of particular interest in theory of function spaces, furthermore, they can be used in boundedness of integral operators of harmonic analysis, when solving boundary value problems of theory of partial differential equations, in studying differential properties of the generalized solution of quasielliptic and hypoelliptic differential equations and in theory of subelliptic equations.

Approbation and application. The results obtained in the dissertation work have been reported in the seminars of the department "General and applied mathematics" of Azerbaijan State University of Oil and Industry (prof. A.R.Aliyev), in the department of "Mathematical analysis" of the Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan (corresponding member of ANAS, prof. V.S.Guliyev), of the department of "Functional analysis" (prof. H.I.Aslanov).

The results of the dissertation work have been reported in the International conference “Modern Problems of Mathematics and Mechanics” held at the Institute of Mathematics and Mechanics (Baku, 2017, 2024), in the International conference “Operators, Functions and Systems of Mathematical Physics” held in Khazar University (Baku, 2018), in Republican conference of doctoral dissertation students and young researchers devoted to 100 years of ASUOI (Baku, 2020), in International scientific conference "Ufa Autumn Mathematical School - 2020" (Ufa, 2020).

Author’s personal contribution. The obtained results and suggestions belong to the author.

Author’s publications. The main results of the dissertation work have been published in her 12 scientific papers given at the end of the work.

The name of the organization where the dissertation was conducted. The dissertation work was completed at the department “General and Applied Mathematics”of the Azerbaijan State Oil and Industry University.

The volume of the dissertation structural sections separately and the general volume. The dissertation work consists of 115 pages, 194504 signs (title page – 372 signs, contents – 1537 sogns, introduction – 58271 signs, first chapter – 84000 signs, second chapter – 49000 signs, conclusion -1324 signs). There are 101 titles of literature in the dissertation.

I express my deep gratitude to my supervisor doctor of mathematical sciences J.J.Hasanov for discussing the results and his valuable remarks.

CONTENT OF THE DISSERTATION WORK

Assume that $\Omega \subseteq R^n$ is an open set, the function $\chi_E(x)$ is a characteristic function in the set E, $B(x,r) = \{y \in R^n :| x - y |< r\}$, $\widetilde{B}(x,r) = B(x,r) \cap \Omega$. Now, let us assume that we are given a measurable function $p(\cdot)$ determined in the set Ω and with the set of values $[1,\infty)$. Assume that the function $p(\cdot)$ satisfies the condition

$$1 < p_- \le p(x) \le p_+ < \infty$$

here

$$p_- := \underset{x\in\Omega}{ess\ inf}\ p(x) > 1, \quad p_+ := \underset{x\in\Omega}{ess\ sup}\ p(x) < \infty.$$

By the space $L^{p(\cdot)}(\Omega)$ we denote a class of functions $f(x)$ measurable in the set Ω and satisfying the condition

$$I_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\, dx < \infty .$$

The function $p'(\cdot)$ is said to be an adjoint function of $p(\cdot)$ if for any $x \in \Omega$ the condition $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$ is satisfied. Then the Holder inequality is in the form

$$\int_\Omega |f(x)||g(x)|\,dx \le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}.$$

The variable exponent Lebesgue space $L^{p(\cdot)}$ coincides with the space

$$\left\{f(x) : \left|\int_\Omega f(y)g(y)dy\right| < \infty \;\; g \in L^{p'(\cdot)}(\Omega)\right\}$$

and the norms

$$\|f\|_{L^{p(\cdot)}} := \sup_{\|f\|_{L^{p'(\cdot)}} \le 1} \left|\int_\Omega f(y)g(y)dy\right|$$

are equivalent.

Variable exponent weighted Lebesgue space $L^{p(\cdot)}_{\omega}(\Omega)$ is a set of measurable functions with the following norm

$$\|f\|_{L^{p(\cdot)}_{\omega}(\Omega)} = \inf\left\{\eta > 0 : \int_\Omega \left(\frac{|f(x)|}{\eta}\right)^{p(x)} \omega(x)dx \le 1\right\}.$$

The class of functions ω satisfying the following condition is denoted by $A_{p(\cdot)}(\Omega)$ and is called a variable exponent Mackenhaupt class:

$$\sup_B |B|^{-1} \|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,r))} \|\omega^{-1}\|_{L^{p'(\cdot)}(\tilde{B}(x,r))} < \infty.$$

$P(\Omega)$ is a boundedly measurable set of functions $p:\Omega\to[1,\infty)$, $P^{log}(\Omega)$ is a set of functions $p\in P(\Omega)$ satisfying the condition

$$|\,p(x)-p(y)\,|\le\frac{A}{-\ln|\,x-y\,|},\quad |\,x-y\,|\le\frac{1}{2},\quad x,y\in\Omega,$$

here the constant $A=A(p)>0$ is independent of *x*, *y*. By $P^{\log}(\Omega)$ we denote a set of functions satisfying the condition $p\in P^{log}(\Omega)$ and $1<p_-\le p_+<\infty$. By $\mathsf{P}_\infty^{log}(\Omega)$ we denote a set of functions satisfying the condition $p\in P^{\log}(\Omega)$ in the unbounded set Ω and the condition

$$|\,p(x)-p(\infty)\,|\le\frac{C}{\ln(e+|\,x\,|)},\quad x\in R^n$$

around the infinity, here $p(\infty)=\lim\limits_{x\to\infty}p(x)$.

Assume that the function $\lambda(x)$ is a measurable function determined in the set Ω with a set of values $[0,n]$. The variable exponent Morrey space $L^{p(\cdot),\lambda(\cdot)}(\Omega)$ is a local integrable space of functions with the finite norm

$$\|f\|_{L^{p(\cdot),\lambda(\cdot)}(\Omega)}=\sup_{x\in\Omega,\,t>0}t^{-\frac{\lambda(x)}{p(x)}}\left\|f\chi_{\tilde{B}(x,t)}\right\|_{L^{p(\cdot)}(\Omega)}.$$

In a similar way, the finite norm of the variable exponent, weighted Morrey space $L_\omega^{p(\cdot),\lambda(\cdot)}(\Omega)$ is in the form:

$$\|f\|_{L_\omega^{p(\cdot),\lambda(\cdot)}(\Omega)}=\sup_{x\in\Omega,\,t>0}t^{-\frac{\lambda(x)}{p(x)}}\left\|f\chi_{\tilde{B}(x,t)}\right\|_{L_\omega^{p(\cdot)}(\Omega)}.$$

The finite norm of the variable exponent generalized Morrey space is in the form: $M^{p(\cdot),\varphi}(\Omega)$

$$\|f\|_{M^{p(\cdot),\varphi}}=\sup_{x\in\Omega,r>0}\frac{r^{-\theta_p(x,r)}}{\varphi(x,r)}\|f\|_{L^{p(\cdot)}(\tilde{B}(x,r))}$$

where $\theta_p(x,r)=\begin{cases}\dfrac{n}{p(x)}, & r\le 1,\\[2mm] \dfrac{n}{p(\infty)}, & r\ge 1\end{cases}.$

Taking in this definition $\varphi(x,r) = r^{\frac{\lambda(x)-n}{p(x)}}$ the space $M^{p(\cdot),\varphi(\cdot)}(R^n)$ transforms into the space $L^{p,\lambda}(R^n)$.

The finite norm of variable exponent generalized weighted local Morrey space $M_{\{0\}}^{p(\cdot),\varphi,\omega}$ is in the form

$$\|f\|_{M_{\{0\}}^{p(\cdot),\varphi,\omega}(\Omega)} = \sup_{t>0} \frac{1}{\varphi(t)\|\omega\|_{L^{p(\cdot)}(\tilde{B}(0,t))}} \|f\|_{L_{\omega}^{p(\cdot)}(\tilde{B}(0,t))}.$$

In this work, for the space $M^{p(\cdot),\varphi}(\Omega)$ not to be trivial, everywhere we will take the function φ as a function satisfying the condition

$$\inf_{t>0} \varphi(t) > 0.$$

The finite norm of variable exponent generalized weighted Morrey space $M^{p(\cdot),\varphi,\omega}$ is in the form

$$\|f\|_{M^{p(\cdot),\varphi,\omega}(\Omega)} = \sup_{x\in\Omega,\, t>0} \frac{1}{\varphi(x,t)\|\omega\|_{L^{p(\cdot)}(\tilde{B}(x,t))}} \|f\|_{L_{\omega}^{p(\cdot)}(\tilde{B}(x,t))}.$$

Note that when p is a constant and φ is a positive constant function, this space coincides with the space $L^{\infty}(\Omega)$.

We now introduce variable exponent local "complementary" generalized Morrey space ${}^{c}M_{\{x_0\}}^{p(\cdot),\omega}(\Omega)$.

Definition 1. *Assume that* $x_0 \in \Omega$, $1 \le p(x) \le p_+ < \infty$. *A variable exponent local "complementary" generalized Morrey space* ${}^{c}M_{\{x_0\}}^{p(\cdot),\omega}(\Omega)$ *is a class of functions with the following finite norm:*

$$\|f\|_{{}^{c}M_{\{x_0\}}^{p(\cdot),\omega}} = \sup_{r>0} \frac{r^{\theta_{p'}(x_0,r)}}{\omega(r)} \|f\|_{L^{p(\cdot)}(\Omega\setminus\tilde{B}(x_0,r))}.$$

For the space ${}^{c}M_{\{x_0\}}^{p(\cdot),\omega}(\Omega)$ to be not trivial, the generalized function should satisfy the condition $\sup_{r>0} \frac{r^{\theta_{p'}(x_0,r)}}{\omega(r)} < \infty$,

Therefore throughout the paper we will accept this condition.

The exact maximum function is determined as

$$M^{\#} f(x) = \sup_{r>0} | B(x,r) |^{-1} \int_{\tilde{B}(x,r)} | f(y) - f_{\tilde{B}(x,r)} | dy$$

here $f_{\tilde{B}(x,t)}(x) = | \tilde{B}(x,t) |^{-1} \int_{\tilde{B}(x,t)} f(y)dy$.

Let φ is a positive and measurable function. Locally integrable class of functions with the finite

$$\|b\|_{BMO_{0,p(\cdot),\varphi}} = \sup_{r>0} \frac{\left\| b(\cdot) - b_{B(0,r)} \right\|_{L_{\varphi}^{p(\cdot)}(B(0,r))}}{\left\| \chi_{B(0,r)} \right\|_{L_{\varphi}^{p(\cdot)}(\Omega)}}$$

a class of functions is called a weighted local space $BMO_{0,p(\cdot),\varphi}$.

Section 2 studies boundedness of manyvariable weighted Hardy, weighted adjoint Hardy operators and their commutators in variable exponent weighted generalized local Morrey spaces.

Assume that the function f is locally integrable in R^n non-negative function the function u is a positive measurable function. We determine the manyvariable weighted Hardy and weighted adjoint Hardi operators as

$$H_u f(x) = | x |^{-n} u(| x |) \int_{|y|<|x|} \frac{f(y)}{u(| y |)} dy, \qquad H_u^* f(x) = u(| x |) \int_{|y|>|x|} \frac{f(y)}{| y |^n u(| y |)} dy.$$

Theorem 1. *Assume that* $p \in \mathrm{P}_{\infty}^{log}(R^n)$, $\omega \in A_{p(\cdot)}(R^n)$, $0 < t < r$, $\gamma \in R$, $\frac{t^{\gamma}}{u(t)} \le C \frac{r^{\gamma}}{u(r)}$ *and the functions* (φ_1, φ_2) *satisfy the condition*

$$\int_0^r \varphi_1(s) \frac{ds}{s} \le C \varphi_2(r) \tag{1}$$

$$\int_0^t \frac{\varphi_1(s)}{u(s) \|\omega\|_{L^{p'(\cdot)}(B(0,s))}} \frac{ds}{s} \le C \frac{\varphi_1(t)}{u(t) \|\omega\|_{L^{p'(\cdot)}(B(0,t))}}. \tag{2}$$

Then the many valued weighted Hardy operator boundedly takes the space $M_{\{0\}}^{p(\cdot),\varphi_1,\omega}(R^n)$ *to the space* $M_{\{0\}}^{p(\cdot),\varphi_2,\omega}(R^n)$.

Theorem 2. *Assume that* $p \in \mathrm{P}_{\infty}^{log}(R^n)$, $\omega \in A_{p(\cdot)}(R^n)$, $0 < r < t$, $\gamma \in R$, $\frac{t^{\gamma}}{u(t)} \leq C \frac{r^{\gamma}}{u(r)}$ *and the functions* (φ_1, φ_2) *satisfy the condition* (1) *and condition*

$$\int_r^{\infty} \frac{\varphi_1(s)}{u(s)} \frac{ds}{s} \leq C \frac{\varphi_1(r)}{u(r)}. \tag{3}$$

Then the many variable weighted adjoint hardy operator unboundedly takes the space $M_{\{0\}}^{p(\cdot),\varphi_1,\omega}(R^n)$ *to the space* $M_{\{0\}}^{p(\cdot),\varphi_2,\omega}(R^n)$.

Determine the commutators of the many variable weighted Hardy and adjoint Hardy operator as follows

$$[b, H_u] f(x) = |x|^{-n} u(|x|) \int_{|z|<|x|} \frac{(b(z) - b(x)) f(z)|}{u(|z|)} dz\,,$$

$$[b, H_u^*] f(x) = u(|x|) \int_{|z|>|x|} \frac{(b(z) - b(x)) f(z)}{|z|^n u(|z|)} dz\,.$$

Theorem 3. *Assume that* $p \in \mathrm{P}_{\infty}^{log}(R^n)$, $\omega \in A_{p(\cdot)}(R^n)$, $0 < t < r$, $\gamma \in R$, $\frac{t^{\gamma}}{u(t)} \leq C \frac{r^{\gamma}}{u(r)}$, $b \in BMO_{0,p'(\cdot),\omega^{-1}}(R^n)$ *and the functions* (φ_1, φ_2) *satisfy the conditions* (1), (2). *Then the commutator* $[b, H_u]$ *of the many variable weighted Hardy operator unboundedly takes the space* $M_{\{0\}}^{p(\cdot),\varphi_1,\omega}(R^n)$ *to the space* $M_{\{0\}}^{p(\cdot),\varphi_2,\omega}(R^n)$.

Theorem 4. *Assume that* $p \in \mathrm{P}_{\infty}^{log}(R^n)$, $\omega \in A_{p(\cdot)}(R^n)$, $0 < r < t$, $\gamma \in R$, $\frac{t^{\gamma}}{u(t)} \leq C \frac{r^{\gamma}}{u(r)}$, $b \in BMO_{0,p'(\cdot),\omega^{-1}}(R^n)$ *and the functions* (φ_1, φ_2) *satisfy the conditions* (1), (3). *Then the commutator* $[b, H_u^*]$ *of the manyvariable weighted adjoint Hardy operator boundedly takes the space* $M_{\{0\}}^{p(\cdot),\varphi_1,\omega}(R^n)$ *to the space* $M_{\{0\}}^{p(\cdot),\varphi_2,\omega}(R^n)$.

In section 3 of chapter I theorems on the boundedness of manyvariable weighted Hardy, weighted adjoint Hardy operators and their commutators in variable exponent weighted generalized local complementary Morrey spaces are studied.

Theorem 5. *Assume that* $p \in \mathrm{P}_{\infty}^{log}(R^n)$, $\omega \in A_{p(\cdot)}(R^n)$ *and the function* $u(r)r^{\gamma}$, $\gamma > 0$ *is an almost increasing function, the functions* (φ_1, φ_2) *satisfy the conditions*

$$\int_0^t \frac{s^n \varphi_1(s)}{\|\omega\|_{L^{p'(\cdot)}(B(0,s))}} \frac{ds}{s} \le C \frac{\varphi_2(t)}{\|\omega\|_{L^{p'(\cdot)}(B(0,t))}}, \tag{4}$$

$$\int_0^t \frac{\varphi_1(s)}{u(s)\|\omega\|_{L^{p'(\cdot)}(B(0,s))}} \frac{ds}{s} \le C \frac{\varphi_1(t)}{u(t)\|\omega\|_{L^{p'(\cdot)}(B(0,t))}} \tag{5}$$

where $t > 0$. *Then the manyvariable weighted Hardy operator unboundedly takes the space* ${}^{c}M_{\{0\}}^{p(\cdot),\varphi_1,\omega}(R^n)$ *to the space* ${}^{c}M_{\{0\}}^{p(\cdot),\varphi_2,\omega}(R^n)$.

Theorem 6. *Assume that the conditions* $p \in \mathrm{P}_{\infty}^{log}(R^n)$, $\omega \in A_{p(\cdot)}(R^n)$, $b \in BMO_{0,p'(\cdot),\omega^{-1}}(R^n)$ *are satisfied, the function,* $u(r)r^{\gamma}$, $\gamma > 0$ *is an almost increasing function and the functions* (φ_1, φ_2) *satisfy the conditions* (4), (5). *Then the commutator* $[b, H_u]$ *of the weighted Hardy operator boundedly takes the space* ${}^{c}M_{\{0\}}^{p(\cdot),\varphi_1,\omega}(R^n)$ *to the space* ${}^{c}M_{\{0\}}^{p(\cdot),\varphi_2,\omega}(R^n)$.

Theorem 7. *Assume that the condition* $p \in P_{\infty}^{\log}(R^n)$, $\omega \in A_{p(\cdot)}(R^n)$ *are satisfied, the function* $u(r)$ *is an almost decreasing function and for* $t > 0$ *the functions* (φ_1, φ_2) *satisfies the conditions*

$$\int_0^t \frac{\varphi_1(s)}{\|\omega\|_{L^{p'(\cdot)}(B(0,s))}} \frac{ds}{s} \le C \frac{\varphi_2(t)}{\|\omega\|_{L^{p'(\cdot)}(B(0,t))}}, \tag{6}$$

$$\int_t^{\infty} \frac{1}{u(s)\|\omega\|_{L^{p(\cdot)}(B(0,s))}} \frac{ds}{s} \le C \frac{1}{u(t)\|\omega\|_{L^{p(\cdot)}(B(0,t))}}. \tag{7}$$

Then the weighted Hardy operator H_u^* *boundedly takes the space* ${}^{c}M_{\{0\}}^{p(\cdot),\varphi_1,\omega}(R^n)$ *to the space* ${}^{c}M_{\{0\}}^{p(\cdot),\varphi_2,\omega}(R^n)$.

Theorem 8. *Assume that the conditions* $p \in P_\infty^{\log}(R^n)$, $\omega \in A_{p(\cdot)}(R^n)$, $b \in BMO_{0,p'(\cdot),\omega^{-1}}(R^n)$ *are satisfied, the function* $u(r)$ *is an almost decreasing function and the functions* (φ_1, φ_2) *satisfy the conditions* (6), (7). *Then the commutator* $[b, H_u^*]$ *of the weighted adjoint operator takes the space* ${}^{c}M_{\{0\}}^{p(\cdot),\varphi_1,\omega}(R^n)$ *to the space* ${}^{c}M_{\{0\}}^{p(\cdot),\varphi_2,\omega}(R^n)$.

Now we give an application of the results obtained in this section.

Let us determine the operator L as

$$Lv = [b, H_u]v - v\,.$$

Consider the following equation:

$$Lv = f\,.$$

Theorem 9. *Assume that the conditions* $p \in \mathrm{P}_\infty^{log}(R^n)$, $\omega \in A_{p(\cdot)}(R^n)$, $b \in BMO_{0,p'(\cdot),\omega^{-1}}(R^n)$ *are satisfied the function* $u(r)r^\gamma$, $\gamma > 0$ *is an almost increasing function and the function* (φ_1, φ_2) *satisfy condition* (1), (2). *Then there exists such a positive constant* $C = C(n, p)$ *that the inequality*

$$\|v\|_{M_{\{0\}}^{p(\cdot),\varphi_2,\omega}(R^n)} \le C\|Lv\|_{M_{\{0\}}^{p(\cdot),\varphi_1,\omega}(R^n)}$$

is satisfied.

In section 4 of chapter I we find necessary and sufficient condition for manyvariable weighted Hardy-Cesaro and weighted adjoint Hardy-Cesaro operators to be bounded in Morrey space. Furthermore, sufficient conditions for the boundedness of the commutators of the manyvariable weighted Hardy-Cesaro and weighted adjoint Hardy-Cesaro operators in Morrey space are found.

Assume that the function f is locally integrable in R^n, the function u is locally integrable in R. Many variable weighted Hardy-

Cesaro and weighted adjoint Hardy-Cesaro operators are determined as follows

$$H_u f(x) = \int_{B(0,1)} | f(x|y|)|\, u(|y|)dy, \quad H_u^* f(x) = \int_{B(0,1)} \left| f\left(\frac{x}{|y|}\right)\right| u(|y|)dy.$$

Theorem 10. *Assume that $1 \le p < \infty$ and $0 \le \lambda < n$. Then a necessary and sufficient condition for the Hardy-Cesaro operator H_u to be bounded in space $L^{p,\lambda}(R^n)$ is*

$$\int_{B(0,1)} |y|^{\frac{\lambda-n}{p}}\, u(|y|)dy < \infty.$$

Theorem 11. *Assume that the function u is a positive measurable function and satisfies the condition*

$$\int_{B(0,1)} u(|y|)dy < \infty \tag{8}$$

Then the Hardy-Cesaro operator H_u is bounded in space $BMO(R^n)$.

Theorem 12. *Assume that $1 \le p < \infty$ and $0 \le \lambda < n$. Then a necessary and sufficient condition for the Hardy-Cesaro operators H_u^* be bounded in space $L^{p,\lambda}(R^n)$ is*

$$\int_{B(0,1)} |y|^{\frac{n-\lambda}{p}}\, u(|y|)dy < \infty.$$

Theorem 13. *Assume that the function u is a positive measurable function and satisfies condition* (8. Then the Hardy operator H_u^* *is bounded in space* $BMO(R^n)$.

We determine the many-variable weighted Hardy-Cesaro and weighted adjoint Hardy-Cesaro operators as follows:

$$H_{u,b} f(x) = \int_{B(0,1)} |b(x|y|) - b(x)||\, f(x|y|)|\, u(|y|)dy,$$

$$H_{u,b}^* f(x) = \int_{B(0,1)} \left| b\left(\frac{x}{|y|}\right) - b(x)\right| \left| f\left(\frac{x}{|y|}\right)\right| u(|y|)dy.$$

Theorem 14. *Assume that* $1 \le p < \infty$, $0 \le \lambda < n$, $b \in BMO(R^n)$ and the function *u is a positive measurable function and satisfies the following condition*

$$\int_{B(0,1)} \left(2 + \left|\ln |y|\right|\right) |y|^{\frac{\lambda - n}{p}} u(|y|) dy < \infty .$$

Then the commutator of the Hardy-Cesaro operator $H_{u,b}$ *is bounded in space* $L^{p,\lambda}(R^n)$.

Theorem 15. *Assume that* $1 \le p < \infty$, $0 \le \lambda < n$, $b \in BMO(R^n)$ and the function *u is a positive measurable function and satisfies the condition*

$$\int_{B(0,1)} \left(2 + \left|\ln |y|\right|\right) |y|^{\frac{n - \lambda}{p}} u(|y|) dy < \infty .$$

Then the commutator od the many variable weighted adjoint Hardy-Cesaro operator $H^*_{u,b}$ *is bounded in space* $L^{p,\lambda}(R^n)$.

Section 1 of chapter II deals with generalized local Orlicz-Morrey and generalized local complementary Orlicz-Morrey spaces and the facts used in this chapter are noted.

Definition 2. If the function $\Phi : [0,+\infty) \to [0,\infty]$ is convex, left continuous and satisfies the conditions $\lim_{r \to +0} \Phi(r) = \Phi(0) = 0$, $\lim_{r \to +\infty} \Phi(r) = \infty$ *the function* Φ *is said to be a Young function.*

In this definition it is seen from the convexity of the function and from the condition $\Phi(0) = 0$ that the Young function is an increasing function. If there exists such $s \in (0,+\infty)$ that $\Phi(s) = +\infty$ then for any $r \ge s$ $\Phi(r) = +\infty$. By Y we denote the set of Φ Young functions satisfying the condition for $0 < \Phi(r) < +\infty,\ 0 < r < +\infty$ if $\Phi \in Y$, then the function Φ is absolutely continuous in the interval $[0,+\infty)$ and is one to one in this interval.

Definition 3. *Assume that the function* f *is locally integrable and for any* $k > 0$ *and the functions set f* satisfying the condition

$$\int_{\mathbb{R}^n}\Phi(k\,|\,f(x)\,|)\,dx < +\infty$$

is said to be Orlicz space. In other words

$$L_\Phi(\mathbb{R}^n) = \left\{f \in L_1^{loc}(\mathbb{R}^n) : \int_{\mathbb{R}^n}\Phi(k\,|\,f(x)\,|)dx < +\infty \text{ for each } k > 0\right\}.$$

For $1 \le p < \infty$ if $\Phi(r) = r^p$, then $L_\Phi(\mathbb{R}^n) = L^p(\mathbb{R}^n)$. If $\Phi(r) = 0\ (0 \le r \le 1)$ and $\Phi(r) = \infty\ (r > 1)$ then $L_\Phi(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$. Under the space $L_\Phi^{loc}(\mathbb{R}^n)$ we understand the set of functions f for all shperes $B \subset \mathbb{R}^n$ from $f\chi_B \in L_\Phi(\mathbb{R}^n)$.

Orlicz space $L_\Phi(\mathbb{R}^n)$ is a Banach space and its finite norm is in the form

$$\|f\|_{L_\Phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n}\Phi\left(\frac{|\,f(x)\,|}{\lambda}\right)dx \le 1\right\}.$$

Assume that the weight function ω is a positive measurable function in R^n. The weighted Orliczspace $L_{\Phi,\omega}(\mathbb{R}^n)$ is in the form

$$L_{\Phi,\omega}(R^n) := \{f : f\omega \in L_\Phi\}$$

and its finite norm is in the form:

$$\|f\|_{L_{\Phi,\omega}} = \|\omega f\|_{L_\Phi}$$

For the Young function Φ and $0 \le s \le +\infty$ let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \quad (\inf \varnothing = +\infty).$$

If $\Phi \in Y$, then the function Φ^{-1} is said to be an inverse function of the function Φ. Note that for the functions Φ^{-1} and Φ the conditions

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r))$$

are satisfied.

If for any $k > 1$ the inequality $\Phi(2r) \le k\Phi(r)$ is satisfied, then we say that the Young function Φ satisfies the condition Δ_2 and is denoted as $\Phi \in \Delta_2$. If the Young function Φ satisfies the condition Δ_2, then $\Phi \in \mathrm{Y}$.

The function

$$\tilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\} & , \quad r \in [0,\infty) \\ +\infty & , \quad r = +\infty. \end{cases}$$

is a complementary function for the Young function Φ.

The complementary function $\tilde{\Phi}$ is also a Young function and $\tilde{\tilde{\Phi}} = \Phi$. If $0 \le r \le 1$ then the complementary function of the function $\Phi(r) = r$ is zero i.e. $\tilde{\Phi}(r) = 0$. If $\Phi(r) = r$ then for $0 \le r \le 1$ the complementary function of the function Φ is zero, i.e. $\tilde{\Phi}(r) = 0$ and $r > 1$ for $\tilde{\Phi}(r) = \infty$. If $1 < p < \infty$, $1/p + 1/p' = 1$ and $\Phi(r) = r^p/p$ then $\tilde{\Phi}(r) = r^{p'}/p'$. Note that a necessary and sufficient condition for $\Phi(r) = e^r - r - 1$ if $\tilde{\Phi}(r) = (1+r)\log(1+r) - r$. It is clear that for $\Phi \in \nabla_2$ is $\tilde{\Phi} \in \Delta_2$. It is clear that for $r \ge 0$, $r \le \Phi^{-1}(r)\tilde{\Phi}^{-1}(r) \le 2r$.

Note that the Young function satisfies the properties $\Phi(\alpha t) \le \alpha \Phi(t)$ for any $0 \le \alpha \le 1$, $0 \le t < \infty$ and the properties $\Phi(\beta t) \ge \beta \Phi(t)$ for any $0 \le t < \infty$.

Definition 5. *(Generalized Orlicz space) Assume that the function* $\varphi(x,r)$ *is a positive, measurable function in* $\mathbb{R}^n \times (0,\infty)$ Φ *is a Young function. By* $M_{\Phi,\varphi}(R^n)$ *we will denote a space of functions from the class* $f \in L_{\Phi}^{loc}(R^n)$ *and with the following finite quasinorm*

$$\| f \|_{M_{\Phi,\varphi}} = \sup_{x \in \mathbb{R}^n, r>0} \varphi(x,r)^{-1} \| f \|_{L_{\Phi}(B(x,r))}.$$

and this space is called generalized Orlicz-Morrey space.

Assume that the function *f* is a locally integrable nonnegative function in R^n. We denote many variable fractional Hardy and weighted adjoint fractional Hardy operators as

$$H_u^{\alpha} f(x) = |x|^{\alpha - n} u(|x|) \int_{|y|<|x|} \frac{f(y)}{u(|y|)} dy, \quad \mathrm{H}_u^{\alpha} f(x) = |x|^{\alpha} u(|x|) \int_{|y|>|x|} \frac{f(y)}{|y|^n u(|y|)} dy$$

here $\alpha \ge 0$.

In this section we found sufficient conditions on the functions Φ, Ψ and φ_1, φ_2 for the boundedness of weighted fractional Hardy and weighted adjoint fractional Hardy operators in the generalized Orlicz-Morrey space.

Theorem 16. *Assume that* Φ, Ψ *are Young functions,* $0<\alpha<n$, $f \in L_{\Phi}^{loc}(R^n)$, $\frac{r^{\beta}}{u(r)} \le C \frac{t^{\beta}}{u(t)}$, $\varphi_1(0,t)/t^{\beta}u(t) \le C\varphi_1(0,r)/r^{\beta}u(r)$ $0<r<t$, $\beta \in R$, (φ_1,φ_2) *and the functions* (Φ,Ψ) *satisfy the following conditions*

$$\int_0^r \frac{t^n \varphi_1(0,t)}{u(t)} \frac{dt}{t} \le C \frac{r^n \varphi_1(0,r)}{u(t)}, \tag{13}$$

$$\int_0^r \frac{s^{\alpha}\varphi_1(0,s)}{\Psi^{-1}(s^{-n})} \frac{ds}{s} \le C \frac{\varphi_2(0,r)}{\Psi^{-1}(r^{-n})} \tag{14}$$

Then weighted fractional Hardy operator H_u^{α} *boundedly takes the space* $M_{\Phi,\varphi_1}^{0,loc}(R^n)$ *to the space* $M_{\Psi,\varphi_2}^{0,loc}(R^n)$.

Theorem 17. *Assume that* Φ, Ψ *are Young functions. The conditions* $0<\alpha<n$, $\frac{r^{\beta}}{u(r)} \le C \frac{t^{\beta}}{u(t)}$ $r^{\alpha}\varphi_1(0,r)\Phi^{-1}(r^{-n}) \le Ct^{\alpha}\varphi_1(0,t)\Phi^{-1}(t^{-n})$, $0<r<t$, $\beta \in R$ *are satisfied, the functions* (φ_1,φ_2) *and* (Φ,Ψ) *satisfy the conditions*

$$\int_r^{\infty} \frac{\varphi_1(0,t)\Phi^{-1}(t^{-n})}{u(t)} \frac{dt}{t} \le C \frac{\varphi_1(0,r)\Phi^{-1}(r^{-n})}{u(r)}$$

$$\int_0^r \frac{t^{\alpha}\varphi_1(0,t)\Phi^{-1}(t^{-n})}{\Psi^{-1}(t^{-n})} \frac{dt}{t} \le C \frac{\varphi_2(0,r)}{\Psi^{-1}(r^{-n})}$$

Then weighted fractional adjoint Hardy operator H_u^{α} *boundedly takes the space* $M_{\Phi,\varphi_1}^{0,loc}(R^n)$ *to the space* $M_{\Psi,\varphi_2}^{0,loc}(R^n)$.

Assume that the functions *f* and *b* are locally integrable nonnegative functions in R^n. The commutator of manyvariable weighted fractional Hardy and weighted adjoint fractional Hardy operator are determined as follows

$$[b,H_u^{\alpha}]f(x) = |x|^{\alpha-n}\, u(|x|) \int_{|y|<|x|} \frac{(b(y)-b(x))f(y)}{u(|y|)} dy,$$

$$[b, \mathrm{H}_u^\alpha] f(x) = |x|^\alpha \; u(|x|) \int_{|y|>|x|} \frac{(b(y) - b(x)) f(y)}{|y|^n \; u(|y|)} dy,$$

here $\alpha \geq 0$.

Theorem 18. *Assume that* Φ, Ψ *are Young functions, the conditions* $0 < \alpha < n$, $b \in BMO_{0,\tilde{\Phi}}(R^n)$, $\frac{r^\beta}{u(r)} \leq C \frac{t^\beta}{u(t)}$, $\varphi_1(0,t)/t^\beta u(t) \leq C \varphi_1(0,r)/r^\beta u(r)$ $0 < r < t$, $\beta \in R$ *are satisfied the functions,* (φ_1, φ_2) *and* (Φ, Ψ) *satisfy conditions* (13), (14) *Then the commutator* $[b, H_u^a]$ *of the weighted fractional operator boundedly takes the space* $M_{\Phi,\varphi_1}^{0,loc}(R^n)$ *to the space* $M_{\Psi,\varphi_2}^{0,loc}(R^n)$.

Theorem 19. *Assume that* Φ, Ψ *are Young functions. The conditions* $0 < \alpha < n$, $\frac{r^\beta}{u(r)} \leq C \frac{t^\beta}{u(t)}$, $0 < r < t$, $\beta \in R$, $b \in BMO_{0,\tilde{\Phi}}(R^n)$, $r^\alpha \varphi_1(0,r) \Phi^{-1}(r^{-n}) \leq C t^\alpha \varphi_1(0,t) \Phi^{-1}(t^{-n})$, $0 < r < t$ *are satisfied, the functions* (φ_1, φ_2) *and* (Φ, ψ) *satisfy the following conditions:*

$$\int_r^\infty \frac{\varphi_1(0,t)}{u(t)} \frac{dt}{t} \leq C \frac{\varphi_1(0,r)}{u(r)},$$

$$\int_0^r \frac{t^\alpha \varphi_1(0,t)}{\Psi^{-1}(t^{-n})} \frac{dt}{t} \leq C \frac{\varphi_2(0,r)}{\Psi^{-1}(r^{-n})}$$

Then the commutator $[b, H_u^a]$ *of weighted adjoint fractional Hardy operator takes the space* $M_{\Phi,\varphi_1}^{0,loc}(R^n)$ *to the space* $M_{\Psi,\varphi_2}^{0,loc}(R^n)$.

In section 3 of chapter II we proved the boundedness of weighted fractional Hardy, weighted adjoint fractional Hardy operators and their commutators in generalized local complementary Orlicz-Morrey spaces.

Theorem 20. *Assume that* Φ, Ψ *are Young functions,* $0 < \alpha < n$ *and* $u(r) r^\gamma$, *the function* $\gamma > 0$ *is an absolutely increasing function, the functions* (φ_1, φ_2) *and* (Φ, Ψ) *satisfy the conditions*

$$\int_0^r \frac{\varphi_1(t)}{t^n u(t)\Phi^{-1}(t^{-n})}\,\frac{dt}{t} \le C\,\frac{\varphi_1(r)}{r^n u(r)\Phi^{-1}(r^{-n})}, \tag{15}$$

$$\int_0^r \frac{s^{\alpha-n}\varphi_1(s)}{\Psi^{-1}(s^{-n})}\,\frac{ds}{s} \le C\,\frac{\varphi_2(r)}{r^n \Psi^{-1}(r^{-n})}, \tag{16}$$

here $t>0$. *Then the operator of manyvariable weighted Hardy operator boundedly takes the space* ${}^cM^{0,loc}_{\Phi,\varphi_1}(R^n)$ *to the space* ${}^cM^{0,loc}_{\Psi,\varphi_2}(R^n)$.

Theorem 21. *Assume that* Φ, Ψ *are Young functions, the conditions* $0<\alpha<n$, $b\in BMO_{0,\tilde{\Phi}}(R^n)$, $\dfrac{r^\beta}{u(r)} \le C\dfrac{t^\beta}{u(t)}$, $\varphi_1(t)/t^\beta u(t) \le C\varphi_1(r)/r^\beta u(r)$ $0<r<t$, $\beta\in R$ *are satisfied, the functions* (φ_1,φ_2) *satisfy conditions* (15), (16). *Then the commutator* $[b, H^\alpha_u]$ *of the weighted fractional Hardy operator boundedly takes the space* ${}^cM^{0,loc}_{\Phi,\varphi_1}(R^n)$ *to the space* ${}^cM^{0,loc}_{\Psi,\varphi_2}(R^n)$.

Theorem 22. *Assume that* Φ, Ψ *are Young functions,* $0<\alpha<n$, *the function* $u(r)$ *is an almost decreasing function. The functions* (φ_1,φ_2) *and* (Φ,Ψ) *satisfy the conditions* (16),

$$\int_t^\infty \frac{\Phi^{-1}(s^{-n})}{u(s)}\,\frac{ds}{s} \le C\,\frac{\Phi^{-1}(t^{-n})}{u(t)}, \tag{17}$$

The the weighted adjoint Hardy operators H^α_u *boundedly takes the space* ${}^cM^{0,loc}_{\Phi,\varphi_1}(R^n)$ *to the space* ${}^cM^{0,loc}_{\Psi,\varphi_2}(R^n)$.

Theorem 23. *Assume that* Φ, Ψ *are Young functions,* $0<\alpha<n$ *the conditionss* $b\in BMO_{0,\tilde{\Phi}}(R^n)$ *are satisfied. The function* $u(r)$ *is an almost decreasing function. The functions* (φ_1,φ_2) *and* (Φ,Ψ) *satisfy the conditions* (16), (17). *Then the commutator* $[b, \mathrm{H}^\alpha_u]$ *of the weighted fractional adjoint Hardy operator boundedly takes the space* ${}^cM^{0,loc}_{\Phi,\varphi_1}(R^n)$ *to the space* ${}^cM^{0,loc}_{\Psi,\varphi_2}(R^n)$.

CONCLUSIONS

The dissertation work is devoted to the boundedness of weighted Hardy, weighted adjoint Hardy operators being from the integral operators of harmonic analysis and their commutators in variable exponent weighted generalized local Morrey, variable degree complementary generalized local Morrey, generalized local Orlicz-Morrey and generalized complmentary local Orlicz-Morrey spaces. The following results have been obtained:

- Necessary and sufficient conditions for the boundedness of the weighted Hardy-Cesaro operator in Morrey space have been found.
- Sufficient conditions for the boundedness of the commutator of the weighted Hardy-Cesaro in Morrey space have been found.
- Necessary and sufficient conditions for the boundedness of the weighted Hardy-Cesaro operator in Morrey space have been found.
- Sufficient conditions for the boundedness of the commutator of the weighted adjoint Hardy-Cesaro operator have been found.
- Boundedness of weighted Hardy operator and its commutator in variable exponent weighted generalized Morrey and generalized complementary Morrey spaces has been proved.
- Boundedness of weighted adjoint Hardy operator and its commutator in variable exponent weighted generalized Morrey and generalized complementary Morrey space has been proved.
- Boundedness of the weighted adjoint fractional Hardy operator and its commutator in generalized Orlicz-Morrey and generalized complementary Orlicz-Morrey spaces has been proved.
- Boundedness of the weighted fractional Hardy operator and its commutator in generalized Orlicz-Morrey and generalized complementary Orlicz-Morrey spaces has been proved.

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