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## ABSTRACT

of the dissertation for the degree of the Doctor of the Science

## DESERCH OF DISCRETE MODELS OF ECONOMIC DYNAMICS OF THE NEUMANN TYPE

Speciality: 3338.01- "System analysis, control and information processing"

Field of science: mathematics

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## GENERAL DESCRIPTION OF WORK

Relevance of the topic. Beginning in the mid-19th century, mathematical methods began to be used in economic research. The creation of mathematical models of the economy served as an impetus for the development of a new field of mathematics mathematical economics. The first works on the study of economic processes using mathematical apparatus appeared at the end of the 19th century and belonged to L. Walras, V. Pareto and others. The further development of these ideas is associated with the names of V . Leontiev and J. Neumann, in whose works simple multidimensional models were developed economic dynamics. Subsequently, in the works of such mathematicians as D. Gale, L.V. Mackenzie, H. Nikaido, L.V. Kantorovich et al., these ideas were significantly advanced in the study of multi-product models of economic dynamics. F. Ramsay, J. Keynes, R. Solow, and others also made a great contribution to the development of mathematical models of economics.

In the first half of the twentieth century, mathematical economics had already emerged as a separate branch of mathematics. Intensive research in various areas of mathematical economics continues to this day.

When studying models of economic dynamics, of particular interest are the study of the behavior of trajectories, the properties of the efficiency of trajectories, determining the growth rate of the model, solving some optimization problems, etc. In the course of studying various models and their modifications, various authors managed to develop approaches and methods applicable to many models of economic dynamics. A unified theory of models of economic dynamics was created and it is reflected in many monographs and textbooks.

This research work also lies within the framework of these problems and is devoted to the study of some theoretical problems of mathematical economics, such as the study of the properties of trajectories of models of economic dynamics of the Neumann type
with discrete time, the study of the dependencies of certain indicators, the application of production functions of various types to the study of production processes, the determination of rates growth and equilibrium conditions in specific models, etc., which make the topic of the dissertation relevant.

It should be noted that the results of studies of economic dynamics models are not only theoretical in nature, but can also be used to build specific production models.

In the proposed work, the main objects of research are onesector and two-sector models of economic dynamics of the Neumann type with discrete time. Models of this type are extremely aggregated models of the economy, but at the same time they reflect the dynamics of relationships between macro indicators and are widely used in the study of patterns of economic development.

The advantage of such models is that they can participate in more complex economic and mathematical models as a component, and the results obtained from the study of such models can be used in the study of more complex dynamic systems. This type of model was studied by F. Ramsey, M. Brown, L. Johansen, E. Phelps, Z. K. Arrow, N. N. Moiseev, V. L. Makarov, A. M. Rubinov and others.

The object of study in the mentioned works was the problem of the existence of an equilibrium state of trajectories, finding optimal stationary trajectories in the sense of one or another optimality criterion, studying the main properties of trajectories, etc.

As an example, we can point out the principle of differential optimization by L.V. Kantorovich, as well as the principle of maximum use of the potential capabilities of the economy by A.M. Rubinov.

Object and subject of research. The object of the study is discrete one-sector and two-sector models of economic dynamics of the Neumann type, which are described using multi-valued mappings. A wide range of issues are studied, including the behavior of effective trajectories, the study of the main properties of trajectories, equilibrium conditions, growth rates, the dynamics of the
interrelations of various macro-indicators, such as consumption, labor, fixed assets, national wealth, etc.

In this work, these issues are studied using optimization problems using the apparatus of production functions, in particular the Cobb-Douglas, Leontief production functions and the function with constant elasticity of substitution (CES).

Goal of the work. The main goal of the work is to study some problems of discrete models of economic dynamics of the Neumann type, incl.

- study of the behavior of trajectories in one-sector and twosector models specified by superlinear multivalued mappings in finite and infinite time intervals.
- determination of characteristic prices and growth rates of the models under consideration.
- determination of types of equilibria in special Neumann-type models.
- determination of the degrees of dependence of some indicators in one-sector and two-sector models.
- solving some optimization problems, including maximizing total consumption, total output and national wealth expenses.

The main provisions of the dissertation submitted for defense.

- effective trajectories of models that allow characterization have been studied and types of effective trajectories have been found,
- the principle of optimality is proposed, according to which consumption is chosen so that the trajectory with a given labor force is efficient.
- the limiting behavior of trajectories under a strict equilibrium state has been studied,
- for a single-sector model, the dependence of the consumption function on the type of production function is determined,
- the conditions for maximizing total consumption, total output and total national wealth are determined,
- necessary and sufficient conditions for the existence of a solution to the consumer problem are found.
- a theorem has been proven about the existence of a solution to the consumer problem without losses with a fixed budget,
- using a lossless mechanism, a trajectory was constructed in one Neumann-type model and the Neumann equilibrium state was determined,
- under certain conditions for the two-sector model, Neumann growth rates, Neumann prices and Neumann equilibrium states are determined
- types of equilibria in the two-sector model are determined,
- necessary and sufficient conditions for the existence of an equilibrium state in a two-sector model were found,
- the uniqueness of Neumann prices in equilibrium has been proven
- a Neumann face was constructed for the model,
- models were built using graph theory with and without transport

Scientific novelty. For a special model of the Neumann type, characteristic prices and growth rates of national wealth are determined. The problem of optimal labor distribution in a multiindustry model has been solved. Using the apparatus of convex analysis, a connection has been established between characteristic prices and the super differential of the corresponding functional in a Neumann-type model.

For a special model of the Neumann type, conditions for the efficiency of trajectories are found and the principle of optimality of trajectories is established.

The problems of determining the dependence of the volume of consumption on the number of labor employed in production and on the means of production for Cobb-Douglas production functions and functions with constant elasticity of substitution (CES) are also studied.

Necessary and sufficient conditions for the existence of an equilibrium state without losses are obtained and the Neumann equilibrium state is determined.

For the two-sector model Z2, the Neumann growth rate and Neumann equilibrium prices are determined, and T - step effective trajectories are also constructed.

Research methods. The main research methods are methods of mathematical modeling of economic processes. In this case, methods of mathematical analysis, the theory of superlinear multivalued mappings, the theory of discrete dynamic systems, mathematical programming, graph theory, convex analysis, etc. are widely used.

Theoretical and practical value. The results obtained in the dissertation work are mainly theoretical in nature. But some results can be applied when studying specific economic models.

Approbation of work. The results of the dissertation were presented at a seminar at the department of "Mathematical theory of modeling control systems" of St. Petersburg State University under the guidance of prof. V.F. Demyanov, at the seminar of the Institute of Socio-Economic Problems of the USSR Academy of Sciences, Leningrad, under the guidance of prof. A.M. Rubinov, at the seminar of the Institute of Applied Mathematics at Baku State University under the leadership of academician. F.A. Aliyev, at the seminar at the department of "Mathematical Cybernetics"of Baku State University under the leadership of prof. K.B.Mansimov and also reported at the following scientific conferences:

- Scientific conference dedicated to the 90th anniversary of academician M.L. Rasulov, Baku, 2006.
- Scientific conference dedicated to the 100th anniversary of Academician A. Huseynov, Baku, 2007.
- Scientific conference dedicated to the 90th anniversary of Professor G.G. Akhmedov, Baku, 2007.
- Republican scientific conference, Sumgait, 2007.
- The $1^{\text {st }}$ International Conference on Control and Optimization with Industrial Applications COIA-2005, Baku, 2005.
- The $2^{\text {nd }}$ International Conference on Control and Optimization with Industrial Applications COIA-2008, Baku, 2008.
- The $3^{\text {rd }}$ International Eurasian Conference on Mathematical Science and Applications IECMSA-2014, Vienna, Austria, 2014.
- The $4^{\text {th }}$ International Eurasian Conference on Mathematical Science and Applications IECMSA-2015, Athens, Greece, 2015.
- The $5^{\text {th }}$ International Eurasian Conference on Mathematical Science and Applications IECMSA-2016, Belgrad, Serbia, 2016.
- The $6^{\text {th }}$ International Eurasian Conference on Mathematical Science and Applications IECMSA-2017, Budapest, Hungary, 2017.
- The $6^{\text {th }}$ International Conference on Control and Optimization with Industrial Applications COIA-2018, Baku, 2018.
- The $7^{\text {th }}$ International Conference on Control and Optimization with Industrial Applications COIA-2020, Baku, 2020.
- The $8^{\text {th }}$ International Conference on Control and Optimization with Industrial Applications COIA-2022, Baku, 2022

Publications. 34 works have been published on the topic of the dissertation, including 21 articles in various journals, a list of which is presented at the end of the abstract.

Name of organization where was performed the work.The work was performed at the Institute of Applied Mathematics at the Baku State University.

Structure and scope of work. The dissertation consists of an introduction, six chapters, main results and a list of references. The total volume of the work is 254903 characters (table of contents- 2863 characters, introduction- 10334 characters, chapter I- 21000 characters, chapter II- 65957 characters, chapter III- 25766 characters, chapter IV -51442 characters, chapter V- 49435 characters, chapter VI- 26437 characters, rezults-1499 characters). Number of characters in the abstract is 64575 .

## SUMMARY OF THE DISSERTATION

The dissertation consists of an introduction, six chapters, main results and a list of references.

The introduction provides a rationale for the relevance and degree of development of the dissertation topic. The main directions of development of the theory of mathematical models of economics are outlined. The purpose and objectives of the research are formulated. A brief overview of results in the study area is provided.

Chapter I presents the basic concepts and definitions used in the dissertation work. It consists of seven paragraphs. The first paragraph presents the basic concepts from the theory of multivalued mappings. In this case, the main attention is paid to the description of superlinear multivalued mappings and their properties. The second paragraph of this chapter is devoted to the description of discrete dynamic systems. Particular attention is paid to special dynamical systems defined using superlinear multivalued mappings. An economic interpretation of a discrete dynamic system is given.

As is known, processes occurring in the economy are described by technological mappings.

The third paragraph of the first chapter is devoted to technological mappings of economic dynamics models studied in this work.

The fourth paragraph of this chapter describes models of various types that are often encountered in practice, including models of the Leontief and Neumann types.

The fifth paragraph provides a description of the equilibrium mechanisms for constructing trajectories. This approach is widely used in the study of models of economic dynamics, and the choice of specific consumption plays a significant role.

The sixth paragraph is devoted to models of expanded reproduction, which were introduced by A, M, Rubinov. They describe the interaction of several sectors, each of which processes its resources into finished products. And finally, in the seventh paragraph, the definition of the Neumann growth rate and the

Neumann equilibrium vector is given. A description of the trajectory is given that admits the characteristic,

Chapter II consists of 9 paragraphs. The first paragraph provides a definition of a Neumann type model and considers a modification of V.L. Makarov's model $Z(\omega)^{1}$. Let there be $n$ technology in the economy. The technology $i$ is described by the $\operatorname{pair}\left(F_{i}, v_{i}\right)$, where $F_{i}$ is the production function and $v_{i}$ is the coefficient of preservation of funds. The volume of fixed assets and the size of the workforce are denoted by and respectively. Thus, the model $Z$ describing the joint functioning of $n$ technologies is given by the sequence $\left(F_{1}, v_{1} ; F_{2}, v_{2} ; \ldots ; F_{n}, v_{n}\right)$. The state of the economy in the model is characterized by the vector $x=$ $\left(K^{1}, \ldots, K^{n}, L^{1}, \ldots, L^{n}, \omega^{1}, \ldots, \omega^{n}\right)$. Here $\omega^{i}$ is specific consumption in $i$ - production. A transition from a state $x_{t}$ at an instant to a state $x_{t+1}$ at an instant $t+1$ is possible if it satisfies the following system of relations:

$$
\begin{gather*}
K_{t+1}^{i} \leq v_{i} K_{t}^{i}+I_{t+1}^{i}, \quad I_{t+1}^{i} \geq 0,  \tag{1}\\
I_{t+1}^{i}+\omega_{t+1}^{i} L_{t+1}^{i} \leq F_{i}\left(K_{t}^{i}, L_{t}^{i}\right), \quad(i=\overline{1, n}),
\end{gather*}
$$

Here $I_{t+1}^{i}$ are investments. Let's fix the sequence $\omega=$ $\left(\omega_{1}, \ldots, \omega_{t}, \ldots\right)$ and consider the model $Z(\omega)$.

In what follows, we consider trajectories $x_{t}=$ $\left(K_{t}^{1}, \ldots, K_{t}^{n}, L_{t}^{1}, \ldots, L_{t}^{n}\right)$ for which the inequalities are valid for each $t$

$$
I_{t+1}^{i}=K_{t+1}^{i}-v_{i} K_{t}^{i}>0, \quad L_{t+1}^{i}>0
$$

We will call such trajectories with the property (A). Let be a $P_{0}, \ldots, P_{t}$ characteristic of some trajectory $x_{0}, \ldots, x_{t}, \ldots$ Let's put

$$
q_{p_{t+1}}(x)=\max _{y \in a(x)} P_{t+1}(y), \quad x \geq 0
$$

Lemma 2.1.1. Let $q_{p_{t+1}}(x)=\max _{y \in a(x)} P_{t+1}(y)$, where $x \geq 0$, then the functional has the following form

$$
\begin{gathered}
q_{p_{t+1}}=v_{1} P_{t+1}^{1} K^{1}+\cdots+v_{n} P_{t+1}^{n} K^{n}+c^{1} F_{1}\left(K^{1}, L^{1}\right)+\cdots+ \\
+c^{n} F_{n}\left(K^{n}, L^{n}\right),
\end{gathered}
$$

${ }^{1}$ Makarov V.L. On dynamic models of the economy and the development of ideas L.V. Kantorovich // Economics and mathematical methods, 1695, vol. 1, no. 5, art. 10-24.
where $c^{i}=\max \left(P_{t+1}^{i}, \frac{q_{t+1}^{i}}{\omega_{t+1}^{i}}\right)$.
Lemma 2.1.2. The superdifferential $\partial q_{p_{t+1}}$ of the functional $q_{p_{t+1}}$ has the form

$$
\partial q_{p_{t+1}}=\left(v_{1} P_{t+1}^{1}, \ldots, v_{n} P_{t+1}^{n}, 0, \ldots, 0\right)+c^{1} \partial \bar{F}_{1}+\cdots+c^{n} \partial \bar{F}_{n} .
$$

Since the trajectory $x_{t}$ admits the characteristic $P_{t}$, the following relations hold:

1. From Lemma 2.1.2 it follows that there are for which

$$
\begin{gathered}
P_{t} \in \partial q_{p_{t+1}} h^{i}=\left(h_{i}^{i}, h_{i+n}^{i}\right) \in \partial F_{i}, \\
P_{t}=\left(v_{1} P_{t+1}^{1}+c^{1} h_{1}^{1}, \ldots, v_{n} P_{t+1}^{n}+c^{n} h_{n}^{n}, c^{1} h_{1}^{1}, \ldots, c^{n} h_{2 n}^{n}\right) ;
\end{gathered}
$$

$$
\text { 2. } P_{t}\left(x_{t}\right)=P_{t+1}\left(x_{t+1}\right) \text {. }
$$

Let be an $\left(x_{t}\right)$ effective trajectory of the model $Z(\omega)$ that admits the characteristic $P_{t}$ and has the property $(A)$. Let's consider the simplest single $n$-product models of the Neumann type

$$
Z^{i}\left(\omega^{i}\right)=\left(\frac{b_{t+1}^{i}}{b_{t}^{i}} F_{i}, \frac{b_{t+1}^{i}}{b_{t}^{i}} v_{i}, \omega_{t}^{i}\right), \quad(i=\overline{1, n})
$$

and the corresponding growth rates of these models will be denoted by $\alpha_{t}^{i}=\frac{b_{t+1}^{i}}{b_{t}^{i}} \max _{\eta>0} \frac{v_{i} \eta+f_{i}(\eta)}{\eta+\omega_{t}^{i}}$, where $\eta=\frac{K}{L} f_{i}(\eta)=F_{i}(\eta, 1)$.

Let's denote $\ell_{t}=\left(b_{t}^{1}, \ldots, b_{t}^{n-1}, 1, b_{t}^{1} \omega_{t}^{1}, \ldots, b_{t}^{n-1} \omega_{t}^{n-1}, \omega_{t}^{n}\right)$. Let us further $b_{t}^{i}$ assume that all the parameter $b_{t+1}^{i}$ in the period $[t, t+1]$ are known and $x=\left(K^{1}, \ldots, K^{n}, L^{1}, \ldots, L^{n}\right)$ is the state of the economy at the moment $t$, which transitions to the state $y=$ ( $K^{1}, \ldots, K^{n}, L^{1}, \ldots, L^{n}$ ). Since, represent the means of production and the consumption $b_{t}^{i} \omega_{t}^{i}$ f und in the $i$-th production at the moment $t$ for the state $x=\left(K^{1}, \ldots, K^{n}, L^{1}, \ldots, L^{n}\right)$ (in value terms), then the values

$$
\begin{gathered}
\ell_{t}(x)=\sum_{i=1}^{n} b_{t}^{i}\left(K^{i}+\omega_{t+1}^{i} L^{i}\right) \text { и } \ell_{t+1}(y)=\psi_{t}(x)= \\
=\sum_{i=1}^{n} b_{t+1}^{i}\left(v_{i} K^{i}+F_{i}\left(K^{i}, L^{i}\right)\right)
\end{gathered}
$$

can be interpreted as national wealth at moments $t$ and $t+1$.
Theorem 2.1.1. For trajectories with the property $(A)$

$$
\alpha_{t}^{1}=\alpha_{t}^{2}=\cdots=\alpha_{t}^{n}=\alpha_{t} .
$$

The second paragraph of Chapter 2 is devoted to the study of effective trajectories of the model $Z(\omega)$. Let us consider the trajectory $\left(x_{t}\right)$ of the model $Z(\omega)$, admitting the characteristic $p_{t}=$ $p_{t}^{n} \ell_{t}$, where

$$
\ell_{t}=\left(b_{t}^{1}, \ldots, b_{t}^{n-1}, 1, b_{t}^{1} \omega_{t}^{1}, \ldots, b_{t}^{n-1} \omega_{t}^{n-1}, \omega_{t}^{n}\right)
$$

Theorem 2.2.2.1) Let the $x_{t}=A_{t} \bar{x}_{t}$ effective trajectory of the model $Z(\omega)$ emanate from a strictly positive point $x_{0}$. Then a) the sequence $x_{t}^{n}$ is an effective trajectory of a Neumann type model $Z^{n}\left(\omega^{n}\right)$, defined by the set $\left(F_{n}, v_{n}, \omega_{t}^{n}\right)$, b) if $\gamma_{t}^{i} \bar{K}_{t+1}^{i} \geq v_{i} \bar{K}_{t}^{i}$ , then $\left(x_{t}^{i}\right)$ is an effective trajectory of the model

$$
Z^{i}\left(\omega^{i}\right)=\left(\frac{b_{t+1}^{i}}{b_{t}^{i}} F_{i}, \frac{b_{t+1}^{i}}{b_{t}^{i}} v_{i}, \omega_{t}^{i}\right)(i=\overline{1, n})
$$

2) If $\left(x_{t}^{i}\right)$ is the effective trajectory of the model $Z^{i}\left(\omega^{i}\right)$ emanating from the point $x_{0}^{i}>0$, then $\left(x_{t}\right)$ is the effective trajectory of the model $Z(\omega)$.

In the third paragraph of Chapter II, it is assumed that the vector $L_{t}=\left(L_{t}^{1}, \ldots, L_{t}^{n}\right), L_{t}^{i} \geq 0, \sum_{i=1}^{n} L_{t}^{i}=1$ is given, where $L_{t}$ is the total number of the workforce. Based on the set $\omega=$ $\left(\omega_{t}^{1}, \ldots, \omega_{t}^{n}\right)$, a model $Z(\omega)$ is built under the assumption that $x_{t}=$ $\left(K_{t}^{1}, \ldots, K_{t}^{n}, L_{t}^{1}, \ldots, L_{t}^{n}\right)$ its state lies on the effective trajectory of the model, which has characteristic. Using a well-known formula, the wage rate $P_{t}$ is established and the state of the model $Z(\omega)$ is determined.

Let us introduce the simplest $2 n$ one-sector models, $Z^{i}\left(L^{i}\right), Z^{i}\left(\omega^{i}\right) \quad i=\overline{1, n}, \quad$ specified by the same set $\left(\frac{b_{t+1}^{i}}{b_{t}^{i}} F_{i}, \frac{b_{t+1}^{i}}{b_{t}^{i}} v_{i}, \omega_{t}^{i}\right)$, in which the coefficients $b_{t}^{i}, i=\overline{1, n}$, are determined by the characteristic prices $P_{t}$ of the model $Z(\omega)$ state $x_{t}$. In the model $Z^{i}\left(L^{i}\right), L_{t}^{i}$ is considered known, and in the
model $Z^{i}\left(\omega^{i}\right)$, specific consumption $\omega_{t}^{i}, i=\overline{1, n}$ is. The controlled parameter in is $\omega_{t}^{i}$. An optimality principle is proposed, according to which it is chosen so that a trajectory $\left(K_{t}, L_{t}\right)$ with a given labor force $L_{t}$ would be effective in the simplest model $(F, v, \omega)$, which is obtained for a given $\omega$.

Using the above optimality principle, it is possible, on the one hand, to determine the state of the model $Z(L)$, and on the other, the state of the effective trajectory of the mode $Z(\omega)$. Let us prove the last statement.

A sequence $\left(K_{t}^{i}, L_{t}^{i}\right)$ with a fixed labor force $L_{t}^{i}$ when selected as a function o $\omega_{t}^{i}$ is the efficient trajectory $L_{t}^{i}$ of the model $Z^{i}\left(\omega^{i}\right)$.

The fourth paragraph of Chapter II discusses one method of distributing labor. It is assumed that the total number of labor force is constant and equal to one. Let's consider the case when the wage rate in each of the industries is the same, i.e. $\omega_{t}^{1}=\omega_{t}^{2}=\cdots=\omega_{t}^{n}=\omega_{t}$.

Thus, it is required to find such a distribution of the total number of labor forces $L_{t+1}, t=1,2, \ldots$, into labor forces $L_{t+1}^{i}\left(\sum_{i=1}^{n} L_{t+1}^{i}=1\right)$, that

$$
\omega_{t+1}^{1}\left(L_{t+1}^{1}\right)=\omega_{t+1}^{2}\left(L_{t+1}^{2}\right)=\cdots=\omega_{t+1}^{n}\left(L_{t+1}^{n}\right)
$$

Theorem 2.4.1. The equation $L_{t+1}(\omega)=1$ has a solution if and only if $\lim _{\omega \rightarrow \bar{s}} L_{t+1}(\omega)<1$; This solution is the only one.

Consequence. If $F_{i}(i=\overline{1, n})$ the Cobb-Douglas function, then the conditions of Theorem 2.4.1 are always satisfied $F_{i}(i=$ $\overline{1, n}$ ).

Theorem 2.4.2. a) If the initial wage rate $\omega_{1}$ belongs to the interval $\left[\bar{\omega}^{1}, \bar{\omega}^{2}\right]$, then for all $\omega_{t} \in\left[\bar{\omega}^{1}, \bar{\omega}^{2}\right]$ and $t$, at the same time, for all $\omega_{t} \rightarrow \bar{\omega}^{2}, \beta_{t}^{2}>1, \beta_{t}^{1}<1, \lim L_{t}^{1}=1, \lim L_{t}^{2}=0$,
b) if $\omega_{1}<\bar{\omega}^{1}$, then $T$ there is a moment in time such that $\bar{\omega}^{1} \leq$ $\omega_{T} \leq \bar{\omega}^{2}$, and increase, $\left(0, \bar{\omega}^{1}\right]$,
c) $\omega_{1}>\bar{\omega}^{2}$ if $\omega_{t}>\bar{\omega}^{2}$ and for all $t$, then decreases, $\omega_{t}$ tending to $\bar{\omega}^{2}$, and $L^{1}=0$.

Theorem 2.4.3. a) If $\omega_{1} \in\left[\bar{\omega}^{1}, \bar{\omega}^{n}\right]$, then for all $\omega_{t} \in\left[\bar{\omega}^{1}, \bar{\omega}^{n}\right]$ and $t, \omega_{t} \rightarrow \bar{\omega}^{n}, L_{t}^{n} \rightarrow 1, L_{t}^{i} \rightarrow 0, i<n$,
b) If $\omega_{t}<\bar{\omega}^{1}$, then there $T$ is a moment in time, and $\bar{\omega}^{1}<$ $\omega_{T}>\bar{\omega}^{n}$ increases $\omega_{t}<\bar{\omega}^{1} T: \bar{\omega}^{1}<\omega_{T}<\bar{\omega}^{n}, \omega_{t}$ in $\left(0, \bar{\omega}^{1}\right]$,
c) If $\omega_{1}>\bar{\omega}^{n}$ and $\omega_{t}>\bar{\omega}^{n}$ for all $t$, then decreases, $\omega_{t}$ tending to $\bar{\omega}^{n}$.

In the second paragraph, the total labor force $L=1$ in the model $Z$ distributed between two industries so as to maximize total consumption, while the constant elasticity of substitution (CES) functions is considered as production functions:

$$
F_{i}\left(K_{t}^{i}, L_{t}^{i}\right)=\left(A_{i} K_{t}^{-\rho_{i}}+B_{i} L_{t}^{-\rho_{i}}\right)^{-\frac{1}{\rho_{i}}} \quad i=1,2
$$

where $\rho_{i}>0$. So, we consider the problem

$$
W_{t+1}^{1}\left(L_{t+1}^{1}\right)+W_{t+1}^{2}\left(L_{t+1}^{2}\right) \rightarrow \max
$$

given that $L_{t+1}^{1}+L_{t+1}^{2}=1$. Here $W_{t+1}^{i}\left(L_{t+1}^{i}\right)$ is the consumption fund in $i$ - production. $L_{t+1}^{i}$ can be expressed as a function of a variable $\eta_{t+1}^{i}$ :

$$
L_{t+1}^{i}\left(\eta_{t+1}^{i}\right)=\frac{M_{t}^{i}}{\Phi_{i}\left(\eta_{t+1}^{i}\right)}
$$

that's fair

$$
W_{t+1}^{i}\left(L_{t+1}^{i}\right)=M_{t}^{i} \frac{f_{i}\left(\eta_{t+1}^{i}\right)-\eta_{t+1}^{i} f_{i}^{\prime}\left(\eta_{t+1}^{i}\right)}{v_{i} \eta_{t+1}^{i}+f_{i}\left(\eta_{t+1}^{i}\right)}
$$

Lemma 2.5.1. Let the following conditions be satisfied
a) $\tilde{\eta}^{2}>\bar{\eta}^{2}$,
b) $L_{\tau}^{2} \leq \tilde{L}_{\tau}^{2}$ for some point in time $\tau$. Then $\tilde{L}_{\tau+1}^{2} \leq \tilde{L}_{\tau}^{2}$

Comment. It is easy to check that the condition is satisfied if $\tilde{\eta}>\bar{\eta}$

$$
A \in\left[\begin{array}{cc}
\frac{1}{\left((1-v)(1+\rho)^{1+\frac{1}{\rho}}\right)^{\rho}}, & \frac{1}{(1-v)^{\rho}}
\end{array}\right]
$$

Theorem 2.5.1. Let the following conditions be satisfied
a) $\quad \eta_{\tau-1}^{1}<\bar{\eta}, \tilde{\eta}^{2}>\eta^{2}$,
b) $\quad \bar{L}_{\tau-1}^{2} \geq \bar{L}_{\tau}^{2}$ and $M_{\tau-1} \geq \Phi_{1}(\bar{\eta})$.

Then $\bar{L}_{t}^{2} \leq \tilde{L}_{t}^{2}$ for all $t>\tau$, and $\tilde{L}_{t}^{2}$ decreases.
In the sixth paragraph this chapter examines a model $\tilde{Z}$ consisting of the simplest $n$ single-product models of economic dynamics and provides a description of the model $\tilde{Z}=\left(F_{i}, v_{i}, \omega_{t}^{i}\right)$. So we denote single-product models and assume that the wage rate in all models is the same at all points in time

$$
\omega_{t}^{1}=\cdots=\omega_{t}^{n}=\omega, \quad t=1,2, \ldots
$$

In the same section, the asymptotic properties of the model $\widetilde{Z}$ trajectories are studied. Through

$$
\begin{equation*}
\alpha^{i}=\max _{K, L \geq 0} \frac{v_{i} K+F_{i}(K, L)}{K+\omega L}=\max _{\eta>0} \frac{v_{i} \eta+f_{i}(\eta)}{\eta+\omega} \tag{2}
\end{equation*}
$$

let us denote the growth rates of the models $\tilde{Z},{ }^{i} i=\overline{1, n}$.
From the properties of the trajectories of a model with a strict equilibrium state it follows

Theorem 2.6.1. For any trajectory there is $\left(x_{t}\right)$

$$
\lim \frac{K_{t}^{1}+\cdots+K_{t}^{n}+\omega L_{t}}{\alpha^{t}}=\lambda>0
$$

If $\lambda>0$, then $\frac{K_{t}^{n}}{L_{t}} \rightarrow \bar{\eta}, \frac{K_{t}^{i}}{\alpha^{t}} \rightarrow 0, i=\overline{1, n-1}$.
Let the function $\hat{g}_{i}$ be defined by the equality $\hat{g}_{i}(\eta)=v_{i} \eta+$ $f_{i}(\eta)(i=\overline{1, n})$. Then it is fair.

Theorem 2.6.2. Let the equalities be satisfied for the sequence $\eta_{t}^{n}$ and $\bar{\eta}$ is the capital-labor ratio at which the maximum in (2) is achieved at. Then

1) $\eta_{t}^{n}$ decreases,
2) $\quad \eta_{t}^{n} \rightarrow \bar{\eta}$, if and for everyone, $\eta_{0}^{n}>\bar{\eta}, \xi_{t}^{n} \leq$ $\min \left(\frac{\hat{g}_{n}\left(\eta_{t}^{n}\right)}{\hat{g}_{n}(\bar{\eta})}, \frac{f_{n}\left(\eta_{t}^{n}\right)}{\omega}\right)$ for all $t$, if $\quad \eta_{0}^{n} \rightarrow \bar{\eta}, \quad$ or $\quad \xi_{t}^{n}>\min \left(\frac{\hat{g}_{n}\left(\eta_{t}^{n}\right)}{\hat{g}_{n}(\bar{\eta})}\right.$, $\left.\frac{f_{n}\left(\eta_{t}^{n}\right)}{\omega}\right)$ at least one $t$, then the sequence becomes $\eta_{t}^{n}$, starting from a certain point, negative.

Let us assume that $\alpha^{n}>\alpha^{i}, i \neq n$ and denote $\alpha=\alpha^{n}, x=$ $(0, \ldots, 0, \bar{K}, \bar{L})$, where, $\bar{K}=\bar{K}^{n}, \bar{L}=\bar{L}^{n}$ and $\bar{K}, \bar{L}$, is selected from the conditions $\bar{K}+\omega \bar{L}=1, \alpha=v_{n} \bar{K}+F_{n}(\bar{K}, \bar{L})$. The capital-labor ratio $\bar{\eta}=\frac{\bar{K}}{\bar{L}}$ at which the maximum in (2) is achieved at $i=\bar{n}$ is called optimal. Since $\alpha=\alpha^{n}=\gamma^{n} \bar{\eta}$, then the optimality of the capitallabor ratio $\alpha=\alpha^{n}=\gamma^{n}$ is equivalent to the equality

$$
\begin{equation*}
\alpha=v_{n}+f_{n}(\bar{\eta}) \tag{3}
\end{equation*}
$$

The seventh paragraph is devoted to the study of the properties of trajectories of a model $\tilde{Z}$ of the form $\left(K_{t}^{1}, \ldots, K_{t}^{n}, 1\right), L_{t}^{1}+\cdots+$ $L_{t}^{n}=1$.

Let $n=2$. Let us define a set $U$ as a collection of pairs $\left(K^{1}, K^{2}\right)$ such that a trajectory of the form $\left(K_{t}^{1}, K_{t}^{2}, 1\right), L_{t}^{1}+L_{t}^{2}=1$ emanates from a point $\left(K^{1}, K^{2}, 1\right)$. Let us note some properties of the set $U$.

1. $U$ stable (in the sense $R_{+}^{2}$ of); if and $\left(K^{1}, K^{2}\right) \geq$ $\left(\widetilde{K}^{1}, \widetilde{K}^{2}\right),\left(\widetilde{K}^{1}, \widetilde{K}^{2}\right) \in U$, then $\left(K^{1}, K^{2}\right) \in U$.
2. $U$ the set budged.
3. The point where and is such that $(0, \lambda \bar{\eta}) \in U, \lambda>$ 1, $\alpha(\omega(\bar{\eta}))=1$.

Let

$$
\begin{equation*}
\left(\widetilde{K}^{1}, \bar{\eta}, 1\right) \in a\left(K^{1}, \bar{\eta}-\varepsilon, 1\right) ; \quad 0<\varepsilon \leq \bar{\eta} \tag{4}
\end{equation*}
$$

where $1=L^{1}+L^{2}=\tilde{L}^{1}+\tilde{L}^{2}$.

Let's $\varepsilon>0$ fix it and let

$$
\begin{equation*}
\mathcal{L}_{\varepsilon}\left(L^{1}\right)=-\left(1-v_{2}\right) \bar{\eta}-v_{2} \varepsilon+F_{2}\left(\bar{\eta}-\varepsilon, 1-L^{1}\right) \tag{5}
\end{equation*}
$$

It is easy to notice that the function $\mathcal{L}_{\varepsilon}\left(L^{1}\right)$ is decreasing and concave. Besides, $\mathcal{L}_{\varepsilon}\left(L^{1}\right) \geq \omega$.

Consider the equality

$$
\begin{equation*}
F_{2}\left(\bar{\eta}-\varepsilon, 1-L^{1}\right)-v_{2} \varepsilon=\left(1-v_{2}\right) \bar{\eta} \tag{6}
\end{equation*}
$$

It is easy to check that (6) has a solution if and only if

$$
F_{2}(\bar{\eta}-\varepsilon, 1)-v_{2} \underset{\sim}{\varepsilon}>\left(1-v_{2}\right) \bar{\eta}
$$

Since and, then $\tilde{L}^{1}=1-\tilde{L}^{2} \omega \tilde{L}^{2}=\mathcal{L}_{\varepsilon}\left(L^{1}\right)$,

$$
\begin{equation*}
F_{1}\left(K^{1}, L^{1}\right)>\omega-\mathcal{L}_{\varepsilon}\left(L^{1}\right)=\mathcal{L}_{\varepsilon}\left(L^{1}\right) \tag{7}
\end{equation*}
$$

Theorem 2.7.1. (7) has a solution for some if and only if

$$
\begin{equation*}
F_{1}^{\prime}\left(K^{1}, 0\right) \geq F_{1}^{\prime}(\bar{\eta}, 1) \tag{8}
\end{equation*}
$$

Note 1. For the Cobb-Douglas production function, condition (8) is always satisfied, since $F_{1}^{\prime}\left(K^{1}, 0\right)=+\infty$.

Note 2. For the CES production function, relation (8) is equivalent to the inequality

$$
B_{1}^{-\frac{1}{\rho_{1}}} \geq B_{2}\left(A_{2} \bar{\eta}^{-\rho_{2}}+B_{2}\right)^{-\frac{1+\rho_{2}}{\rho_{2}}}
$$

There is great interest in Neumann-type models of economic dynamics causes asymptotic behavior of trajectories of various classes. In the eighth section, the asymptotic behavior of trajectories with an average growth rate $\alpha$ is studied. These trajectories are of both independent and applied interest: in some cases, they make it possible to describe the asymptotic behavior of optimal trajectories. Let us give its description.

Let $Z$ - a convex cone lying in $R_{+}^{n} \times R_{+}^{n}$ and such that $P_{r} Z \cap$ $\operatorname{int} R_{+}^{n} \neq \emptyset$. Let us call the Neumann growth rate of the cone $Z$ the number

$$
\alpha=\sup _{(x, y) \in Z} \min _{i \in I} \frac{y^{i}}{x^{i}}
$$

where $I=\{1,2, \ldots, n\}$.
We call a sequence $\left(x_{k}, y_{k}\right)$ of elements of a cone $Z$ Neumannian

$$
\left(x_{k}, y_{k}\right)=\min _{i \in I} \frac{y^{i}}{x^{i}}
$$

Let us introduce a set of indices. $I_{Z} \subset I$ into consideration. Number $i \in I_{Z}$ if and only if $\left(x_{k}, y_{k}\right)$ there is a Neumann sequence such that $y_{k}^{i}>0(k=1,2, \ldots)$.

Let $Z$ be the Neumann-Gale model. A cone generates a finite sequence $Z_{1}, \ldots, Z_{2}, Z_{n}$ of cones as follows. Let's $Z_{1}=Z$ put, let's denote $R_{+}^{n}=K_{1}$. Thus, $Z_{1} \subset K_{1} \times K_{1}$. If $I^{1}=I_{Z_{1}}=I$, then the process is over; if $I^{1} \neq I$, then consider the face $K_{2}$ of the cone $R_{+}^{n}$ spanned by unit vectors with numbers from $I \backslash I^{1}$, and define $Z_{2}$ it as the projection of the cone onto the face $K_{2} \times K_{2}$ of the cone $R_{+}^{n} \times$ $R_{+}^{n}$.

If $I^{2}=I_{Z_{2}}=I \backslash I^{1}$, then the process is over; otherwise, consider the face $K_{3}$ of the cone spanned by unit vectors with numbers from, and denote $Z_{3}$ by the projection $Z_{2}$ onto the face $K_{3} \times K_{3}$ of the cone $R_{+}^{n} \times R_{+}^{n}$. If $I^{3} \equiv I_{Z_{3}}=I \backslash\left(I^{1} \cup I^{2}\right)$, then we build a cone $Z_{4}$, etc. This process will end at some step $N$.

As a result, we have constructed cones $Z_{j}$ and sets of indices $I^{j}(j=1,2, \ldots, N), I^{j} \cap I^{j_{1}}=\emptyset, I^{j} \equiv I_{Z_{j}}\left(j \neq j_{1}\right)$ and Let us denote by the Neumann growth rate $\alpha_{j}$ of the cone $Z_{j}$. We will call the number the quasi-growth rate $\alpha_{j}$ of the model. It is known that $\alpha_{j-1}>\alpha_{j}$.

The highway is the conical shell of the set $M_{\alpha}$ of all points $\alpha$ of all trajectories $x y$ with an average tempo that are limiting in angular distance. The angular distance between points is called the quantity $\left\|\frac{x}{\|x\|}, \frac{y}{\|y\|}\right\|$.

Let us denote by the conical hull the sets $A_{z}$ of all points of all optimal trajectories that are limiting in angular distance.

The Neumann-Gale model $Z \subset R_{+}^{n} \times R_{+}^{n}$, which has quasitempos, and its dual model $Z^{\prime}$ are considered.

Lemma 2.8.1. For any quasi-tempo $\alpha_{j}$, any number $\lambda>\alpha_{j}$, any index $i \in \cup_{j}^{N} I_{\mu}$ and any trajectory $X=\left(x_{t}\right)$ there is a limit

$$
\lim \lambda^{-t} x_{t}^{i}=0
$$

Theorem 2.8.2.Let be the Neumann-Gale model $Z$ and there be a trajectory $X$ and an infinite set of time moments $\tau$ such that the sets of indices $I=\{1,2, \ldots, n\}$ can be divided into three subsets $J_{1}, J_{2}, J_{3}$ as follows

$$
\begin{gather*}
x_{t}^{i}=0 \text { for anyone } i \in J_{1}, t \in \tau  \tag{9}\\
0<c_{1} \leq x_{t}^{i} \leq c_{2}<\infty \text { for anyone } i \in J_{2}, t \in \tau  \tag{10}\\
\lim x_{t}^{i}=+\infty \text { for anyone } i \in J_{3} . \tag{11}
\end{gather*}
$$

Then for any point satisfying the conditions $x \in R_{+}^{n}$

$$
\begin{gathered}
x^{i}=0 \text { for anyone } i \in J_{1} \cup J_{2}, \\
x^{i}>0 \text { for anyone } i \in J_{3} .
\end{gathered}
$$

there $X_{1}$ is a trajectory that satisfies conditions (9)-(11) and has among its points limiting the angular distance the point $x$.

The ninth paragraph of the second chapter is devoted to the study of the main properties of the trajectories of models of economic dynamics. Let the simulated economy consist of $n$ industries, each producing different products. Phase space of the model - cone $\left(R_{+}^{n}\right)^{n}$. Vector $x=\left(x^{\prime}, \ldots, x^{n}\right) \in\left(R_{+}^{n}\right)^{n}$ is the state of the model, vector $x^{i}=\left(x^{i j}\right) \in R_{+}^{n}$ - vector of resources at disposal $i$ oh industry. Production activities $i$ - th industry is described using a superlinear continuous production function $\Phi^{i}: R_{+}^{n} \rightarrow R_{+}$and diagonal matrix $A^{i}$, on the diagonal of which there are numbers $v^{i j} \in$ $[0,1](j=\overline{1, n})$. It is assumed that $\Phi^{i}(x)>0$ then and only when $x \in K$, where $K=\left\{y \in R^{n}: y_{i}>0(i=\overline{1, n})\right\}$. The production display is defined as follows:

$$
\begin{aligned}
& a(x)=\left\{y=\left(y^{i}\right) \in\left(R_{+}^{n}\right)^{n} \mid 0 \leq y^{i} \leq A^{i} x^{i}+d^{i}\right. \\
& \left.d^{i}=\left(d^{i j}\right) \geq 0, \sum_{i=1}^{n} d^{i j} \leq \Phi^{j}\left(x^{j}\right), j=\overline{1, n}\right\}
\end{aligned}
$$

where $x=\left(x^{\prime}, \ldots, x^{n}\right) \in\left(R_{+}^{n}\right)^{n}$.
Consider the superlinear normal mapping $b: R_{+}^{n} \rightarrow \pi\left(R_{+}^{n}\right)$, whose Neumann growth rate is equal to $\alpha$, and there is a state of
equilibrium $(\alpha,(x, \alpha x), p)$ such that $p \in K$. In the future we will need

Lemma 2.9.1. Let $\left(x_{t}\right)$ - display trajectory $b$ with an average growth rate $\alpha ; \Omega$ (respectively $\Omega_{1}$ ) - set of limit points of the sequence $\left\{\alpha^{-t} x_{t}\right\}$ (respectively $\left\{\alpha^{-t}\left(x_{t}, x_{t+1}\right)\right\}$ ). Then for anyone $y_{0} \in \Omega$ (respectively $\left.\left(y_{0}, y_{1}\right) \in \Omega_{1}\right)$ there is a sequence $\left\{y_{t}\right\}(t=$ $0, \pm 1, \ldots)$ such that $y_{t} \in \alpha^{t} \Omega, y_{t+1} \in b\left(y_{t}\right)$.

Lemma 2.9.4. For anyone $x \in R^{n}$, integer $m \geq 0$ performed

$$
\begin{aligned}
& \left\|B^{m} x-\left(r^{i}(p)\right)^{-1} \alpha^{m} x^{i} r(p)\right\| \leq \\
& \leq\left\|A^{i}\right\|^{m}\left\|x-\left(r^{i}(p)\right)^{-1} x^{i} r(p)\right\| .
\end{aligned}
$$

Theorem 2.9.2. Display trunk $a^{\prime}$ is the beam $\{\lambda p \mid \lambda \geq 0\}$.
The third Chapter consists of five paragraphs and is devoted to the study of the consumption function in single-product models of economic dynamics. The first paragraph of this chapter describes the model. The model is given by the relations

$$
0 \leq K \leq v \bar{K}+I, \quad I \geq 0, \quad I+\omega L \leq F(\bar{K}, \bar{L})
$$

and it is assumed that $F_{t}=F, v_{t}=v$ and a constant $s$ rate of accumulation is given, independent of the size of the labor force. Let, $L=\beta \cdot \bar{L}$, where $\beta$ is the growth rate of the labor force. Then

$$
W(L)=(1-s) F\left(\bar{K}, \frac{1}{\beta} L\right)
$$

and specific consumption $\omega$ is expressed through the accumulation rate $s$ by the equality

$$
\begin{gather*}
\omega=\frac{W(L)}{L}=(1-s) F\left(\frac{\bar{K}}{\bar{L}} \cdot \frac{\tilde{L}}{L}, \frac{1}{\beta}\right)=(1-s) F(\bar{\eta}, 1) \frac{1}{\beta}= \\
=(1-s) \frac{1}{\beta} f(\bar{\eta}) \tag{12}
\end{gather*}
$$

The second paragraph of the third Chapter is devoted to the study of the dependence of the consumption function then $W(L)$ on the type of production function $F_{t}=F$. Since for everyone, then

$$
\begin{equation*}
\omega=\frac{f(\eta)-\eta f^{\prime}(\eta)}{v+f^{\prime}(\eta)} \tag{13}
\end{equation*}
$$

where is the root of the equation $\eta=\eta(L)$

$$
\begin{equation*}
\eta=\frac{M}{L}-\frac{f(\eta)-\eta f^{\prime}(\eta)}{v+f^{\prime}(\eta)} \tag{14}
\end{equation*}
$$

Here $M$ the quantity denoted by is national wealth $v \bar{K}+$ $F(\bar{K}, \bar{L})$ at the moment $t$. We assume that the function $f$ is three times continuously differentiable, and $f^{\prime}(\eta)>0 f^{\prime \prime}(\eta)<0 \eta>0$ for $\eta>0$

Theorem 3.2.1. Let $F(K, L)=A K^{r} L^{1-r}, 0<r<1$ be the Cobb-Douglas function and $\omega$ the specific consumption is calculated using formulas (12) and (13). Then $W$ is an increasing concave function, and $\lim _{L \rightarrow+\infty} W(L) \rightarrow+\infty$.

Let now be a function (CES). Further. Let us introduce the designation. Then for we get

$$
F(K, L)=\left(A K^{-\rho}+B L^{-\rho}\right)^{-\frac{1}{\rho}} \rho>1, Y=A+B \eta^{\rho}
$$

It is assumed that,

$$
W^{\prime}(L)=\mu\left(v B \eta^{\rho}-v A \rho-A \rho Y^{-\frac{1}{\rho}}\right)
$$

where $\rho>0$.
That's why

$$
\operatorname{Sign} W^{\prime}(L)=\operatorname{Sign}\left(v B \eta^{\rho}-v A \rho-A \rho Y^{-\frac{1}{\rho}}\right)
$$

Lemma 3.2.1. The equation $g(\eta)=v B \eta^{2}-v A \rho-$ $A \rho\left(A+B \eta^{\rho}\right)^{-\frac{1}{\rho}}=0$ has a single root $\bar{\eta}_{1}$ on the positive semi-axis, and at $\eta>\bar{\eta}_{1} g(\eta)>0$ and at $\eta<\bar{\eta}_{1} g(\eta)<0$.

Theorem 3.2.2. Let $F(K, L)=\left(A K^{-\rho}+B L^{-\rho}\right)^{-\frac{1}{\rho}}$, where $\rho>$ 1 and specific consumption $\omega$ is calculated using formulas (12), (13). Then:

1) the function $W(L)$ has a single inflection point $\bar{L}_{2}$ at which it changes concavity to convexity, while $\bar{L}_{2}>\bar{L}_{1}$;
2) $\lim _{L \rightarrow \infty} W(L)=0$.

The third paragraph of the third Chapter is devoted to the study of the dependence of the volume of consumption on the means of production. Let $\bar{L}$ and $L$ and $\omega$ be given and calculated using formulas (12), (13). It is shown that in this case, consumption $\omega$ at the moment $t+1$ will depend on the volume of funds $M=v \bar{K}+$ $F(\bar{K}, \bar{L})$. Since $\bar{L}$, and $L$ are fixed, consumption depends on $\bar{K}$, From equation (14) $M$ can be expressed as a function:

$$
M(\eta)=L \frac{u(\eta)}{\beta(\eta)}
$$

Note that $M$ the increasing function, and

$$
\lim _{\eta \rightarrow+0} M(\eta)=0, \quad \lim _{\eta \rightarrow+\infty} M(\eta)=\infty .
$$

We have

$$
W(M)=L \cdot \omega(M)=L \frac{\delta(\eta(M))}{\beta(\eta(M))} .
$$

Let's assume that $F(K, L)=\left(A K^{-\rho}+B_{i} L^{-\rho}\right)^{-\frac{1}{\rho}}$ the CES function. Let us introduce the following notation

$$
\begin{gathered}
d(\eta)=d_{1}(\eta)-d_{2}(\eta) \\
d_{1}(\eta)=\left(-v B \eta^{\rho}+v A \rho+A \rho Y^{-\frac{1}{\rho}}\right)\left(1+v A Y^{\frac{1}{\rho}-1}\right) \\
d_{2}(\rho)=v(1+\rho) B^{2} \eta^{2 \rho}\left(1+v Y^{\frac{1}{\rho}}\right) Y^{-1}
\end{gathered}
$$

Obviously. $W^{\prime \prime}=0$, if and only if $d_{1}(\eta)=d_{2}(\eta)$. You can easily calculate and.

Similar to theorem 3.2.2, it can be shown that there is a unique point $\bar{M}=M\left(\bar{\eta}_{2}\right)$ where $\bar{\eta}_{2}$ is the root of the equation $d(\eta)=0$ such that $W^{\prime \prime}(M)=0$.

From the results obtained it follows that

$$
\lim _{M \rightarrow+\infty} W(M)=L \lim _{\eta \rightarrow+\infty} \frac{\delta(\eta)}{v}=\frac{L}{v} \lim _{\eta \rightarrow+\infty} \delta(\eta)
$$

It is easy to check that in the case when that $F(K, L)=$ $A K^{r} L^{1-r}$ is the Cobb-Douglas function, $\lim _{\eta \rightarrow+\infty} \delta(\eta)=+\infty$. If $F(K, L)$ is a function with constant elasticity of comment (CES), then
$\lim _{\eta \rightarrow+\infty} \delta(\eta)=B^{-\frac{1}{\rho}}$. Indeed, let $F$ be the Cobb-Douglas function. Since in this case $F f(\eta)=A \eta^{\rho}$, it follows from (3.2.3) that

$$
\lim _{\eta \rightarrow+\infty} \delta(\eta)=\lim _{\eta \rightarrow+\infty}\left(A \eta^{\rho}-\eta A \rho \eta^{\rho-1}\right)=\lim _{\eta \rightarrow+\infty} A \eta^{\rho}(1-\rho)=+\infty
$$

For the CES function we have:

$$
\begin{gathered}
\lim _{\eta \rightarrow+\infty} \delta(\eta)=B \lim _{\eta \rightarrow+\infty} \frac{\left(\eta^{\rho}\right)^{1+\frac{1}{\rho}}}{\left(A+B \eta^{\rho}\right)^{1+\frac{1}{\rho}}}= \\
\quad=B \lim _{\eta \rightarrow+\infty} \frac{1}{\left(\frac{A}{\eta^{\rho}}+B\right)^{1+\frac{1}{\rho}}}=B^{-\frac{1}{\rho}}
\end{gathered}
$$

Thus, for the Cobb-Douglas function, consumption and $W(M)$ is an increasing concave function, and in the case $\lim _{M \rightarrow+\infty} W(M)=$ $+\infty$ where $F$ is a function with constant elasticity of substitution, consumption $W(M)$ is an increasing function that has a single inflection point $\bar{M}$, at which it changes convexity to concavity, and $\lim _{M \rightarrow+\infty} W(M)<+\infty$.

In the fourth paragraph of the third Chapter, several indicators are maximized, including total consumption, total output and total national wealth.

$$
\begin{align*}
& \sum_{i=1}^{n} W_{i}\left(\ell_{i}\right) \rightarrow \max  \tag{15}\\
& \sum_{i=1}^{n} F_{i}\left(K^{i}, \ell_{i}\right) \rightarrow \max  \tag{16}\\
& \sum_{i=1}^{n} v_{i} K^{i}+F_{i}\left(K^{i}, \ell_{i}\right) \rightarrow \max \tag{17}
\end{align*}
$$

given that, $0 \leq \ell_{i} \leq L \sum_{i=1}^{n} \ell_{i}=L$. Here $W_{i}\left(\ell_{i}\right)$ is the consumption fund, $F_{i}$ is output, $F_{i}\left(K^{i}, \ell_{i}\right)+v_{i} K^{i}$ is national wealth in the th
model under the assumption that specific consumption is selected according to formulas (12), (13).

Theorem 3.4.1. Let $F_{i} i=\overline{1, n}$ be the Cobb-Douglas function. Then in problems (15)-(17) the vector $\bar{\ell}_{i}$ belongs to the interior of the cone $R_{+}^{n}(\ell \geq 0)$.

Theorem 3.4.2. Consider the models $\left(F_{1}, v_{1}\right)$ and $\left(F_{2}, v_{2}\right)$, where $F_{i}$ are the CES functions

$$
F_{i}(K, L)=\left(A_{i} K^{-\rho_{i}}+B_{i} L^{-\rho_{i}}\right)^{-\frac{1}{\rho_{i}}}, \quad(i=1,2)
$$

and $\rho_{i}>0$. Then total consumption $W_{i}\left(\ell_{i}\right)$ reaches its maximum on the segment $[0, L]$ at the point $\ell_{j}=L$ if and only if $L \leq \bar{L}_{j}, W_{j}^{\prime}(L) \geq$ $\frac{1}{v_{i}} B_{i}^{-\frac{1}{\rho_{i}}}>L$ and total output reaches its maximum at the point $L$ if and only if $F_{j}^{\prime}(L) \geq B_{i}^{-\frac{1}{\rho_{i}}}, i \neq j$. Here $\bar{L}_{j}$ is the only point of maximum of the function $W_{j}$ on the positive semi-axis.

Theorem 3.4.3. For any production function $F$ in problem (17) $\bar{\ell}_{i}<1$.

In the fifth paragraph of the third Chapter, the question of the dependence of the volume of funds on the size of the labor force at a constant rate of accumulation is studied within the framework of the simplest single-product model.

Let us assume that in (1) equality is realized and $s$ constant rate of accumulation is given, independent of the size of the labor force, i.e. $I_{t+1}=s F\left(K_{t}, L_{t}\right)$. Then

$$
\begin{equation*}
K_{t+1}=v K_{t}+s F\left(K_{t}, L_{t}\right) \tag{18}
\end{equation*}
$$

Let's introduce the function

$$
\bar{g}\left(K_{t}\right)=v K_{t}+s F\left(K_{t}, L_{t}\right)
$$

Then

$$
K_{t+1}=\bar{g}\left(K_{t}\right)
$$

Let us first consider the case when for all. $\operatorname{Let} L_{t}=\operatorname{Lt} \bar{g}(K)=$ $v K+s F(K, L)$ where

$$
\begin{equation*}
K-\bar{g}(K)=K\left(1-v-s F\left(1, \frac{L}{K}\right)\right), \quad K \neq 0 \tag{19}
\end{equation*}
$$

Due to concavity $\bar{g}$, the equation

$$
\begin{equation*}
K=\bar{g}(K) \tag{20}
\end{equation*}
$$

has at most two roots, one of them being the point $[0,+\infty)$. Let us assume that there $\bar{K}_{0}=0$ is a solution to equation (20) that is different from zero. Then from (19) it follows that the relations

$$
\begin{equation*}
K>\bar{K}(K<\bar{K}), \quad \bar{g}(K)<K(\bar{g}(K)>K) \tag{21}
\end{equation*}
$$

are equivalent.
Proposition 3.5.1. Let $\tilde{g}$ some function be defined on $(0,+\infty)$, increase and the equation $K=\tilde{g}(K)$ have a unique solution $\bar{K}$, and relations (21) are equivalent. Let, further, the sequence $K_{t}$ satisfy the equalities $K_{t+1}=\tilde{g}\left(K_{t}\right)(t=0,1, \ldots)$ and $K_{0}>0$. Then $K_{t} \rightarrow \bar{K}$, and if $K_{0}>0$, then $K_{t}$ increases, if $K_{0}>\bar{K}$, then $K_{t}$ decreases.

Theorem 3.5.1. If $\bar{K}_{0}=0$ the only root of the equation $K=$ $\tilde{g}(K)$ on $[0,+\infty)$ and for the sequence $K_{t}$ satisfies the equality, $K_{t+1}=\tilde{g}\left(K_{t}\right)$, then $K_{t}$ decreases and $K_{t} \rightarrow 0$.

Theorem 3.5.2. Let the sequence $K_{t}$ be constructed using formula (18) for a certain initial volume of funds $K_{0}$, and the equation $K=\tilde{g}_{L_{i}}(K)$ has a positive solution starting from $i=\tau$. Then $K_{t} \rightarrow \bar{K}_{L^{\prime}}$, and if $K_{i}>\bar{K}_{L_{i}}$ for all $i \geq \tau$, then $K_{t}$ decreases, but if exists $m \geq \tau: K_{m}<\bar{K}_{L_{m}}$, then the sequence $K_{t}$ decreases until the moment $m$, after which it increases

Corollary 1. If the equation $K=\tilde{g}_{L^{\prime}}(K)$ does not have a positive solution, then the sequence $K_{t}$ constructed using (18) decreases and $K_{t} \rightarrow 0$

Corollary 2. Let $\tau=m=0$ in theorem 3.5.2. Then $K_{t} \rightarrow \bar{K}$, and 1) if $K_{t}<\bar{K}_{L_{0}}\left(\right.$ i.e. $m=0$ ), then $K_{t}$ increases; 2) if $K_{0}>\bar{K}_{L_{0}}$ and $m \neq 0$ a), then $K_{t}$ decreases until $m$, after which it increases; b) $m$ does not exist, then $K_{t}$ decreases.

Theorem 3.5.3. Let $K_{t}$, under the assumption made above, the sequence be constructed using formula (18) for a certain volume of funds $K_{0}$, and the equation $K=\bar{g}_{L_{i}}(K)$ has a positive solution for all $i=0,1$

Then $K_{t} \rightarrow \bar{K}_{L^{\prime \prime}}$, and if $K_{0}>\bar{K}_{L_{0}}$, then $K_{t}$ decreases, if $K_{0}<$ $\bar{K}_{L_{0}}$ and a) $K_{i}<\bar{K}_{L_{i}}$ for any $i$, then $K_{t}$ increases; b) there $\tau$ : $K_{L_{\tau}}>$ $\bar{K}_{L_{\tau}}$ is a moment, then $K_{t}$ it increases to, after which it decreases

Let us denote by the $\bar{K}^{1}$ root of the equation $K^{1}=v K^{1}+$ $F\left(K^{1}, L^{\prime}\right)$, where $K^{1}=\bar{K}^{1}$ do we have

$$
\frac{1-v}{s}=\frac{F_{1}\left(\bar{K}^{1}, L^{\prime}\right)}{\bar{K}^{1}}=F_{1}\left(1, \frac{L^{\prime}}{\bar{K}^{1}}\right)=Q_{1}\left(\xi_{1}\right)
$$

where $\xi_{1}=\frac{L^{\prime}}{\bar{K}^{1}}$ and therefore $Q_{1}^{-1}\left(\frac{1-v}{s}\right)=\xi_{1}$. Therefore $\xi_{1} \bar{K}^{1}=\tilde{\theta}_{1}$ where

$$
\tilde{\theta}_{1}=\frac{1}{Q_{1}^{-1}\left(\frac{1-v}{s}\right)}
$$

Similarly, $\bar{K}^{2}=\tilde{\theta}_{2}\left(L-L^{\prime}\right)$, where $\bar{K}^{2}$ is the root of the equation

$$
\begin{gathered}
K^{2}=v K^{2}+s F_{2}\left(K^{2}, L-L^{\prime}\right), \\
\tilde{\theta}_{2}=\frac{1}{Q_{2}^{-1}\left(\frac{1-v}{s}\right)} \\
Q_{2}\left(\xi_{2}\right)=F_{2}\left(1, \frac{L-L^{\prime}}{\bar{K}^{2}}\right),
\end{gathered} \quad \xi_{2}=\frac{L-L^{\prime}}{\bar{K}^{2}} .
$$

Theorem 3.5.4. Let $\bar{K}^{1}, \bar{K}^{2}$ it be positive. Then the maximum in the problem is achieved at one of the ends.

Chapter 1V consists of six paragraphs. Since in the future reproduction models will mainly be studied, the first paragraph provides a description of the reproduction model with Leontief-type production functions. These models were introduced by A.M. Rubinov ${ }^{2}$.

The simulated economy consists of $n$ industries, each producing one product, with different industries producing different

[^0]products. Phase space of the model - cone $\left(R_{+}^{n}\right)^{n}$. The vector $X=$ $\left(x^{1}, \ldots, x^{n}\right) \in\left(R_{+}^{n}\right)^{n}$ is called the state of the model, $x^{k \cdot}=$ $\left(x^{k 1}, \ldots, x^{k n}\right) \in R_{+}^{n}$ - the state of the $k-$ th industry. The production activity of the $k$ - th industry during the period $[t, t+1]$ is described using the production function $F_{t}^{k}: R_{+}^{n} \rightarrow R_{+}$and the diagonal safety matrix $B_{t}^{k}$, on the diagonal of which there are numbers $v_{t}^{k i} \in$ $[0,1](i=\overline{1, n})$.

It is assumed that

$$
F_{t}^{k}(x)=\min _{i=1, n} \frac{x^{i}}{c_{t}^{i k}}\left(x \geq 0, \quad c_{t}^{i j} \geq 0, \quad i, j=\overline{1, n}, \quad t=0,1, \ldots\right)
$$

Thus, if the $k$-th industry at the moment $t$ has a vector of resources $x$, then at the moment $t+1$ after the production process it will have a vector of resources $B_{t}^{k} \cdot x$ and a newly produced product $F_{t}^{k}(x)$ in quantity.

Let us introduce a superlinear operator $(B \cdot F)_{t}:\left(R_{+}^{n}\right)^{n} \rightarrow R_{+}^{n}$ and $(B \cdot F)_{t}^{k}: R_{+}^{n} \rightarrow R_{+}^{n}$

$$
\begin{gathered}
(B F)_{t}^{k}(x)=B_{t}^{k} \cdot x+\left(0, \ldots, 0, F_{t}^{k}(x), 0, \ldots, 0\right)\left(x \in R_{+}^{n}\right) \\
(B F)_{t}(X)=\sum_{k=1}^{n}(B F)_{t}^{k}\left(x^{k \cdot}\right)=\sum_{k=1}^{n} B_{t}^{k} \cdot x^{k \cdot}+\left(F_{t}^{1}\left(x^{1 \cdot}\right), \ldots, F_{t}^{n}\left(x^{n \cdot}\right)\right) \\
\left(X=\left(x^{1 \cdot}, \ldots, x^{n \cdot}\right) \in\left(R_{+}^{n}\right)^{n}\right)
\end{gathered}
$$

Model with production display

$$
\begin{equation*}
a_{t}(X)=\left(0,(B F)_{t}(X)\right) \quad\left(x \in\left(R_{+}^{n}\right)^{n}\right) \tag{23}
\end{equation*}
$$

denote by $Z$.
Note that the production display of the industry at the moment has the form $a_{t}^{k} k t$

$$
a_{t}^{k}(x)=\left(0, \quad(B F)_{t}^{k}(X)\right)\left(x \in R_{+}^{n}\right)
$$

The second paragraph of Chapter 1 V is devoted to equilibrium mechanisms for constructing trajectories of economic dynamics models. Let at some moment $t$ be given the security matrix $B^{k}$ and $F^{k}(x)=\min _{i=\overline{1, n}} \frac{x^{i}}{c^{i k}}(k=\overline{1, n})$. Let us denote the vector of prices $\ell=\left(\ell^{1}, \ldots, \ell^{n}\right)$ at the moment $t$. The total wealth of industries
at these prices and the vector of resources $x^{\cdot}=\left(x^{1^{\cdot}}, \ldots, x^{n \cdot}\right)$ at the moment $t$ will be denoted by

$$
\begin{align*}
& U^{k}(\ell, x)=\max \left\{[\ell, y] \mid y \in a^{k}(x)\right\}= \\
& =\left[\ell, B^{k} x\right]+\ell^{k} F^{k}(x), \quad(k=\overline{1, n}) \tag{24}
\end{align*}
$$

Usually called the industry utility function. Let $U^{k}(k=\overline{1, n})$, $\Lambda=\left(\lambda^{1}, \ldots, \lambda^{n}\right)$, where $\lambda^{k}$ are the given budgets of the $k$-th industry, be the vector $y$ of allocated resources. Consider the equilibrium model

$$
M=\{y, U(\ell), \Lambda\}
$$

with fixed budgets.
Let $x_{t}=\left(x_{t}^{1^{\cdot}}, \ldots, x_{t}^{n \cdot}\right)$ the state of the economy be known at the moment $t$, Based on these data, the vector $y_{t+1}=\left(y_{t+1}^{1}, \ldots, y_{t+1}^{n}\right)$ to be distributed is determined at the moment $t+1$.

$$
y_{t+1}^{i}=\sum v_{t}^{k i} x_{t}^{k i}+F_{t}^{i}\left(x_{t}^{i \cdot}\right)(i=\overline{1, n})
$$

Attitude

$$
\begin{equation*}
\mu^{k}(\ell, x)=\frac{U^{k}(\ell, x)}{[P, x]} k=\overline{1, n}, x \in R_{+}^{n} \tag{25}
\end{equation*}
$$

we will call the growth rate of the total wealth of the $k$-th industry in the state $x$.

The same paragraph provides a description of the consumer's task. Let the production mapping $a$ be specified at the moment $t$ :

$$
\begin{align*}
& a(x)=\left\{\tilde{x}=\left(\tilde{x}^{1 \cdot}, \ldots, \tilde{x}^{n \cdot}\right) \in\left(R_{+}^{n}\right)^{n} \mid 0 \leq \sum_{i=1}^{n} \tilde{x}^{i \cdot} \leq \sum_{i=1}^{n} B^{k} \tilde{x}^{k \cdot}+\right. \\
& \left.+\left(F^{1}\left(x^{1 \cdot}\right), \ldots, F^{n}\left(x^{n \cdot}\right)\right), \quad x^{k \cdot}=\left(x^{k 1}, \ldots, x^{k n}\right), \quad k=\overline{1, n}\right\} \tag{26}
\end{align*}
$$

Let the vector be a solution to the problem of the $k-$ th consumer:

$$
U^{k}(\ell, x) \rightarrow \max , x \in \tilde{V}=\{x \geq 0,[P, x] \leq 1\}, k \in I .(27)
$$

Then the equilibrium vector $x^{k \cdot}$ has the form:

$$
x^{k \cdot}=\lambda^{k} \cdot \bar{x}^{k^{\cdot}}(k \in I)
$$

The task is the following: does there exist a model $M$ with data $v^{k i}$ and $c^{i k}$, in which the set $\left(P, x^{1^{*}}, \ldots, x^{n}, y\right)$ is an equilibrium, then
find it, that is, indicate such and, that this set is the equilibrium state in the model $\{y, U(\ell), \Lambda\}$. This problem with $2 n$ unknowns is $\ell^{i}$ and $\lambda^{i}$.

Let us introduce the notation $x \in R_{+}^{n}$

$$
\begin{gather*}
I_{1}(x)=\left\{i \in I \mid x^{i}=0\right\}, \\
I_{2}(x)=\left\{i \in I \mid x^{i}>0\right\},  \tag{28}\\
R^{k}(x)=\left\{i \in I \left\lvert\, \frac{x^{i}}{c^{i k}}=\min _{j \in I} \frac{x^{j}}{c^{j k}}\right.\right\}, \quad(k \in I), \\
Q^{k}(x)=I \mid R^{k}(x), \quad(k \in I) .
\end{gather*}
$$

Lemma 4.2.1. Let $x$ be the solution to the problem of the $k$-th consumer. Then, if $I_{1}(\bar{x}) \neq 0$, then $I_{1}(\bar{x}) \subset R^{k}(\bar{x})$.

The utility function of the industry at a point has the form

$$
U^{k}(\ell, x)=\sum_{j \in I} \ell^{j} \cdot v^{k j} \cdot x^{j}+\ell^{k} \min _{j \in I} \frac{x^{j}}{c^{j k}}, \quad(k \in I)
$$

where $\ell=\left(\ell^{1}, \ldots, \ell^{n}\right)$ is a given price vector. Let's introduce the vector

$$
\ell_{v}^{k}=\left(\ell^{1} \cdot v^{k 1}, \ldots, \ell^{n} \cdot v^{k n}\right), \quad k \in I .
$$

In this case, the expression $U^{k}(\ell, x)$ will take the form

$$
U^{k}(\bar{x})=U^{k}(\ell, \bar{x})=\left[\ell_{v}^{k}, \bar{x}\right]+\ell^{k} \min _{j \in I} \frac{\bar{x}^{j}}{c^{j k}}, \quad(k \in I) .
$$

To study the problem of the $k-$ th consumer, we apply necessary and sufficient extremum conditions, according to which a maximum is reached at a point $\bar{x}$ if and only if

$$
\left(U^{k}\right)^{\prime}(\bar{x}, g) \leq 0, \quad \forall g \in G_{\bar{x}}(V)(k \in I)
$$

where $G_{\bar{x}}(V)$ the cone is defined by the formula

$$
G_{\bar{x}}(V)=\left\{g \in R^{n} \mid[P, g]=0, \quad g^{i} \geq 0, \quad \forall i \in I_{1}(\bar{x})\right\}
$$

It is well known that

$$
\left(U^{k}\right)^{\prime}(\bar{x}, g)=q^{k}(g)
$$

where

$$
q^{k}(g)=\left[\ell_{v}^{k}, g\right]+\ell^{k} \cdot \min _{j \in R^{k}(\bar{x})} \frac{g^{j}}{c^{j k}}, \quad g \in R^{n}
$$

Let us introduce the notation

$$
\tilde{q}^{k}(g)=\ell^{k} \cdot \min _{j \in R^{k}(\bar{x})} \frac{g^{j}}{c^{j k}}, \quad(k \in I)
$$

Thus, if a maximum is reached at a point $\bar{x}$, then

$$
\begin{gathered}
q^{k}(g) \leq 0, \quad \forall g \in G_{\bar{x}}(V) \\
=\left\{g \in R^{n} \mid[P, g]=0, \quad g^{i} \geq 0, \quad \forall i \in I_{1}(\bar{x})\right\}
\end{gathered}
$$

where $I_{1}(\bar{x})$ the set is defined by formula (25).
Lemma 4.2.2. The following conditions are equivalent:

$$
\begin{array}{ll}
\text { 1. } & q^{k}(g) \leq 0,, \forall g \in \Omega, \\
\text { 2. } & \exists \mu^{k}>0,, \mu^{k} \cdot P \in \partial q^{k}
\end{array}
$$

where $\partial q^{k}$ is the superdifferential of the function $q^{k}$.
Lemma 4.2.3. The superdifferential $q^{k}(g)\left(g \in R^{n}\right)$ of the function $q^{k}$, defined above, has the form

$$
\partial q^{k}=\ell_{v}^{k}+\partial \bar{q}^{k}, \quad(k \in I)
$$

where $\ell_{v}^{k}=\left(\ell^{1} \cdot v^{k 1}, \ldots, \ell^{n} \cdot v^{k n}\right)$,

$$
\tilde{q}^{k}(g)=\ell^{k} \cdot \min _{j \in R^{k}(\bar{x})} \frac{g^{i}}{c^{i k}}
$$

and

$$
\begin{aligned}
\partial \tilde{q}^{k} & =\left\{f=\ell^{k} \cdot\left(f^{1}, \ldots, f^{n}\right) \mid \exists \alpha^{i} \geq 0, \sum_{j \in R^{k}(\bar{x})} \alpha^{i}=1,\right. \\
f^{i} & \left.=\frac{\alpha^{i}}{c^{i k}}, \quad i \in R^{k}(\bar{x}), \quad f^{i}=0, \quad i \in Q^{k}(\bar{x})\right\}, \quad(k \in I)
\end{aligned}
$$

Lemma 4.2.4. The number $\mu^{k}(k \in I)$ defined in Lemma 4.2.2 is equal to:

$$
\mu^{k}=\frac{\ell^{k}+\sum_{i \in R^{k}(\bar{x})} \ell^{i} \cdot v^{k i} \cdot c^{i k}}{\sum_{i \in R^{k}(\bar{x})} P^{i} \cdot c^{i k}}(k \in I) .
$$

Theorem 4.2.1. Let $P=\left(P^{1}, \ldots, P^{n}\right)$ a strictly positive vector be given, the index $k \in I$ and the number $\mu^{k}$ defined in Lemma 4.2.4.

A vector $\bar{x}$ is a solution to the $k$-th consumer problem that satisfies the relation if $I_{1}(\bar{x})=\varnothing$ if and only if

1. $\left\{\begin{array}{c}\ell^{i} \cdot v^{k i}=\mu^{k} \cdot P^{i}, \quad \forall i \in Q(\bar{x}), \\ \ell^{j} \cdot v^{k j} \leq \mu^{k} \cdot P^{j}, \quad \forall j \in R^{k}(\bar{x}),\end{array}\right.$
2. $P \in \frac{1}{\mu^{k}} \cdot\left(\ell_{v}^{k}+\partial \tilde{q}^{k}\right)$,
where $\ell_{v}^{k}, \partial \tilde{q}^{k}$, are defined in Lemma 4.2.3.
Comment. If $R^{k}(\bar{x})=I=\{1,2, \ldots, n\}$, then the number $\mu^{k}$ defined above is the maximum growth rate of the total wealth of the $k$ - th industry

Proposition 4.2.1. Under any conditions $\in I$
1.

$$
q^{k}(g) \leq 0 \text { for all } g \in G_{\bar{x}}(V)
$$

2. 

$$
q_{m}^{k}(\tilde{g}) \leq 0 \text { for all } \tilde{g} \in T_{m}
$$

are equivalent.
Theorem 4.2.2. Let a strictly positive vector $P=\left(P^{1}, \ldots, P^{n}\right)$ be given

1. If the vector $\bar{x}$ for which

$$
I_{1}(\bar{x}) \neq \varnothing
$$

is the maximum point in the problem of the $k-$ th consumer, then $m \in Q^{k}(\bar{x})$ when the following relations are satisfied:

$$
\begin{cases}\frac{\ell^{i}}{P^{i}} v^{k i} \leq \frac{\ell^{m}}{P^{m}} v^{k m}, \quad \forall i \in R^{k}(\bar{x})  \tag{*}\\ \frac{\ell^{j}}{P^{j}} v^{k i}=\frac{\ell^{m}}{P^{m}} v^{k m}, \quad \forall j \in Q^{k}(\bar{x})\end{cases}
$$

when the $m \in R^{k}(\bar{x})$ are satisfied

$$
\left\{\begin{array}{c}
\frac{\ell^{i}}{P^{i}} v^{k i} \leq \frac{\ell^{j}}{P^{j}} v^{k j}, \quad \forall i \in R^{k}(\bar{x}), \quad j \in Q^{k}(\bar{x})  \tag{**}\\
\frac{\ell^{j}}{P^{j}} v^{k i}=\frac{\ell^{m}}{P^{m}} v^{k m}, \quad \forall i \in R^{k}(\bar{x})
\end{array}\right.
$$

2. Let $I_{1}(\bar{x}) \neq \emptyset$ and for somem hold (*) (if $m \in Q^{k}(\bar{x})$ ) or (**) (if $m \in Q^{k} \overline{(x)}$ then the vector $\bar{x}$ is a solution to the problem of the $k-$ th consumer.

The third paragraph of Chapter 1 V examines lossless equilibrium problems in models of economic dynamics with a fixed budget.

Let $I=\{1,2, \ldots, n\}$ and $\bar{x}^{k}$ be the maximum point in the consumer problem.

Definition. We call an equilibrium $\left\{P, \bar{x}^{1}, \ldots, \bar{x}^{n}, \Lambda, y\right\}$ an equilibrium without losses if for all $k \in I$

$$
R^{k}\left(\bar{x}^{k \cdot}\right)=I
$$

Definition. Prices $P=\left(p^{1}, \ldots, p^{n}\right)$ determined in equilibrium without losses will be called equilibrium prices without losses.

Let us consider the problem of the $k$-th consumer without losses. To study the problem, we apply necessary and sufficient extremum conditions, according to which $\bar{x}$ a maximum is reached at a point if and only if

$$
\left(U^{k}\right)^{\prime}(\bar{x}, g) \leq 0 \quad \text { for all } g \in G_{\bar{x}}(V)
$$

where $G_{\bar{x}}(V)=\left\{g \in R^{n} \mid[P, g]=0, g^{i} \geq 0, \forall i \in I_{1}(\bar{x})\right\}$.
As is known, $\left(U^{k}\right)^{\prime}(\bar{x}, g)=q^{k}(g)$, where

$$
q^{k}(g)=\left[\ell_{v}^{k}, g\right]+\ell^{k} \min _{j \in R^{k}(\bar{x})} \frac{g^{i}}{c^{i k}}(k \in I)
$$

Then in our case (without losses) we obtain that the necessary and sufficient conditions for optimality $\bar{x}$ in the industry $k$ take the form:

$$
\begin{gathered}
q^{k}(g)=\left[\ell_{v}^{k}, g\right]+\ell^{k} \cdot \min _{j \in I} \frac{g^{i}}{c^{i k}} \leq 0 \\
\forall g \in \Omega=\left\{g \in R^{n} \mid[P, g]=0\right\}
\end{gathered}
$$

Lemma 4.3.1. The number $\mu^{k}(k \in I)$ defined in Lemma 4.2.4 in the case without losses $\left(R^{k}(\bar{x})=I\right)$ coincides with the maximum growth rate of the total wealth of the ith industry $k$ and is equal to

$$
\mu^{k}=\frac{\ell^{k}+\left[\ell_{v}^{k}, c^{\cdot k}\right]}{\left[P, c^{\cdot k}\right]}, \quad(k \in I)
$$

where $\ell_{v}^{k}=\left(\ell^{1} \cdot v^{k 1}, \ldots, \ell^{n} \cdot v^{k n}\right)$.
Theorem 4.3.1. Let $P=\left(p^{1}, \ldots, p^{n}\right)$ a strictly positive vector be given, an index $k \in I$ and a number $\mu^{k}$ defined by formula (25). The vector $\bar{x}$ is the solution to the problem

$$
\begin{equation*}
U^{k}(\ell, x) \rightarrow \max , x \in V=\{x \geq 0 \|[P, x]=1\} \tag{29}
\end{equation*}
$$

satisfying the relation

$$
R^{k}(\bar{x})=I
$$

then and only when

1. $\quad \ell^{j} \cdot v^{k j} \leq \mu^{k} \cdot p^{j}, \quad \forall j \in R^{k}(\bar{x})$,
2. $p \in \frac{1}{\mu^{k}}\left(\ell_{v}^{k}+\partial \tilde{q}^{k}\right)$.

Comment. Given $\mu^{k}$, the equality from Lemma 4.3.1 can be considered as a system of $n$ linear equations with respect to variables - the coordinates of the equilibrium price vector $p$ without losses:

$$
\left[P, c^{\cdot k}\right]=\frac{1}{\mu^{k}} \cdot\left(\ell^{k}+\left[\ell_{v}^{k}, c^{\cdot k}\right]\right)(k \in I)
$$

where $c^{\cdot k}=\left(c^{1 k}, \ldots, c^{n k}\right)$ and $\ell_{v}^{k}=\left(\ell^{1} \cdot v^{k 1}, \ldots, \ell^{n} \cdot v^{k n}\right)$, vice versa, at given prices $p$ the maximum growth rate $\mu^{k}$ of the total wealth of the $k(k \in I)$-th industry is uniquely determined from the equality in lemma 4.3.1.

Lemma 4.3.2. Let the given numbers $\ell^{i}>0, \mu^{k}>0, v^{j i}>$ $0, c^{i j}>0(i, j, k \in I)$ and $\left(P, x^{1 \cdot}, \ldots, x^{n \cdot}\right)$ be the lossless equilibrium in the model $U^{j}$ with utility functions, budgets $\lambda^{j}=\left[p, x^{j}\right]$ and distributed vector $y=\sum_{i=1}^{n} x^{i \cdot}(i \in I)$.

Relation 2) in Theorem 4.3.1 is satisfied for: $\forall k \in I$ if and only if for any $v^{j i} \geq 0$ and $u^{j}(i, j \in I)$ satisfying the equalities

$$
\sum_{j \in I} \mu^{j}\left(v^{j i}+u^{j} c^{i j}\right)=0, \quad \forall i \in I
$$

inequality holds

$$
\sum_{j \in I}\left(\sum_{i \in I} v^{j i} \cdot \ell^{i} \cdot v^{j i}+u^{j}\left(\ell^{j}+\sum_{i \in I} \ell^{i} \cdot v^{j i} \cdot c^{i j}\right)\right) \leq 0
$$

Lemma 4.3.3. Let the numbers $v^{j i} \geq 0, c^{j i} \geq 0,(i, j \in I)$ and the determinant $|c| \neq 0$ of the matrix be given. The following conditions are equivalent:

1. numbers $v^{j i} \geq 0, u^{j}(i, j \in I)$, are such that the conditions of Lemma 4.3.2 are satisfied;
2. $\quad \ell^{i} \mu^{j}(i, j \in I)$ numbers such that for $\forall i, j \in I$ is satisfied
3. $\quad \ell^{i} \cdot v^{j i}+\frac{1}{|c|} \sum_{k \in I}(-1)^{i+k+1} \frac{\mu^{j}}{\mu^{k}}\left(\ell^{k}+\sum_{m \in I} \ell^{m} \cdot v^{k m}\right.$. $\left.c^{m k}\right)\left|c_{i}^{k}\right| \leq 0$,
where $c_{i}^{k}$ is the $(n-1) \times(n-1)$ order matrix obtained from the matrix $c$ by removing the $k-$ th column and $i-$ th row.

Theorem 4.3.4. Let the numbers $v^{j i} \geq 0, c^{i j}>0(i, j \in I)$ be such that and $\max _{j \in I} v^{j i}>0$. Equilibrium prices without losses for given, and some $v^{j i}, c^{i j}, \ell^{i}>0,(i \in I)$ exist if and only if the inequality in the second paragraph of Lemma 4.3.3 is satisfied; in this case, the coefficients $\mu^{k}(k \in I)$ and equilibrium prices $P$ are related by the formula

$$
\mu^{k}\left[P, c^{\cdot k}\right]=\ell^{k}+\left[\ell_{v}^{k}, c^{\cdot k}\right](k \in I)
$$

where $c^{\cdot k}=\left(c^{1 k}, \ldots, c^{n k}\right) \ell_{v}^{k}=\left(\ell^{1} \cdot v^{k 1}, \ldots, \ell^{n} \cdot v^{k n}\right)$.
Let's enter the numbers

$$
d_{i}^{k j}=\left\{\begin{array}{c}
\ell^{i} v^{j i}+(-1)^{i+j+1} \frac{\left|c_{i}^{j}\right|}{|c|}\left(\ell^{j}+\sum_{m \in I} \ell^{m} \cdot v^{j m} \cdot c^{m j}\right) \\
\text { if } k=j(i, j, k \in I) \\
(-1)^{i+k+1} \frac{\left|c_{i}^{k}\right|}{|c|}\left(\ell^{k}+\sum_{m \in I} \ell^{m} \cdot v^{k m} \cdot c^{m k}\right) \\
\text { if } k \neq j,
\end{array}\right.
$$

and vector

$$
\tilde{d}^{k j}=\left(\begin{array}{c}
-d_{1}^{k j} \\
\vdots \\
\vdots \\
-d_{n}^{k j}
\end{array}\right)(k, j \in I)
$$

Proposition 4.3.1. The number $\mu^{j}(j \in I)$, exists if and only if there $k_{0} \in I$ is an index and such that $\beta^{k}$

$$
\beta^{k_{0}}>0, \quad \sum_{k=1}^{n^{2}} \beta^{k} \tilde{d}^{k j} \geq 0 \quad(j \in I)
$$

where $\widetilde{d}^{k j}$ defined above.
In the fourth paragraph of Chapter 1 V , using an equilibrium mechanism without losses, trajectories are constructed in the model $Z$ , which is specified by the mapping

$$
\begin{gathered}
a(x)=\left\{\tilde{x}=\left(\tilde{x}^{1}, \ldots, \tilde{x}^{n \cdot}\right) \in\left(R_{+}^{n}\right)^{n} \mid \sum_{k \in I}^{n} \tilde{x}^{k i} \leq \sum_{k \in I}^{n} v^{k i} x^{k i}+\right. \\
+\min _{j \in I} \frac{x^{i j}}{c^{j i}}, \quad x^{k}=\left(x^{k 1}, \ldots, x^{k n}\right), \quad k \in I, \quad v^{k i} \in[0,1] \\
\left.c^{i j}>0,(i, j \in I)\right\} .
\end{gathered}
$$

Recall that the equilibrium $\left(P, x^{1}, \ldots, x^{n}, y\right)$ is the se, and if equilibrium exists, then the vector $x^{k}$ is necessarily proportional to the vector $c^{\cdot k}(k \in I)$. If $y \notin \operatorname{cone}\left\{c^{\cdot i} \mid i \in I\right\}$, then equilibrium certainly does not exist.

Therefore, it is advisable to provide a necessary and sufficient condition for the existence of equilibrium prices without losses. According to Fang Zi's theorem ${ }^{3}$ the necessary conditions for the non-existence of equilibrium prices without losses can be formulated as

[^1]\[

$$
\begin{equation*}
\sum_{j \in I} v^{j i}=-\sum_{j \in I} u^{j} c^{i j}, \quad \forall i \in I \tag{a}
\end{equation*}
$$

\]

for which the inequality hold

$$
\begin{equation*}
\sum_{j \in I}\left(\sum_{i \in I} v^{j i} \ell^{i} v^{j i}+u^{j}\left(\ell^{j}+\sum_{i \in I} \ell^{i} v^{j i} c^{i j}\right)\right)>0 \tag{b}
\end{equation*}
$$

Let us introduce sets of indices

$$
\begin{gathered}
J_{1}(i)=\left\{j \in I \mid v^{j i}=\max _{k \in I} v^{k i}\right\}, \quad \forall i \in I, \\
J_{2}(i)=\left\{j \in I \mid \ell^{j}+v^{j i} c^{i j}=\min _{k \in I}\left(\ell^{k}+v^{k i} c^{i k}\right)\right\}, \quad \forall i \in I .
\end{gathered}
$$

Let the vectors $\ell=\left(\ell^{1}, \ldots, \ell^{n}\right)$ be normalized by the relation:

$$
\sum \ell^{i}=1, \quad \ell^{i}>0, \quad(i \in I)
$$

Occurs
Lemma 4.4.1. Let the conditions be met

$$
\sum \ell^{i}=1, \quad \ell^{i}>0, \quad(i \in I)
$$

and inequality

$$
\frac{\max _{i \in I} v^{j i}}{\left|J_{1}(i)\right|} \cdot \sum_{j \in J_{1}(i)} c^{i j}>\min _{j \in I}\left(\ell^{j}+v^{j i} c^{i j}\right), \quad i \in I
$$

where $\left|J_{1}(i)\right|$ is the number of elements in the set of indices $J_{1}(i)$. In this case, there are numbers $u^{j}(i, j \in I)$ and satisfying condition (a) for which (c) holds.

Theorem 4.4.1. Let the numbers $v^{j i} \geq 0, \ell^{i}>0(i, j \in I)$, be such that $\max v^{j i}>0, \sum_{i \in I} \ell^{i}=1 \mu^{j}=1(j \in I)$. If equilibrium prices without losses for given $v^{j i}, \ell^{i}$ and some $c^{i j}>0(i, j \in I)$ exist, then an inequality of the form holds:

$$
\min _{j \in I}\left(\frac{\max _{j \in I} v^{j i}}{\left|J_{1}(i)\right|} \sum_{j \in J_{1}(i)} c^{i j}-\min _{j \in I}\left(\ell^{i}+v^{j i} c^{i j}\right)\right) \leq 0 .
$$

Let's consider the Neumann-Gale model and define the Neumann equilibrium state in this model $Z$. Recall that the nondegenerate case is that

$$
\begin{equation*}
\alpha \sum_{k \in I} \bar{x}^{k i}=\sum_{k \in I} v^{k i} \bar{x}^{k i}+\min _{j \in I} \frac{\bar{x}^{i j}}{c^{j i}}, \quad \forall i \in I \tag{30}
\end{equation*}
$$

Theorem 4.4.2. A Neumann equilibrium state satisfying equality (30) can be constructed using a lossless equilibrium model for $\ell=\frac{1}{\alpha} P, \mu^{k}=1(k \in I)$.

The fifth paragraph of Chapter 1 V examines the effective trajectories of the model $Z$.

Let be $x_{t}=\left(x_{t}^{1 \cdot}, \ldots, x_{t}^{n \cdot}\right)$ the effective trajectory of this model. This trajectory admits the characteristic $L_{t}=\left(\ell_{t}^{1}, \ldots, \ell_{t}^{n}\right)$. Here $\ell_{t}^{i \cdot} \in$ $R_{+}^{n}$. We can assume that $L_{t}=\left(\ell_{t}, \ldots, \ell_{t}\right)$, where $\ell_{t}=\left(\ell_{t}^{1}, \ldots, \ell_{t}^{n}\right) \in$ $R_{+}^{n}$

It is believed that for all $\ell_{t}^{i}>0$, where $\ell_{t}^{i}$ is the price of the $i$-th product at time $t$. Let's define the functions $U_{t}^{k}\left(\ell_{t+1}, x\right)$

$$
\begin{equation*}
U_{t}^{k}\left(\ell_{t+1}, x\right)=\left[\ell_{t+1}, B_{t}^{k} \cdot x\right]+\ell_{t+1}^{k} \cdot F_{t}^{k}(x) \tag{31}
\end{equation*}
$$

Recall that the number

$$
\begin{equation*}
\mu_{t}^{k}(x)=\frac{U_{t}^{k}\left(\ell_{t+1}, x\right)}{\left[\ell_{t}, x\right]}, \quad k \in I \tag{32}
\end{equation*}
$$

is the growth rate of the total wealth of the $k-$ th industry in the state $x$ at prices $\ell_{t+1}$ and $\ell_{t}$.

Let there be equality

$$
R\left(\bar{x}_{t}^{k \cdot}\right)=I \quad \text { for all } \quad k \text { and } t
$$

The equilibrium mechanism in the case of $R\left(\bar{x}_{t}^{k \cdot}\right)=$ $I(k \in I) t=1,2, \ldots$, will be called an equilibrium mechanism without losses.

We will say that in the model $Z$ it is possible to construct a trajectory $\left(x_{t}\right)_{t=1}^{\infty}$ using an equilibrium mechanism without losses if at any moment $t$ of time the vectors $\bar{x}_{t}^{k \cdot}$ that make up the state $x_{t}$ of
this trajectory fall into a conical shell spanned by vectors of the form $c^{\cdot k}(k \in I)$.

Let $\bar{x}$ be the Neumann equilibrium vector.
Lemma 4.5.1. Let $\left(x_{t}\right)_{t=1}^{\infty}$ the trajectory of the model $Z$ be given, which has the following properties: the vector $x_{t}$ is constructed as part of the equilibrium in the model $M=\left(\{y\}, U_{t}\left(\ell_{t+1}\right), \Lambda_{t}, V\right)$, where $V=\left(R_{+}^{n}, \ldots, R_{+}^{n}\right), y=(B F)_{t-1}\left(x_{t-1}\right)$, and the vector of budgets $\Lambda_{t}=\left(\lambda_{t}^{1}, \ldots, \lambda_{t}^{n}\right)$ is chosen so $\mu_{t}^{k}=1$ that the growth rate is, while the equilibrium prices $\ell_{t}=\left(\ell_{t}^{1}, \ldots, \ell_{t}^{n}\right)$ coincide with, and the budgets $\lambda_{t}^{k}$ are associated with the equalities $\lambda_{t}^{k}=\left[\ell_{t}, x_{t}^{k \cdot}\right]$. Then $L_{t}=\left(\ell_{t}, \ldots, \ell_{t}\right)$ is the characteristic of the trajectory $\left(x_{t}\right)$.

Note 1. The characteristic $\left(L_{t}\right)\left(L=\left(\ell_{t}, \ldots, \ell_{t}\right)\right)$ of the trajectory $\left(x_{t}\right)$ is constructed inductively using the formula

$$
\beta_{t+1}=c_{t+1}^{-1}\left(E+c_{t}^{v}\right) \beta_{t}, \quad t=1,2, \ldots
$$

and $\ell_{t+1} \gg 0$ and the vectors $x_{t}^{k \cdot}$ that make up the state $x_{t}$ of this trajectory are determined by the formula

$$
x_{t}^{k \cdot}=\frac{\lambda_{t}^{i}}{\left[\ell_{t}, c_{t}^{\cdot k}\right]} \cdot c_{t}^{\cdot k}, \quad k \in I .
$$

Note 2. If the sequence $\left(\ell_{t}\right)$ that is inductively constructed for the initial vector $\ell$ is not a characteristic of the trajectory $\left(x_{t}\right)$, then for some $t$ the equality does not hold:

$$
(B F)_{t}\left(x_{t}\right)=\sum_{k=1}^{n} x_{t+1}^{i \cdot}=\sum_{k=1}^{n} \lambda_{t+1}^{i} \bar{x}_{t+1}, \quad t=1,2, \ldots
$$

or the condition is violated

$$
\ell_{t+1} \gg 0, \quad t=1,2, \ldots
$$

Theorem 4.5.1. Let the model $Z$ have only one fund-forming industry and numbers $v^{i}(i \in I)$ such that

$$
v^{i}<\frac{c^{1 i}}{\sum_{k=2}^{n} c^{1 k} \cdot c^{k i}}
$$

Then from any initial state $x_{1} \neq \lambda \bar{x},(\lambda \geq 0)$ it is impossible to construct an effective trajectory $\left(x_{t}\right)_{t=1}^{\infty}$ using an equilibrium mechanism without losses.

The sixth paragraph of this chapter is devoted to the asymptotic behavior of the trajectories of reproduction models.

When studying the formulated reproduction model, a model with fixed budgets is used, which has the form

$$
\mathfrak{M}=(\{y\}, U, \wedge),
$$

here $y \gg 0$ is some element of the cone $\mathrm{R}_{+}{ }^{n}$, where $U=$ $U^{i}, \ldots, U^{n}$ are the utility functions $U^{i}$ defined by the

$$
U^{i}\left(\bar{f}, x^{i}\right)=\left[\bar{f}, B^{i} x^{i}\right]+\bar{f}^{i} F^{i}\left(x^{i}\right), \quad(i=\overline{1, n}) .
$$

In essence, $U^{i}(i=\overline{1, n})$ the functions represent the cost of all the funds hat the corresponding industry has, at prices $\bar{f}=$ $\left(\bar{f}^{1}, . ., \bar{f}^{n}\right)$.

The set of vectors $\left(\rho, \bar{x}^{1} \bar{x}^{n}\right), \ldots$, forms an equilibrium state in the model $\mathfrak{M}$, if the vectors $\bar{x}^{i}$ are solutions to problems

$$
U^{i}\left(\bar{f}, x^{i}\right) \rightarrow \max \quad(i=\overline{1, n})
$$

under conditions $\left[\rho, x^{i}\right]=\lambda_{i}, x^{i} \geq 0$, and in addition, the relations are satisfied

$$
\sum_{i}^{n} \bar{x}^{i}=y, \rho \geq 0
$$

Let us introduce the function $\left(f, x^{i}\right)=\beta_{i} \nabla_{x} U^{i}\left(f, x^{i}\right)$ into consideration. As is known, the equilibrium vector $\tilde{X}=$ $\left(\tilde{x}^{1}, \ldots, \tilde{x}^{n}\right)$ is a solution to the equation $\Psi(f, X)=0$, where $\Psi=\left(\psi_{1}, \ldots, \psi_{n}\right)$ is a mapping into $\left(R_{+}{ }^{n}\right)^{n}$ itself having coordinate functions

$$
\psi_{i}(X)=\tau_{i}\left(f, x^{i}\right)-\tau_{i+1}\left(f, x^{i+1}\right)
$$

at

$$
1 \leq i \leq n-1, \quad \psi_{n}(X)=\sum_{1}^{n} x^{i}-y
$$

We study the system $\Psi(f, X)=0$ in order to find out how the growth rate of the trajectory constructed using the model changes

The following lemma is true.
Lemma 4.6.1. The function $\Psi(f, x)$ is differentiable with respect to $X$. In this case, $\frac{\partial \Psi}{\partial X}$ it coincides with the following block matrix
where $\varphi_{i}$ is the matrix of second partial derivatives $A_{i}=$ $\nabla_{x}{ }^{2} \varphi_{i}\left(\tilde{x}^{i}\right)$ of the function $\varphi_{i}$ calculated at the point $\tilde{x}^{i}$.

Theorem 4.6.1. The rates of change in the states $y^{i}$ of the $i-$ th $(i=1, n-1)$ industry are expressed linearly through the coordinates of the vector $y^{n}$.

Theorem 4.6.2. Let $M^{i}=\max \left\{\frac{x_{1}^{i}}{x_{1}^{n}}, \ldots, \frac{x_{n}^{i}}{x_{n}^{n}}\right\}$. If there are sufficiently small ones $\Delta_{i}>0 j=\overline{1, n}$ such that for all $j=\overline{1, n}$, then for any sufficiently small ones $\left|\frac{x_{j}^{i}}{x_{j}^{n}}-M^{i}\right|<\Delta_{i}$ the following equality is true

$$
\left|y_{k}^{i}-\frac{a_{12}^{n} q_{n}}{a_{12}^{i} q_{i}} y_{k}^{n}\right|<\varepsilon_{i} \quad(i=\overline{2, n-1}, k=\overline{1, n})
$$

Consequence. Under the conditions of the theorem, the signs of the coordinates $y^{i} a \leq i \leq n-1$ of the vectors $y^{n}$ at coincide with the signs of the coordinates of the vector $y^{1}$. The
coordinates of the vector have signs opposite to the signs of the corresponding coordinates of other vectors.

Lemma 4.6.7. Given a matrix $C=\left(C_{i j}\right)_{i, j=1}^{n}$ such that

$$
C_{i j}= \begin{cases}\frac{(3-n) x_{i}^{2}}{x_{1} x_{2}}, & i=j \\ \frac{x_{i} x_{j}}{x_{1} x_{2}}, & i \neq j\end{cases}
$$

where $x_{0}=\left(x_{1} x_{2}, \ldots, x_{1} x_{2}\right)$ is a vector with positive coordinates. Then for $x_{i}$ non-negative matrix $\mathcal{A}=C+\mu \mathrm{I}$ where $\quad \mu=(n-3) \max _{k=1, n} \frac{x_{k}^{2}}{x_{1} x_{2}} \quad$ and vector $x_{0}=$ $\left(x_{1} x_{2}, \ldots, x_{1} x_{2}\right)$ the inequality holds

$$
\mathcal{A} x_{0} \leq \xi x_{0} .
$$

Lemma 4.6.8. When $n>3$ for $1 \leq i \leq n$ is a fair estimate

$$
\left\|C_{i}^{-1}\right\| \leq \frac{\xi_{i}+\mu_{i}}{2(n-2)\left|q_{i} a_{12}^{i}\right|}
$$

where

$$
\mu_{i}=(n-3) \max \frac{\left(x_{k}^{i}\right)^{2}}{x_{1}^{i} x_{2}^{i}}, \quad \xi_{i}=\max _{k=1, n} \sum_{j=1}^{n} \bar{a}_{k j}^{i}
$$

here $\bar{a}_{k j}^{i}$ are the elements of matrices $\mathcal{A}^{i}$ constructed similarly to the matrix $\mathcal{A}$ from Lemma 4.6.7.

Theorem 4.6.4. For there is an estimate $y^{k} \quad(k=\overline{1, n})$

$$
\begin{aligned}
& \left\|y^{k}\right\| \leq \frac{\bar{L}^{2}}{\left(q_{p} a_{12}^{p}\right)^{2}}\left\|K^{-1}\right\|\left\{\beta_{1}\left\|\frac{\partial F^{1}}{\partial x^{1}}\right\|+\right. \\
= & \frac{\delta}{2}[k(k-1)+(n-k)(1+n-k)] .
\end{aligned}
$$

Chapter V examines the model $Z^{2}$, which is a special case of the model $Z$ studied in previous chapters. A two-sector model $Z^{2}$ of
the economy is considered, the first sector of which produces means of production, the second - consumer goods. The equilibrium mechanism for constructing trajectories is studied in relation to the model. It turns out under what conditions equilibrium mechanisms, determined by Leontief production functions, generate effective trajectories.

The first paragraph of Chapter V provides a description of the model $Z^{2}$. The vector $X_{t}=\left(X_{t}^{1}, X_{t}^{2}\right) \in\left(R_{+}^{2}\right)^{2}$ is the state of the model; here $X_{t}^{i}=\left(K_{t}^{i}, L_{t}^{i}\right) \in R_{+}^{2}, K_{t}^{i}$ - fixed assets, $L_{t}^{i}, i(i=1,2)-$ labor force in the $i$-th sector. The production activity of the $i$-th sector at the moment $t$ is described using the production function $F_{t}^{i}: R_{+}^{2} \rightarrow R_{+}$and safety coefficients $0 \leq v_{t}^{i} \leq 1(i=1,2)$. The wage rate (specific consumption) $\omega_{t}>0$ is considered to be known, coinciding in the first and second sector.

The transition from state $X_{t}=\left(K_{t}^{1}, L_{t}^{1}, K_{t}^{2}, L_{t}^{2}\right)$ to state $X_{t+1}=$ ( $K_{t+1}^{1}, L_{t+1}^{1}, K_{t+1}^{2}, L_{t+1}^{2}$ ) is carried out using a system of inequalities

$$
\left\{\begin{array}{c}
K_{t+1}^{1}+K_{t+1}^{2} \leq v_{t}^{1} \cdot K_{t}^{1}+v_{t}^{2} \cdot K_{t}^{2}+F_{t}^{1}\left(K_{t}^{1}, L_{t}^{1}\right) \\
\omega_{t+1}\left(L_{t+1}^{1}+L_{t+1}^{2}\right) \leq F_{t}^{2}\left(K_{t}^{2}, L_{t}^{2}\right) \\
K_{t}^{i} \geq 0, \quad L_{t}^{i} \geq 0, \quad(i=1,2)
\end{array}\right.
$$

where

$$
\begin{aligned}
F_{t}^{i}\left(K_{t}^{i}, L_{t}^{i}\right) & =\min \left(\frac{K_{t}^{i, j}}{C_{t}^{i, j}}, \frac{L_{t}^{i, j}}{C_{t}^{j, i}}\right), \quad(i, j=1,2) \\
C_{t}^{j i} & >0, \quad(i, j=1,2)
\end{aligned}
$$

It is clear that the model $Z^{2}$ coincides with the model $Z$ for $n=$ 2 and matrices $B_{t}^{i}$ having the form

$$
B_{t}^{i}=\left(\begin{array}{cc}
v_{t}^{i} & 0 \\
0 & 0
\end{array}\right)(i=1,2)
$$

When studying the model $Z^{2}$, the simplest superlinear mapping $a$ of the form is used

$$
\begin{gathered}
a(K, L)= \\
=\left\{\left(K^{\prime}, L^{\prime}\right) \mid K^{\prime} \geq 0, L^{\prime} \geq 0, K^{\prime}+\omega L^{\prime} \leq v K+\min \left(\frac{K}{c^{1}}, \frac{L}{c^{2}}\right),(K, L \geq 0)\right\}
\end{gathered}
$$

where $c^{1}>0, c^{2}>0$.
The Neumann-Gale model $\tilde{Z}$, defined by the mapping $a$, will be written in the form $\tilde{Z}=\left(v, \omega, c^{1}, c^{2}\right)$.

Consider the function

$$
f(\eta)=\min \left(\frac{\eta}{c^{1}}, \frac{1}{c^{2}}\right), \quad(\eta>0)
$$

and enter the number

$$
\beta=\frac{c^{1}}{c^{2}}
$$

Occurs
Lemma 5.1.1. Let

$$
g(\eta)=\frac{v \eta+f(\eta)}{\eta+\omega}, \quad(\eta>0)
$$

where $v, \omega$ are some constants. Then, if $\omega \cdot v<\frac{1}{c^{2}}$ then the function $g$ reaches a maximum on the interval $(0,+\infty)$, and at a single point $\bar{\eta}=\beta$. If $\omega \cdot v>\frac{1}{c^{2}}$ then the function $g$ strictly increases on this interval. In the case, $\omega \cdot v=\frac{1}{c^{2}}$ the function $g$ reaches a maximum on the interval $(0,+\infty)$, at all points $\bar{\eta} \geq \beta$, and $\max _{\eta>0} g(\eta)=v$

Proposition 5.1.1. Let's consider the model $\tilde{Z}=\left(\omega, c^{1}, c^{2}, v\right)$. Let the point $\bar{X}=(\bar{K}, \bar{L})$ be that

$$
\max _{\substack{K \geq 0, K+\omega \cdot L \neq 0}} \frac{v \cdot K+\min \left(\frac{K}{c^{1}}, \frac{L}{c^{2}}\right)}{K+\omega \cdot L}=\frac{v \cdot \bar{K}+\min \left(\frac{\bar{K}}{c^{1}}, \frac{\bar{L}}{c^{2}}\right)}{\bar{K}+\omega \cdot \bar{L}}
$$

and $\bar{K}+\omega \cdot \bar{L}=1$. Let's put it $\bar{p}=(1, \omega)$. Then the triple $(\alpha(\tilde{Z}), \bar{x}, \bar{p})$ forms the equilibrium state of the model $\left(\omega, v, c^{1}, c^{2}\right)$, and

$$
\text { 1. if } \omega v<\frac{1}{c^{2}} \text {, then } \bar{K}=\frac{c^{1}}{c^{1}+\omega \cdot c^{2}} \overline{, L}=\frac{c^{2}}{c^{1}+\omega \cdot c^{2}},
$$

> 2. if $\omega v=\frac{1}{c^{2}}$, then $\frac{\bar{K}}{\bar{L}}>\frac{c^{1}}{c^{2}}$
> 3. if $\omega v>\frac{1}{c^{2}}$, then $\bar{K}=1, \bar{L}=0$.

The second paragraph of the fifth Chapter is devoted to the study of the stationary case of the model $Z^{2}$, described in §5.1: $v_{t}^{i}=$ $v^{i} \omega_{t}=\omega, c_{t}^{i j}=c^{i j}, t=0,1, \ldots$, for all $(i, j=1,2)$. In this case, the model $Z^{2}$ turns into a Neumann-Gale model $\bar{Z}^{2}$, which we denote by. It is determined by the mapping

$$
\begin{gather*}
a\left(K_{1}, L_{1}, K_{2}, L_{2}\right)=\left\{\left(K_{1}^{1}, L_{1}^{1}, K_{2}^{1}, L_{2}^{1}\right) \mid 0 \leq K_{1}^{1}+K_{2}^{1} \leq\right. \\
\leq v^{1} K_{1}+v^{2} K_{2}+\min \left(\frac{K_{1}}{c^{11}}, \frac{L_{1}}{c^{21}}\right)  \tag{33}\\
\left.0 \leq \omega\left(L_{1}^{1}+L_{2}^{1}\right) \leq \min \left(\frac{K_{2}}{c^{12}}, \frac{L_{2}}{c^{22}}\right), \quad K_{i} \geq 0, \quad L_{i} \geq 0, \quad(i \neq 1,2)\right\} .
\end{gather*}
$$

Let us introduce models $Z_{1}(b), Z_{2}(b)$ for some $b>0$, defined respectively by sets

$$
\begin{equation*}
\left(v^{1}, b \omega, c^{11}, c^{21}\right), \quad\left(v^{2}, b \omega, \frac{1}{b} c^{12}, \frac{1}{b} c^{22}\right) \tag{34}
\end{equation*}
$$

and numbers

$$
\begin{equation*}
\beta^{1}=\frac{c^{11}}{c^{21}}, \quad \beta^{2}=\frac{c^{12}}{c^{22}} \tag{35}
\end{equation*}
$$

To each of the sets (34) we associate the numbers $\alpha_{i}(b), \bar{\eta}^{i}$ by Lemma 5.1.1

$$
\bar{\eta}^{1}=\beta^{1}, \quad \alpha_{1}(b)=\frac{1+v^{1} c^{11}}{c^{11}+b \omega c^{21}}
$$

and

$$
\bar{\eta}^{2}=\beta^{2}, \quad \alpha_{2}(b)=\frac{b+v^{2} c^{12}}{c^{12}+b \omega c^{22}}
$$

## Lemma 5.2.1. Let

$$
\begin{equation*}
v^{2}-v^{1}<\frac{1}{c^{11}}, \quad \beta^{1} \neq \beta^{2} \tag{36}
\end{equation*}
$$

Then the equation is where $b=V(b)$

$$
V(b)=\left(1+v^{1} c^{11}\right) \frac{c^{12}+b \omega c^{22}}{c^{11}+b \omega c^{21}}-v^{2} c^{12}
$$

has only one positive solution $\bar{b}$. In this case, the relation $b<V(b)$ is equivalent to inequality $0<b<\bar{b}$.

Proposition 5.2.1. Let the conditions $\beta^{1}>\beta^{2}$ of Lemma 5.2.1 be satisfied and the sequence $b_{t}$ satisfies the equalities $b_{t+1}=$ $V\left(b_{t}\right)(t=0,1,2, \ldots)$ and $b_{0}>0$. Then $b_{t} \rightarrow \bar{b}$, and if $b_{0}<\bar{b}$, then $b_{t}$ increases, if $b_{0}>\bar{b}$, then $b_{t}$ decreases.

Theorem 5.2.1. Let $\omega v^{2}<\frac{1}{c^{22}}$, conditions (33) be satisfied and the number $\bar{b}$ defined as in Lemma 5.2.1. The number $\alpha=\alpha_{1}(\bar{b})=$ $\alpha_{2}(\bar{b})$ is then the model's $\bar{Z}_{2}$ Neumann growth rate; the functional $\bar{p}=(1, \bar{b} \cdot \omega, 1, \bar{b} \cdot \omega)$ is Neumann equilibrium prices; the Neumann equilibrium vector $\bar{x}=\left(\bar{K}^{1}, \bar{L}^{1}, \bar{K}^{2}, \bar{L}^{2}\right)$ is defined ratios:

$$
\frac{\bar{K}^{1}}{\bar{L}^{1}}=\beta^{1}, \quad \frac{\bar{K}^{2}}{\bar{L}^{2}}=\beta^{2}, \quad \frac{\bar{L}^{1}}{\bar{L}^{2}}=\frac{\left(\frac{1}{c^{22}}-\omega v^{2}\right) \cdot c^{12}}{\left(\bar{b}+v^{2} c^{12}\right) \omega}
$$

where, $\beta^{1}, \beta^{2}$ are defined by formula (35).
Let us describe, in relation to the model $\tilde{Z}_{2}$, the equilibrium mechanism for constructing model $Z$ trajectories, discussed in §3.1. Let the vector of prices $\ell=\left(\ell^{1}, \ell^{2}\right), \ell^{1}>0, \ell^{2}>0$ be given. Without loss of generality, we believe that it will be convenient for us to write the price of funds $\ell^{2}$ in the form $\ell^{2}=b \omega$, thus the price vector $\ell$ has the form $\ell=(1, b \omega)$, where $\ell>0$ and $b$ is the price of a consumption unit. Consider the expected total wealth of divisions at these prices and resource vector $x=(K, L)$ :

$$
\begin{aligned}
& U^{1}(\ell, x)=v^{1} K+\min \left(\frac{K}{c^{11}}, \frac{L}{c^{21}}\right) \\
& U^{2}(\ell, x)=v^{2} K+\min \left(\frac{K}{c^{12}}, \frac{L}{c^{22}}\right)
\end{aligned}
$$

Further, $U^{1}$ and $U^{2}$ act as functions of the utility of the units. Let be the vector $y=(K, L)$ of distributed resources; $\lambda^{1}, \lambda^{2}-$ given
department budgets. Let us consider an equilibrium model $M$ with fixed budgets, determined by the quantities $(K, L)$.

Let's put

$$
\begin{gathered}
p=\left(p^{1}, p^{2}\right), \quad \beta=\frac{K}{L}, \quad \beta^{1}=\frac{c^{11}}{c^{21}} \\
\beta^{2}=\frac{c^{11}}{c^{22}}, \quad u^{1}=\frac{1}{v^{1} c^{21}}, \quad u^{2}=\frac{b}{v^{2} c^{22}} .
\end{gathered}
$$

Recall that a set $\left(p, x^{1}, x^{2}\right)$ is an equilibrium state (see $\left.\S 1.5\right)$ in the model $M$, if

$$
x^{1}+x^{2}=y
$$

and the vector $x^{i}$ is the solution to the problem
$U^{i}(\ell, x) \rightarrow$ max given that, $x \geq 0,\left[p, x^{i}\right] \leq \lambda^{i},(i=1,2)$.
Proposition 5.2.3. Let $p=\left(p^{1}, p^{2}\right)$ be some prices. Then

1) if $p^{1}=0, p^{2}>0$, then the equilibrium state $\left(p, x^{1}, x^{2}\right)$ does not exist;
2) if $p^{2}=0, p^{1}>0$, then the set $\left(p, x^{1}, x^{2}\right)$, where

$$
\begin{gathered}
p^{1}=\frac{1}{\beta \mathcal{L}}\left(\lambda^{1}+\lambda^{2}\right), \quad p^{2}=0, \quad x^{1}=\frac{\lambda^{1}}{p^{1}}\left(1, \frac{1}{\beta^{1}}\right) \\
x^{2}=\frac{\lambda^{2}}{p^{2}}\left(1, \frac{1}{\beta^{2}}\right)
\end{gathered}
$$

(here $\beta, \beta^{1}, \beta^{2}$, defined above) is an equilibrium state if and only if

$$
\mu>0 \text { and } \beta=\beta^{1}=\beta^{2}
$$

Proposition 5.2.4. Let $u^{2} \leq \frac{p^{2}}{p^{1}} \leq u^{1}\left(p^{1}>0, p^{2}>0\right)$ where the numbers, $u^{1}, u^{2}, \beta, \beta^{1}, \beta^{2}$, are defined above. Then the set $\left(p, x^{1}, x^{2}\right)$ is an equilibrium state in the model $M$ if and only if

$$
\begin{gathered}
\text { 1) } x^{1}=\frac{\lambda^{1}}{p^{1} \beta^{1}+p^{2}}\left(\beta^{1}, 1\right), \quad x^{2}=\frac{\lambda^{2}}{p^{1}}(1,0), \\
p^{1}=\frac{\lambda^{2}}{\left(\beta-\beta^{1}\right) \mathcal{L}}, \quad p^{2}=\frac{1}{\mathcal{L}}\left(\lambda^{1}-\frac{\lambda^{2} \cdot \beta^{1}}{\beta-\beta^{1}}\right)
\end{gathered}
$$

$$
\text { 2) } \beta>\beta^{1}, \quad \vartheta_{1}(\beta)<\mu<\vartheta_{2}(\beta) \text {. }
$$

Here

$$
\vartheta_{1}(\beta)=\frac{\beta^{1}+u^{2}}{\beta-\beta^{1}}, \quad \vartheta_{2}(\beta)=\frac{\beta^{1}+u^{1}}{\beta-\beta^{1}} .
$$

Proposition 5.2.5. Let $u^{1} \leq \frac{p^{2}}{p^{1}} \leq u^{2},\left(p^{1}>0, p^{2}>0\right)$, where numbers $u^{1}, u^{2}, \beta^{1}, \beta^{2}$, are defined above be satisfied and

$$
\begin{gathered}
p=\left(p^{1}, p^{2}\right), \quad \beta=\frac{K}{L}, \quad \beta^{1}=\frac{c^{11}}{c^{21}}, \\
\beta^{2}=\frac{c^{11}}{c^{22}}, \quad u^{1}=\frac{1}{v^{1} c^{21}}, \quad u^{2}=\frac{b}{v^{2} c^{22}} .
\end{gathered}
$$

Then the set if ( $p, x^{1}, x^{2}$ ) is an equilibrium state in the model $M$ if and only if

$$
\begin{array}{ll}
\text { 1) } x^{1}=\frac{\lambda^{1}}{p^{2}}(1,0), & x^{2}=\frac{\lambda^{2}}{p^{1} \beta^{2}+p^{2}}\left(\beta^{2}, 1\right), \\
p^{1}=\frac{\lambda^{1}}{\left(\beta-\beta^{2}\right) \mathcal{L}}, & p^{2}=\frac{1}{\mathcal{L}}\left(\lambda^{2}-\frac{\lambda^{1} \cdot \beta^{2}}{\beta-\beta^{2}}\right), \\
\text { 2) } \quad \beta>\beta^{2}, & \vartheta_{3}(\beta)<\mu<\vartheta_{4}(\beta) .
\end{array}
$$

Here

$$
\vartheta_{3}(\beta)=\frac{\beta-\beta^{2}}{\beta^{2}+u^{2}}, \quad \vartheta_{4}(\beta)=\frac{\beta-\beta^{2}}{\beta^{2}+u^{1}} .
$$

Proposition 5.2.6. Let $\left(p^{1}>0 \frac{p^{2}}{p^{1}}<\min \left(u^{1}, u^{2}\right), p^{2}>0\right)$, where the parameters $u^{1}, u^{2}, \beta, \beta^{1}, \beta^{2}$ are satisfied. Then the set ( $p, x^{1}, x^{2}$ ) is an equilibrium state in the model $M$ if and only if

$$
\begin{array}{cl}
x^{1}=\frac{\lambda^{1}}{p^{1} \beta^{1}+p^{2}}\left(\beta^{1}, 1\right), & x^{2}=\frac{\lambda^{2}}{p^{1} \beta^{2}+p^{2}}\left(\beta^{2}, 1\right), \\
p^{1}=\frac{1}{\mathcal{L}}\left(\frac{\lambda^{1}}{\beta-\beta^{2}}-\frac{\lambda^{2}}{\beta^{1}-\beta}\right), & p^{2}=\frac{1}{\mathcal{L}}\left(\frac{\lambda^{2} \beta^{1}}{\beta^{1}-\beta}-\frac{\lambda^{1} \cdot \beta^{2}}{\beta-\beta^{2}}\right),
\end{array}
$$

and one of two conditions is met:

$$
\vartheta_{5}(\beta)<\mu<\vartheta_{6}(\beta) \text { by } \beta^{2}<\beta<\beta^{1},
$$

or

$$
\vartheta_{6}(\beta)<\mu<\vartheta_{5}(\beta) \text { by } \beta^{1}<\beta<\beta^{2},
$$

where

$$
\begin{gathered}
\vartheta_{5}(\beta)=\frac{\beta-\beta^{2}}{\beta^{1}-\beta} \cdot \frac{\beta^{1}+\min \left(u^{1}, u^{2}\right)}{\beta^{2}+\min \left(u^{1}, u^{2}\right)}, \\
\vartheta_{6}(\beta)=\frac{\beta-\beta^{2}}{\beta^{1}-\beta} \cdot \frac{\beta^{1}}{\beta^{2}} .
\end{gathered}
$$

Proposition 5.2.7. Let $p^{2}=0$, the parameters $\beta, \beta^{1}, \beta^{2}, u^{1}, u^{2}$, have the form as above. Then the set $\left(p, x^{1}, x^{2}\right)$ is a semi-equilibrium state if and only if

$$
\begin{array}{cl}
p^{1}=\frac{1}{\beta \mathcal{L}} \cdot\left(\lambda^{1}+\lambda^{2}\right), & p^{2}=0, \\
x^{1}=\left(K^{1}, L^{1}\right): \quad K^{1}=\frac{\lambda^{1}}{p^{1}}, \quad L^{1} \geq \frac{\lambda^{1}}{p^{1} \cdot \beta^{1}}, \\
x^{2}=\left(K^{2}, L^{2}\right): \quad K^{2}=\frac{\lambda^{2}}{p^{1}}, \quad L^{2} \geq \frac{\lambda^{2}}{p^{1} \cdot \beta^{2}},
\end{array}
$$

and one of the following conditions is met:

$$
\begin{gathered}
\mu>0 \quad \text { by } \quad \beta<\min \left(\beta^{1}, \beta^{2}\right) ; \\
\mu \geq \vartheta_{6}(\beta) \quad \text { by } \quad \beta^{2}<\beta<\beta^{1} ; \\
\mu \leq \vartheta_{6}(\beta) \quad \text { by } \quad \beta^{1}<\beta<\beta^{2} ;
\end{gathered}
$$

where

$$
\mu=\frac{\lambda^{1}}{\lambda^{2}}, \quad \vartheta_{6}(\beta)=\frac{\beta-\beta^{2}}{\beta^{1}-\beta} \cdot \frac{\beta^{1}}{\beta^{2}} .
$$

Theorem 5.2.3. The model $M$ can have half-equilibria only of the type $\tilde{E}^{1}, \tilde{E}^{2}, \tilde{E}^{3}, \widetilde{E}^{4}, \widetilde{E}^{5}$, and the half-equilibrium $\tilde{E}^{1}$ is realized if and only

$$
\beta<\min \left(u^{1}, u^{2}\right), \quad \mu>0,
$$

or

$$
\beta^{2}<\beta<\beta^{1}, \quad \mu \geq \frac{\beta-\beta^{2}}{\beta^{1}-\beta} \cdot \frac{\beta^{1}}{\beta^{2}}
$$

or

$$
\beta^{1}<\beta<\beta^{2}, \quad \mu \leq \frac{\beta^{2}-\beta}{\beta-\beta^{1}} \cdot \frac{\beta^{1}}{\beta^{2}}
$$

where $\beta=\frac{\mathcal{K}}{\tilde{\mathcal{L}}}$;
half-equilibrium $\tilde{E}^{2}$ is realized if and only if

$$
\tilde{\beta}>\beta^{1}, \quad \frac{\beta^{1}+u^{2}}{\tilde{\beta}-\beta^{1}}<\mu<\frac{\beta^{1}+u^{1}}{\tilde{\beta}-\beta^{1}}
$$

where $\tilde{\beta}=\frac{\mathcal{K}}{\tilde{\mathcal{L}}} \tilde{\mathcal{L}} \leq \mathcal{L}$;
half-equilibrium $\tilde{E}^{3}$ is realized if and only if

$$
\tilde{\beta}>\beta^{2}, \quad \frac{\tilde{\beta}-\beta^{2}}{\beta^{2}+u^{2}}<\mu<\frac{\tilde{\beta}-\beta^{2}}{\beta^{2}+u^{1}}
$$

where $\tilde{\beta}=\frac{\mathcal{K}}{\tilde{\mathcal{L}}} \tilde{\mathcal{L}} \leq \mathcal{L}$;
half-equilibrium $\tilde{E}^{4}$ is realized if and only if

$$
\begin{gathered}
\beta^{2}<\tilde{\beta}<\beta^{1} \\
\frac{\tilde{\beta}-\beta^{2}}{\beta^{1}-\tilde{\beta}} \cdot \frac{\beta^{1}+\min \left(u^{1}, u^{2}\right)}{\beta^{2}+\min \left(u^{1}, u^{2}\right)}<\mu<\frac{\tilde{\beta}-\beta^{2}}{\beta^{1}-\tilde{\beta}} \cdot \frac{\beta^{1}}{\beta^{2}}
\end{gathered}
$$

or

$$
\begin{gathered}
\beta^{1}<\tilde{\beta}<\beta^{2} \\
\frac{\beta^{2}-\tilde{\beta}}{\tilde{\beta}-\beta^{1}} \cdot \frac{\beta^{1}}{\beta^{2}}<\mu<\frac{\beta^{2}-\tilde{\beta}}{\tilde{\beta}-\beta^{1}} \cdot \frac{\beta^{1}+\min \left(u^{1}, u^{2}\right)}{\beta^{2}+\min \left(u^{1}, u^{2}\right)^{\prime}}
\end{gathered}
$$

where $\tilde{\beta}=\frac{\mathcal{K}}{\tilde{\mathcal{L}}} \tilde{\mathcal{L}} \leq \mathcal{L}$
half-equilibrium $\tilde{E}^{5}$ is realized if and only if , $\beta>0 \mu>0$
where

$$
\beta=\frac{\mathcal{K}}{\tilde{\mathcal{L}}} .
$$

Fair

Theorem 5.2.4. The model $M$ can only have half-equilibria of the type $\tilde{\tilde{E}}^{1}, \tilde{\tilde{E}}^{2}, \tilde{\tilde{E}}^{3}, \tilde{\tilde{E}}^{4}$, and $\tilde{\tilde{E}}^{1}=\left(p, x^{1}, x^{2}\right)$. In this case, halfequilibrium, where

$$
\begin{gathered}
x^{1}=\frac{\lambda^{1}}{p^{1}}\left(1, \frac{1}{\beta^{1}}\right), \quad x^{2}=\frac{\lambda^{2}}{p^{1}}\left(1, \frac{1}{\beta^{2}}\right), \\
p^{1}=\frac{1}{\beta \mathcal{L}} \cdot\left(\lambda^{1}+\lambda^{2}\right), \quad p^{2}=0
\end{gathered}
$$

is realized if and only if

$$
\beta=\beta^{1}=\beta^{2}, \quad \mu>0
$$

where $\beta=\frac{\mathcal{K}}{\mathcal{L}}$
half-equilibrium $\tilde{\tilde{E}}^{2}=\left(p, x^{1}, x^{2}\right)$ where

$$
\begin{gathered}
x^{1}=\frac{\lambda^{1}}{p^{1} \beta^{1}+p^{2}}\left(\beta^{1}, 1\right), \quad x^{2}=\frac{\lambda^{2}}{p^{1}}(1,0), \\
p^{1}=\frac{\lambda^{2} \tilde{\tilde{\beta}}}{\widetilde{\mathcal{K}}\left(\tilde{\tilde{\beta}}-\beta^{1}\right)}, \quad p^{2}=\frac{\tilde{\tilde{\beta}}}{\widetilde{\mathcal{K}}}\left(\lambda^{1}-\frac{\lambda^{2} \beta^{1}}{\tilde{\tilde{\beta}}-\beta^{1}}\right)
\end{gathered}
$$

is realized if and only if

$$
\tilde{\tilde{\beta}}>\beta^{1}, \quad \frac{\beta^{1}+u^{2}}{\tilde{\tilde{\beta}}-\beta^{1}} \leq \mu \leq \frac{\beta^{1}+u^{1}}{\tilde{\tilde{\beta}}-\beta^{1}}
$$

where $\tilde{\tilde{\beta}}=\frac{\widetilde{\mathcal{K}}}{\mathcal{L}} \widetilde{\mathcal{K}} \leq \mathcal{K}$;
half-equilibrium $\tilde{\tilde{E}}^{3}=\left(p, x^{1}, x^{2}\right)$, where

$$
\begin{gathered}
x^{1}=\frac{\lambda^{1}}{p^{1}}(1,0), \quad x^{2}=\frac{\lambda^{2}}{p^{1} \beta^{2}+p^{2}}\left(\beta^{2}, 1\right), \\
p^{1}=\frac{\lambda^{1} \tilde{\tilde{\beta}}}{\widetilde{\mathcal{K}}\left(\tilde{\tilde{\beta}}-\beta^{2}\right)}, \quad p^{2}=\frac{\tilde{\tilde{\beta}}}{\widetilde{\mathcal{K}}}\left(\lambda^{2}-\frac{\lambda^{1} \beta^{2}}{\tilde{\tilde{\beta}}-\beta^{2}}\right)
\end{gathered}
$$

is realized if and only if

$$
\tilde{\tilde{\beta}}>\beta^{2}, \quad \frac{\tilde{\tilde{\beta}}-\beta^{2}}{\beta^{2}+u^{2}} \leq \mu \leq \frac{\tilde{\tilde{\beta}}-\beta^{2}}{\beta^{2}+u^{1}}
$$

where $\tilde{\tilde{\beta}}=\frac{\widetilde{\mathcal{K}}}{\mathcal{L}} \widetilde{\mathcal{K}} \leq \mathcal{K}$;
half-equilibrium $\tilde{\tilde{E}}^{4}=\left(p, x^{1}, x^{2}\right)$ where

$$
\begin{gathered}
x^{1}=\frac{\lambda^{1}}{p^{1} \beta^{1}+p^{2}}\left(\beta^{1}, 1\right), \quad x^{2}=\frac{\lambda^{2}}{p^{1} \beta^{2}+p^{2}}\left(\beta^{2}, 1\right), \\
p^{1}=\frac{\tilde{\tilde{\beta}}}{\widetilde{\mathcal{K}}}\left(\frac{\lambda^{1}}{\tilde{\tilde{\beta}}-\beta^{2}}-\frac{\lambda^{2}}{\beta^{1}-\tilde{\tilde{\beta}}}\right), \quad p^{2}=\frac{\tilde{\tilde{\beta}}}{\widetilde{\mathcal{K}}}\left(\frac{\lambda^{2} \beta^{1}}{\beta^{1}-\tilde{\tilde{\beta}}}-\frac{\lambda^{1} \beta^{2}}{\tilde{\tilde{\beta}}-\beta^{2}}\right)
\end{gathered}
$$

is realized if and only if

$$
\begin{gathered}
\beta^{2}<\tilde{\tilde{\beta}}<\beta^{1} \\
\frac{\tilde{\tilde{\beta}}-\beta^{2}}{\beta^{1}-\tilde{\tilde{\beta}}} \cdot \frac{\beta^{1}+\min \left(u^{1}, u^{2}\right)}{\beta^{2}+\min \left(u^{1}, u^{2}\right)} \leq \mu \leq \frac{\tilde{\tilde{\beta}}-\beta^{2}}{\beta^{1}-\tilde{\tilde{\beta}}} \cdot \frac{\beta^{1}}{\beta^{2}} .
\end{gathered}
$$

If

$$
\begin{gathered}
\beta^{1}<\tilde{\tilde{\beta}}<\beta^{2} \\
\frac{\beta^{2}-\tilde{\tilde{\beta}}}{\tilde{\tilde{\beta}}-\beta^{1}} \cdot \frac{\beta^{1}}{\beta^{2}} \leq \mu \leq \frac{\tilde{\tilde{\beta}}-\beta^{2}}{\beta^{1}-\tilde{\tilde{\beta}}} \cdot \frac{\beta^{1}+\min \left(u^{1}, u^{2}\right)}{\beta^{2}+\min \left(u^{1}, u^{2}\right)}
\end{gathered}
$$

where $\tilde{\tilde{\beta}}=\frac{\widetilde{\mathcal{K}}}{\mathcal{L}} \widetilde{\mathcal{K}} \leq \mathcal{K}$.
Effective trajectories of the model $Z^{2}$ are discussed in the third paragraph of Chapter V.

Let's consider a trajectory $\left(x_{t}\right)_{t=0}^{\infty}$ whose vectors $x_{t}=$ $\left(K_{t}^{1}, L_{t}^{1}, K_{t}^{2}, L_{t}^{2}\right)\left(K_{t}^{i}>0, L_{t}^{i}>0, i=1,2\right)$ are strictly positive. Recall that a trajectory $\left(x_{t}\right)_{t=0}^{\infty}$ is optimal (effective) if it admits the characteristic $\left(\tilde{\ell}_{t}\right)$.

Let the trajectory $\left(x_{t}\right)_{t=0}^{\infty}$ admit the characteristic $\left(\tilde{\ell}_{t}\right)$. Then there are numbers $b_{t}^{1} \geq 0, b_{t}^{2} \geq 0$ such that

$$
\tilde{\ell}_{t}=\left(b_{t}^{1}, b_{t}^{2}, \quad b_{t}^{1}, \quad b_{t}^{2}\right)
$$

Let, $\ell_{t}^{\Delta}=\left(b_{t}^{1}, b_{t}^{2}\right), \widetilde{\ell}_{t}=\left(\ell_{t}^{\Delta}, \ell_{t}^{\Delta}\right)$, i.e. product prices do not depend on the division where they are considered. Let us assume that $b_{t}^{1}>0$ for all $t$ and write the vector $\ell_{t}^{\Delta}$ in the form

$$
\ell_{t}^{\Delta}=b_{t}^{1} \ell_{t}
$$

$$
\begin{equation*}
\ell_{t}=\left(1, \quad b_{t} \omega_{t}\right) . \tag{37}
\end{equation*}
$$

Let's put

$$
\begin{gathered}
U_{t}^{1}\left(\ell_{t+1}, x\right)=v_{t}^{1} K+\min \left(\frac{K}{c_{t}^{11}}, \frac{L}{c_{t}^{21}}\right), \\
U_{t}^{2}\left(\ell_{t+1}, x\right)=v_{t}^{2} K+b_{t+1} \min \left(\frac{K}{c_{t}^{12}}, \frac{L}{c_{t}^{22}}\right),
\end{gathered}
$$

where $c_{t}^{i j}>0(i, j=1,2 ; \quad t=0,1,2, \ldots)$ are the given numbers.
Theorem 5.3.1. Let be the trajectory $\left(x_{t}\right)_{t=0}^{\infty}$ of the model $Z^{2}$ admitting the characteristic $\left(\tilde{\ell}_{t}\right)$. Let us assume that, $K_{t}^{i}>$ $0, L_{t}^{i}>0(i=1,2)$ and the numbers $b_{t}$ are defined by equality (34). Then

$$
b_{t+1}=V_{t}\left(b_{t}\right) .
$$

Lemma 5.3.2. If for everyone $t=0,1,2, \ldots$

$$
\begin{gathered}
v_{t}^{2}-v_{t}^{1}<\frac{1}{c_{t}^{11}} \\
\beta_{t}^{1}>\beta_{t}^{2}, \\
d_{t}^{2} \leq c_{2}=\text { const },
\end{gathered}
$$

Theorem 5.3.2. Let assumptions (37) and the conditions of Lemma 5.3.2 be satisfied. Then the sequence $\left\{b_{t}\right\}$ converges to $\bar{b}$ where $\bar{b}$ is defined in Lemma 5.2.1.

The following paragraphs discuss the same type of reproduction model as in previous chapters, but with arbitrary production functions. It is shown that in the two-sector model there are prices different from the Neumann ones at which a balanced growth of divisions is possible. A description of the Neumann face in the case of non-degeneracy is also given.

The fourth paragraph of Chapter V provides a description of the two-sector model of economic dynamics. The first sector produces means of production, the second sector produces consumer goods.

The two-sector model $Z$ is specified using the production mapping $a(x)$ :

$$
a\left(K_{1}, L_{1}, K_{2}, L_{2}\right)=\left\{\left(K_{1}^{\prime}, L_{1}^{\prime}, K_{2}^{\prime}, L_{2}^{\prime}\right) \mid K_{1}^{\prime} \leq v_{1} K_{1}+F_{1}\left(K_{1}, L_{1}\right) u_{1},\right.
$$

$$
\begin{aligned}
& K_{2}^{\prime} \leq v_{2} K_{2}+F_{1}\left(K_{1}, L_{1}\right) u_{2}, \quad u_{1}+u_{2} \leq 1, \quad L_{1}^{\prime} \leq F_{2}\left(K_{2}, L_{2}\right) v_{1}, \\
& \left.L_{2}^{\prime} \leq F_{2}\left(K_{2}, L_{2}\right) v_{2}, \quad \omega_{1} v_{1}+\omega_{2} v_{2} \leq 1\right\} .
\end{aligned}
$$

It is assumed that the production function is given on a cone $R_{+}^{2}$ and is non-negative and superlinear there. Besides, $F_{i}(K, 0)=$ $F_{i}(0, L)=0$. Under these assumptions, the mapping $a(x)$ is superlinear and, therefore, $Z$ a Neumann-Gale model.

Let us introduce the notation

$$
\begin{gathered}
f_{i}(\eta)=F_{i}(\eta, 1), \quad(i=1,2), \quad \eta=\frac{K}{L} \\
p=\left(p^{11}, p^{12} \omega_{1}, p^{21}, \quad p^{22} \omega_{2}\right)
\end{gathered}
$$

where $p^{1 i}(i=1,2)$ is the price of a unit of funds in the first and second sectors has the same meaning $p^{i 2}(i=1,2)$

When studying a two-sector model $Z$, we will need singleproduct models $Z_{1}$ and $Z_{2}$, which are specified by production mappings $a_{1}(x)$ and $a_{2}$, accordingly, defined on the cone $R_{+}^{2}$ using the formulas:

$$
\begin{gathered}
a_{1}\left(K_{1}, L_{1}\right)=\left\{\left(K_{1}^{\prime}, L_{1}^{\prime}\right) \mid 0 \leq K_{1}^{\prime} \leq v_{1} K_{1}+u_{1}, u_{1} \geq 0\right. \\
u_{1}+\widetilde{\omega}_{1} L_{1}^{\prime} \leq \tilde{F}_{1}\left(K_{1}, L_{1}\right), \widetilde{\omega}_{1}=\frac{p^{12}}{p^{11}} \omega_{1} \\
\left.\tilde{F}_{1}\left(K_{1}, L_{1}\right)=\max \left(1, \frac{p^{21}}{p^{11}}\right) F_{1}\left(K_{1}, L_{1}\right)\right\} \\
a_{2}\left(K_{2}, L_{2}\right)=\left\{\left(K_{2}^{\prime}, L_{2}^{\prime}\right) \mid 0 \leq K_{2}^{\prime} \leq v_{2} K_{2}+u_{2}, u_{2}\right. \\
\geq 0, \quad u_{2}+\widetilde{\omega}_{2} L_{2}^{\prime} \leq \widetilde{F}_{2}\left(K_{2}, L_{2}\right) \\
\left.\widetilde{\omega}_{2}=\frac{p^{22}}{p^{21}} \omega_{2}, \quad \widetilde{F}_{2}\left(K_{2}, L_{2}\right)=\max \left(\frac{p^{12}}{p^{21}}, \frac{p^{22}}{p^{21}}\right) F_{2}\left(K_{2}, L_{2}\right)\right\}
\end{gathered}
$$

Let us consider the models $Z_{i}(i=1,2)$, for $\widetilde{\omega}_{i} v_{i} \leq \tilde{s}_{i}$, where $\widetilde{s}_{i}=\lim _{\eta \rightarrow+\infty} \tilde{f}_{i}(\eta), \tilde{f}_{i}(\eta)=\widetilde{F}_{i}(\eta, 1)$. It is easy to verify that the Neumann growth rates $\alpha_{i}$ of these models are calculated by the formula

$$
\begin{equation*}
\alpha_{i}=\max _{K, L>0} \frac{v_{i} K+\tilde{F}_{i}(K, L)}{K+\widetilde{\omega}_{i} L}=\max _{\eta>0} \frac{v_{i} \eta+\tilde{f}_{i}(\eta)}{\eta+\widetilde{\omega}_{i}}(i=1,2) \tag{38}
\end{equation*}
$$

Lemma 5.4.1. Let vector $p=\left(p^{11}, p^{12} \omega_{1}, p^{21}, p^{22} \omega_{2}\right)$ the price vector $a(x)$ be the production display of the model $Z$. Then the relation is true for the $x=\left(K_{1}, L_{1}, K_{2}, L_{2}\right)$ :

$$
\begin{gathered}
\max [p, y]=p^{11} v_{1} K_{1}+p^{21} v_{2} K_{2}+\max \left(p^{11}, p^{21}\right) F_{1}\left(K_{1}, L_{1}\right)+ \\
+\max \left(p^{12}, p^{22}\right) F_{2}\left(K_{2}, L_{2}\right)
\end{gathered}
$$

Proposition 5.4.1. Equality is fair

$$
\alpha=\max \left(\alpha_{1}, \alpha_{2}\right)
$$

where $\alpha, \alpha_{1}, \alpha_{2}$ are the Neumann growth rates of the models $Z, Z_{1}, Z_{2}$, , respectivel.

Theorem 5.4.1. Let. $\bar{x}=\left(\bar{K}_{1}, \bar{L}_{1}, \bar{K}_{2}, \bar{L}_{2}\right)$ be the equilibrium vector, $p=\left(p^{11}, p^{12} \omega_{1}, p^{21}, p^{22} \omega_{2}\right),\left(p^{11}>0, p^{21}>0\right)$ be the equilibrium prices, number $\alpha>0$ be the Neumann growth rate $\bar{K}_{i}+$ $\omega_{i} \bar{L}_{i}=1(i=1,2), \omega_{i} v_{i} \leq s_{i}, s_{i}=\lim _{\eta \rightarrow+\infty} f_{i}\left(\eta_{i}\right)$

Then

1) if $\alpha_{1}>\alpha_{2}, p^{22}>0$, then the equilibrium vector of the model $Z$ has the form $\bar{x}=(1,0,0,0)$, and the vector $\bar{x}_{1}=(1,0)$ is the equilibrium vector in the single-product model $Z_{1}$.
2) if $\alpha_{2}>\alpha_{1}, p^{12}>0$, then $\bar{x}=(0,0,1,0)$, and $\bar{x}_{2}=(1,0)$ is the equilibrium vector in the model $Z_{2}$.
3) if $\alpha=\alpha_{1}=\alpha_{2}$, then the Neumann equilibrium vector $\bar{x}=$ ( $\bar{K}_{1}, \bar{L}_{1}, \bar{K}_{2}, \bar{L}_{2}$ ) is determined by the relations

$$
\frac{\overline{\bar{K}}_{i}}{\bar{L}_{i}}=\bar{\eta}_{i}, \quad \frac{\bar{L}_{1}}{\bar{L}_{2}}=\frac{\left(\alpha-v_{2}\right) \bar{\eta}_{2}}{\alpha \omega_{1}}
$$

where $\bar{\eta}_{i}$ is the point at which the maximum in (35) is achieved.
In this case, there are Neumann prices of the form $p=$ $\left(1, b \omega_{1}, 1, b \omega_{2}\right)$, where $b$ is the solution to the equation $\alpha_{1}(b)=$ $\alpha_{2}(b)$.

In the fifth section of Chapter V, non-degenerate Neumann equilibrium is studied. Neumann equilibrium is characterized by the fact that $\alpha_{1}(p)=\alpha_{2}(p)$ where

$$
\begin{aligned}
& \alpha_{1}(p)=g\left(b_{1}, c\right), \\
& \alpha_{2}(p)=h\left(b_{2}, d\right)
\end{aligned}
$$

where $b_{1}, b_{2}, c, d$, are defined in the first paragraph of Chapter V.
It would be interesting to consider such prices

$$
p=\left(p^{11}, p^{12} \omega_{1}, p^{21}, p^{22} \omega_{2}\right)\left(p^{11}>0, p^{21}>0\right)
$$

not necessarily Neumann ones, which $\alpha_{1}(p)=\alpha_{2}(p)$. Let us find out at what prices $p$ the equality $\alpha_{1}(p)=\alpha_{2}(p)$ is valid. Let's introduce new variables

$$
q_{1}=\frac{p^{12}}{p^{11}}, \quad q_{2}=\frac{p^{22}}{p^{21}}, \quad q_{3}=\frac{p^{12}}{p^{21}}
$$

Let's look at the functions

$$
\begin{array}{lll}
\beta_{1}(q)=g\left(b_{1}, c\right), & \text { где } \quad b_{1}=q_{1}, & c=\max \left(1, \frac{q_{1}}{q_{3}}\right) \\
\beta_{2}(q)=h\left(b_{2}, d\right), & \text { где } \quad b_{2}=q_{2}, & d=\max \left(q_{3}, q_{2}\right)
\end{array}
$$

Let's put:

$$
c=h\left(q_{3}\right),
$$

where $h\left(q_{3}\right)=h\left(q_{2}, q_{3}\right)$.
Theorem 5.5.1. Let $\omega_{i} v_{i}<s_{i}(i=1,2)$

1) The set of points $\left(q_{1}, q_{3}\right)$ represents the set of solutions to the inequality

$$
\beta_{1}\left(q_{1}, q_{3}\right) \geq c
$$

2) If $\left(q_{1}, q_{3}\right)$ satisfies the strict inequality

$$
\beta_{1}\left(q_{1}, q_{3}\right)>c
$$

then $q_{2}$ there are two values at which the conditions are met

$$
\beta_{1}\left(q_{1}, q_{3}\right)=\beta_{2}\left(q_{2}, q_{3}\right)
$$

3) If $\left(q_{1}, q_{3}\right)$ they satisfy the equality

$$
\beta_{1}\left(q_{1}, q_{3}\right)=c,
$$

then $q_{1}=q_{3}$ there is only one solution - Neumann prices.
Comment. For each $q_{3}$ equation $q_{3}$

$$
\beta_{2}\left(q_{2}, q_{3}\right)=\beta_{1}\left(q_{1}, q_{3}\right)
$$

it makes sense to consider for $q_{1}$ such that $\beta_{1}\left(q_{1}, q_{3}\right)=a \geq d\left(q_{3}\right)$. Since $\inf d\left(q_{3}\right)=v_{1}<c$, then $q_{1}, q_{3}$ there are such, that $\beta_{1}\left(q_{1}, q_{3}\right)<c$. For these $\left(q_{1}, q_{3}\right)$, there is no such thing for $q_{2}$ which the equation has a solution.

If $\beta_{1}\left(q_{1}, q_{3}\right)=a=c$, then $q_{2}$ there is a unique solution to the equation. In this case, $p=\left(p^{11}, p^{12} \omega_{1}, p^{21}, p^{22} \omega_{2}\right)$ are the Neumann prices, where $q_{1}=\frac{p^{12}}{p^{11}}, q_{2}=\frac{p^{22}}{p^{21}}, q_{3}=\frac{p^{12}}{p^{21}}$ and for $\beta_{1}\left(q_{1}, q_{3}\right)=a>c$, then $q_{2}$ there are two solutions to the equation Based on the above, we find that there is no such function that would express one of the vector $q=\left(q_{1}, q_{2}, q_{3}\right)$ coordinates in terms of the other two, since it has two values.

The sixth paragraph of the fifth Chapter is devoted to Neumann faces, i.e. we will be interested only in the non-degenerate case, namely $\alpha_{1}=\alpha_{2}$

As is known, many

$$
N_{\alpha}=K \cap H_{p},
$$

where $H_{p}=\{(x, y) \mid[p, y]=\alpha[p, x]\}$, $p$ are Neumann prices, is called the Neumann edge of a given equilibrium state, where $K$ is the cone of the model $Z$. Let's construct a Neumann face $N_{\alpha}$ for the model $Z$. Let, $x=\left(K_{1}, L_{1}, K_{2}, L_{2}\right) p=\left(p^{11}, p^{12} \omega_{1}, p^{21}, p^{22} \omega_{2}\right)=$ (1, $b \omega_{1}, 1, b \omega_{2}$ ) be Neumann prices.

A-priory

$$
N_{\alpha}=\{(x, y) \in Z \mid[p, y]=\alpha[p, x]\} .
$$

Theorem 5.6.1. Let $\alpha=\alpha_{1}=\alpha_{2}, p=\left(1, b \omega_{1}, 1, b \omega_{2}\right)$, be Neumann prices. Then the Neumann edge has the following form:

$$
=\left\{(x, y) \mid x=\left(\lambda\left(\bar{K}_{1}, \bar{L}_{1}\right), \mu\left(\bar{K}_{2}, \bar{L}_{2}\right)\right), \forall y \in \tilde{a}(x), \lambda \geq 0, \mu \geq 0\right\} .
$$

The sixth Chapter examines models of the economic dynamics of production and exchange of the Neumann type, defined on a graph. The model is defined by a digraph, each vertex of which is associated with some superlinear multivalued mapping that describes the technological capabilities of some economic unit. The existence of an arc from vertex to vertex means the possibility of transporting products from one vertex to another. A dual model is calculated and with its help the characteristics of optimal trajectories are found.

The first paragraph explores a model of economic dynamics that describes the time behavior of an economic system $S$ consisting of a finite number of production units $s_{i}, i=\overline{1, m}$. The case is considered when transport costs are not taken into account. Each production unit is specified by a Neumann-Gale model defined by a superlinear set-valued mapping $a_{i}: R_{+}^{n} \rightarrow \pi\left(R_{+}^{n}\right)$. Therefore, it is assumed that the phase space is the same for all models. Exchange of products is allowed between some of these production units. To formally describe the model, we introduce the graph $P(J, Q)$, where $P(J, Q)$ is a loop-free graph, $J$ is a set of vertices, and $Q$ is a multivalued mapping $J$ in $J$. More precisely, $Q(j)$ this is a set of vertices $k \in J$ such that there is an arc from vertex $j$ to vertex $k$ and $Q^{-1}(j)$ a set to vertex $k \in J$. By an arc from vertex $j$ to vertex $k w e$ mean an ordered. The presence of an arc means the possibility of transportation from. The model deals with the same products as models, but it is more convenient to distinguish between products corresponding to different vertices of the graph.

In this regard, we will assume that the phase space of the model coincides with the cone $\left(R_{+}^{n}\right)^{m}$. If $X=\left(x_{1}, \ldots, x_{m}\right) \in\left(R_{+}^{n}\right)^{m}$, then the element $x_{i} \in R_{+}^{n}$ is interpreted as a set of products available at the production site $i$. The mappings $A$ and $B$, describing production and exchange respectively, are defined as follows:

$$
\begin{gathered}
A(X)=\left\{Y \mid Y=\left(y_{1}, \ldots, y_{m}\right), y_{i} \in a_{i}\left(x_{i}\right), i=\overline{1, m}\right\} \\
B(Y)=\left\{Z \mid Z=\left(z_{1}, \ldots, z_{m}\right), z_{i}=y_{i}+\sum_{j \in Q(i)} u_{j i}-\sum_{k \in Q^{-1}(i)} u_{i k},\right. \\
\left.u_{i j} \geq 0, \sum_{k \in Q^{-1}(i)} u_{i k} \leq y_{i}\right\} .
\end{gathered}
$$

Vector $u_{j i}$ describes products transported in an $\operatorname{arc}(j, i)$. It is easy to verify that the mappings $A$ and $B$ are superlinear.

If $M \in R_{+}^{n} \times R_{+}^{n}$ the Neumann-Gale model, then the dual model $M^{\prime}$ is defined as follows

$$
M^{\prime}=\left\{(f, g) \in\left(R_{+}^{n}\right)^{*} \times\left(R_{+}^{n}\right)^{*} \mid[f, x] \geq[g, y] \text { for all }(x, y) \in M\right\}
$$

Let us give a description of the model dual to $S$. For this purpose, we describe the mappings $A^{\prime}$ and $B^{\prime}$ dual to $A$ and $B$, respectively. In what follows we will denote the cone $\left(R_{+}^{n}\right)^{m}$ by $K$. Then $K^{*}=\left(\left(R_{+}^{n}\right)^{m}\right)^{*}$. We denote the elements of the cones $K$ and $K^{*}$ by capital letters, and the projections of these elements onto the $i$ - th factor by the corresponding lowercase letters with an index $i$. Thus, if $\in K$ then $X=\left(x_{1}, \ldots, x_{m}\right)$. We will denote $[F, X]$ the value of the functionality $F \in K^{*}$ on the element $X \in K$. In other words

$$
[F, X]=\sum_{i}\left[f_{i}, x_{i}\right]
$$

Proposition 6.1.1. Equality is fair

$$
A^{\prime}(F)=\left\{G \in K^{*} \mid g_{i} \in a_{i}^{\prime}\left(f_{i}\right), i=\overline{1, m}\right\}, F \in K^{*}
$$

Lemma 6.1.1. If $H \in K^{*}, Z \in B(Y)$, where

$$
\begin{gathered}
z_{i}=y_{i}+\sum_{j \in Q(i)} u_{j i}-\sum_{k \in Q^{-1}(i)} u_{i k} \\
{[H, Z]=\sum_{i}\left[h_{i}, y_{i}\right]+\sum_{i=1}^{m} \sum_{k \in Q^{-1}(i)}\left[h_{k}-h_{i}, u_{i k}\right]}
\end{gathered}
$$

Proposition 6.1.3. Let $H \in B^{\prime}(G), \bar{Z} \in B(\bar{Y})$ it be, and

$$
\bar{z}_{i}=\bar{y}_{i}+\sum_{j \in Q(i)} \bar{u}_{j i}-\sum_{k \in Q^{-1}(i)} \bar{u}_{i k}, i=\overline{1, m}
$$

Equality $[H, \bar{Z}]=[G, \bar{Y}]$ occurs if and only if

$$
\left[g_{i}-h_{i}, \bar{y}_{i}-\sum_{k \in Q^{-1}(i)} \bar{u}_{i k}\right]=0,\left[g_{i}-h_{k}, \bar{u}_{i k}\right]=0,(i, k) \in Q .
$$

Let us formulate the results obtained in the form of a theorem.
Theorem 6.1.1. In order for the price trajectory to be a characteristic of the trajectory, it is necessary and sufficient for the following conditions to be met:

$$
\begin{gathered}
g_{i}(t) \in a_{i}^{\prime}\left(f_{i}(t-1)\right), \\
{\left[f_{i}(t-1), x_{i}(t-1)\right]=\left[g_{i}(t), y_{i}(t)\right], i=\overline{1, m},} \\
g_{l}(t) f_{j}(t), j \in Q^{-1}(l) \cup\{l\}\left[f_{i}(t)-g_{i}(t), \bar{y}_{i}(t)-\right. \\
\left.\sum_{k \in Q(i)} \bar{u}_{i k}(t)\right]=0, i=\overline{1, m}, \\
{\left[f_{i}(t)-g_{k}(t), \bar{u}_{i k}(t)\right]=0, \quad(i, k) \in Q, t=\overline{1, T} .}
\end{gathered}
$$

Theorem 6.1.2. Let the triple $(\alpha, \bar{X}, F)$ be the equilibrium state of the model $S$, and let the elements $\bar{u}_{j i}$ and $g_{l}$, and functionals $g_{l}$ be determined by relations

$$
\bar{y}_{i}(x) \in a_{i}\left(\bar{x}_{i}\right), \sum_{k \in Q^{-1}(i)} u_{i k} \leq \bar{y}_{i}, i=\overline{1, m}
$$

and

$$
g_{l}(t) \in a_{l}^{\prime}\left(f_{l}\right), \quad l=\overline{1, m}, \frac{1}{\alpha} f_{j} \leq g_{l}, j \in Q^{-1}(l) \cup\{l\},
$$

respectively. Then

$$
\begin{gathered}
{\left[g_{i}-\frac{1}{\alpha} f_{i}, \bar{x}_{i}\right]=0, i=\overline{1, m},} \\
{\left[g_{i}-\frac{1}{\alpha} f_{i}, \bar{u}_{j i}\right]=0, j \in Q(i), i=\overline{1, m},} \\
{\left[f_{i}, \bar{u}_{j i}\right]=\left[f_{j}, \bar{u}_{j i}\right], j \in Q(i), i=\overline{1, m} .}
\end{gathered}
$$

The following statements are true.

Proposition 6.1.4. Let $Q=\left(g^{1}, g^{2} \ldots . g^{m}\right) \in\left(R_{+}^{n}\right)^{m}$
Then $A^{*}(Q)=$.
Let us introduce the following notation. Let $H=$ $\left(h^{1}, \ldots . h^{m}\right) \in\left(R_{+}^{n}\right)^{m}$. Let's put

$$
q^{i}(H)=\operatorname{Sup}_{i \in G(j)}\left(\left(c^{j i}\right) T^{*} h^{i}\right)
$$

Here the supremum of vectors $\left(c^{j i}\right)^{*} h$ is calculated coordinatewise ( ${ }^{*}$-sign of matrix transposition). Let further

$$
K(H)=\left(q^{1}(H), \ldots . q^{m}(H)\right)
$$

In the second paragraph of the sixth Chapter models of reproduction and exchange on the graph are considered, taking into account transport costs. The effective trajectories of such models are studied. In this case, the simplest equilibrium mechanisms are used.

In we take into account the mapping $B$ that describes the exchange relation in the simulated system:

$$
\begin{aligned}
& \mathrm{B}(Y)=\left\{Z=\left(Z^{1}, \ldots, Z^{m}\right) \mid Z^{k}=\sum_{j \in \Gamma^{-1}(k)} C^{j k} u^{j k},\right. \\
& k=1,2, \ldots, m ; u^{i j} \geq 0 \text { in front of everyone }(i, j) \in \Gamma, \\
& \left.\quad \sum_{j \in \Gamma(i)} u^{i j}=y^{i}, \quad i=1,2, \ldots, m\right\} .
\end{aligned}
$$

Production capabilities of the entire system are given by the mapping $A$ defined on the cone $\left(R_{+}^{n}\right)^{m}$. If $x=\left(x^{1}, \ldots, x^{m}\right) \in\left(R_{+}^{n}\right)^{m}$, then

$$
A(x)=a^{1}\left(x^{1}\right) \times a^{2}\left(x^{2}\right) \times \ldots \times a^{m}\left(x^{m}\right)
$$

in other words

$$
A(x)=\left\{y=\left(y^{1}, \ldots, y^{1}\right) \mid y^{i} \in a^{i}\left(x^{i}\right), i=\overline{1, m}\right\}
$$

The functioning of the entire system comes down to production and exchange. If production activity is considered first,
and then exchange, then the functioning of the system is described by composition $a=B \circ A$

$$
a(x)=\bigcup_{y \in A(x)} B(y), \quad x \in\left(R_{+}^{n}\right)^{m}
$$

If exchange occurs first, and then production, then composition $b=A \circ B$ is considered

$$
b(y)=\bigcup_{z \in B(y)} A(z), \quad y \in\left(R_{+}^{n}\right)^{m}
$$

Under natural assumptions, an optimal trajectory in the sense $F$ of admitting the characteristic, then there exists such a sequence

$$
\begin{gathered}
F_{0}, F_{1}, \ldots, F_{T} \text {, what } F_{T}=F \\
{\left[F, X_{0}\right]=\cdots=\left[F_{T}, X_{T}\right],\left[F, \tilde{X}_{0}\right] \geq \cdots \geq\left[F_{T}, \tilde{X}_{T}\right]}
\end{gathered}
$$

for any $T$-steppertrajectories $\tilde{X}_{0}, \ldots, \tilde{X}_{T}$ where $F$ satisfy the condition.
The following statements are true.
Proposition 6.2.2. Let $H=\left(h^{1}, \ldots . h^{m}\right) \in\left(R_{+}^{n}\right)^{m}$ it be then $B^{*}(H)=K(H)+\left(R_{+}^{n}\right)^{m}, \quad$ in other words, $B^{*}(H)=$ $\left\{Q=\left(g^{1}, \ldots g^{m}\right) \mid g^{k} \geq q^{k}(H), k=\overline{1, m}\right\}$,

$$
\left[F, X_{T}\right]=\max \left[F, \tilde{X}_{T}\right]
$$

Proposition 6.2.3. Let $\bar{Z} \in B(\bar{Y}) \quad$ where, $\quad \bar{Z}=$ $\left(\bar{z}^{1}, \bar{z}^{2}, \ldots, \bar{z}^{m}\right), \bar{Y}=\left(\bar{y}^{1}, \bar{y}^{2}, \ldots, \bar{y}^{m}\right)$ and the elements $\bar{u}^{j i}(i, j \in$ $G)$ are such that

$$
\bar{y}^{j}=\sum_{i \in G(j)} \bar{u}^{j i}, \bar{z}^{i}=\sum_{j \in G^{-1}(k)} C^{j i} \bar{u}^{j i}
$$

Let further, if $Q \in B^{*}(H)$, where $Q=\left(g^{1}, g^{2}, \ldots, g^{m}\right), H=$ $\left(h^{1}, h^{2}, \ldots, h^{m}\right)$. Then the equality $[\bar{H}, \bar{Z}]=[Q, \bar{Y}]$ is valid if and only if

$$
\left[q^{i}-\left(C^{j i}\right)^{*} h^{i}, \bar{u}^{j i}\right]=0 q^{i}-\left(C^{j i}\right)^{*} h^{i}, \bar{u}^{j i}
$$

for all $(j, i) \in G$.

Theorem 6.2.1. The sequence $F_{0}, F_{1}, \ldots, F_{T}$, where $F_{t}=$ $\left(f_{t}^{1}, \ldots, f_{t}^{m}\right)$ is a characteristic of the trajectory $X_{0}, X_{1}, \ldots, X_{T}$ if and only if

$$
\begin{aligned}
& \text { 1. } f_{t-1}^{1} \in\left(a_{t}^{i}\right)^{*}\left(q\left(F_{t}\right)\right), \quad t=1,2, \ldots, T ; \quad i=\overline{1, m} \\
& \text { 2. }\left[q^{i}\left(F_{t}\right)-\left(c^{i j}\right)^{*} f_{i}^{j}, u_{t}^{j i}\right]=0, \quad(\forall(j, i) \in \Gamma)
\end{aligned}
$$

Definition. The equilibrium state of the model $(U, X)$ is the set (Z,H), where $Z=\left(z^{1}, z^{2}, \ldots, z^{m}\right), H=\left(h^{1}, h^{2}, \ldots, h^{m}\right)$ that here $z^{i}$ the vector of resources, $h^{i}$ is the vector of prices, and $z^{i}$ is a solution to the problem $\frac{u^{i}(Z)}{\left[h^{i}, Z\right]} \rightarrow \max$ under the condition $\mathrm{Z} \geq 0$ and there $u^{j i}(j, i \in G$ are vectors with the property that

$$
\sum_{j \in G(i)} u^{j i}=x^{j}, \sum_{j \in G^{-1}(i)}^{m} C^{j i} u^{j i}=z^{i}
$$

In this case, the vector $H=\left(h^{1}, h^{2}, \ldots, h^{m}\right)$ is related to vectors $u^{j i}$ by relations of the type $\left[q^{j}(H)-\left(C^{j i}\right)^{*} h, u^{j i}\right]=0$, where $q^{j}(H)$ is the vector defined by formula (4).

Let there be a graph ( $\mathrm{J}, \mathrm{G}$ ) equipped with a matrix system $C^{j i}(j, i \in G)$, and each $i$ is associated with a vector of resources $x^{i}$ and a utility function $U^{i}$. Using the characteristic theorem, it is possible, under certain assumptions, to prove the existence in the model ( $\mathrm{U}, \mathrm{X}$ ) of an equilibrium $(Z, H)$, which has the additional property that the value of the problem $\max \frac{U^{i}(Z)}{\left[\mathrm{h}^{i}, Z\right]}$ coincides with either zero or one for all $i$.

Let us assume that the vectors $\mathrm{x}^{i}$ are strictly positive and consider a one-step trajectory of the model $Z$; starting from point X and maximizing the price vector $Q$ on the set $\mathrm{b}(\mathrm{X})$; here, as above, $\mathrm{b}=\mathrm{A} \circ \mathrm{B}$, the mapping B is defined using the graph $(\mathrm{J}, \mathrm{G})$ and matrices according to formula (1), and the mapping A is defined using the mapping $a^{i}$ according to formula (2). Let the
indicated trajectory have the form $(\mathrm{X}, Y)$. Then there is a vector Z with the property that $Z \in \mathrm{~B}(\mathrm{X}), Y \in \mathrm{~A}(\mathrm{Z})$.

According to well-known theorems, there $F$ is a price vector such that the pair $(F, Q)$ is a characteristic of the trajectory $(\mathrm{X}, Y)$. There $H$ is a vector such that

$$
F \in B^{*}(H), H \in A^{*}(Q) \quad \text { and } \quad[F, \mathrm{X}]=[H, \mathrm{Z}]=
$$

$[Q, \mathrm{y}]$.
It is clear that the pair $[\mathrm{Z}, H]$ is an equilibrium.

## MAIN RESULTS

- Characteristic prices have been found for some Neumanntype models.
- For some trajectories with certain properties, a theorem on the growth rate is proven and a description of effective trajectories is given.
- The principle of optimality for maximizing consumption is proposed and two ways of distributing labor are proposed.
- For trajectories with strict equilibria, a theorem on asymptotics has been proven.
- The conditions for introducing new technologies have been determined.
-For the Cobb-Douglas and CES production functions, the dependence of the consumption function on the type of production functions is determined.
- The dependence of the consumption function on the means of production is determined.
-The conditions for maximizing some macro-indicators were found.
- Trajectories were studied at a constant accumulation rate.
- A necessary and sufficient maximum condition for the utility function is proved and superdifferentials are found.
- A necessary and sufficient condition for the existence of a solution to the consumer problem has been proven.
- A necessary and sufficient condition for the existence of a solution to the consumer problem in equilibrium without losses has been proven.
- A necessary and sufficient condition for the existence of equilibrium prices without losses has been proven.
- The connection between the Neumann equilibrium condition and lossless equilibrium is determined.
- Conditions have been determined under which constructing an effective trajectory from the starting point is impossible.
- For the two-sector model, using the equilibrium mechanism, the Neumann growth rate, equilibrium prices and equilibrium vectors are found.
- Necessary and sufficient conditions for equilibrium in the two-sector model have been proven.
- The types of equilibrium and semi-equilibrium for the twosector model are determined.
- Neumann faces were found for the non-degenerate case.
- Effective trajectories of models of economic dynamics of production and exchange on graphs were constructed, taking into account and without taking into account transport costs


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