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ABSTRACT

of the Dissertation for the degree of Doctor of Philosophy

DIRECT AND INVERSE PROBLEMS OF SCATTERING FOR THE SCHRÖDINGER EQUATION WITH ADDITIONAL GROWING POTENTIAL

Speciality: 1211.01 – Differential equations
Field of science: Mathematics

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Baku – 2024

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BAKI DÖVLƏT UNİVERSİTETİ p.h.ş.
ELMİ KATİB
SCIENTIFIC SECRETARY
İmzanı təsdiq edirəm
"13" 02 2024

GENERAL CHARACTERISTICS OF THE RESEARCH

Research issue rationale and development rate. About 250 years ago, D. Bernoulli and L. Euler's research on the vibrating string equation laid the foundation for the study of the Sturm-Liouville operator $L(y) = -y'' + q(x)y$ (this operator is also called the one-dimensional Schrödinger operator when considered in the infinite range). Using the ideas proposed by D. Bernoulli, L. Euler, and J. Dalamber, the spectral theory of linear differential operators began to emerge in the scientific works of J. Fourier. Transformation operators played a special role in the development of the spectral theory of Sturm-Liouville operators. These operators originated from the ideas of the theory of generalized shift operators founded by J. Delsart and B. M. Levitan. Transformation operators for arbitrary Sturm-Liouville equations were established by A.Y. Povzner. V. A. Marchenko used transformation operators in studying the spectral properties of the singular Sturm-Liouville operator and finding the asymptotics of the spectral function. B. Y. Levin established a new type of transformation operator that preserves the asymptotics of solutions at infinity. The peculiarity of this transformation operator was that its existence was not derived from the ideas of the theory of generalized shift operators.

One of the important directions in the development of the spectral theory of differential operators is related to inverse spectral problems with various applications in physics and mechanics. The inverse problem of spectral analysis is understood as the recovery of a linear operator or boundary value problem according to certain spectral characteristics. Let's note that spectrum, spectral function, scattering data, etc. are spectral characteristics. can be taken. Depending on the choice of spectral characteristics, the inverse problem can be in different settings. Inverse problems of spectral analysis for the Sturm-Liouville operator have been studied by many authors in different settings. In the work of G. Borg and N. Levinson, it was proved that one spectrum is not enough to restore the Sturm-Liouville operator. Using the transformation operator, V.A. Marchenko proved the uniqueness of the solution of the inverse problem for the Sturm-Liouville operator with respect to the spectral function. The complete solution of the last inverse problem is given in the work of I.M. Gelfand and B.M. Levitan. Algorithms for restoring the Sturm-Liouville operator to a spectral function and two spectra with a different method proposed by M.G. Crane are given. The inverse problem for the Sturm-Liouville operator with respect to two spectra was completely solved in the work of M.G. Gasimov and B.M. Levitan. In general, the inverse spectral problem for the Sturm-Liouville set of operators given by boundary conditions was studied in the work of H.M. Huseynov and I.M. Nabiyev. Inverse spectral problems for the Sturm-Liouville operator with non-self-adjoint periodic coefficients were studied in the works of M.G. Gasimov, L.A. Pastur and V.A. Tkachenko, R.F. Efendiyev and others.

Starting from the end of the 50s of the 20th century, the interest in the study of inverse problems that appeared in the theory of scattering began to increase. In quantum mechanics, the scattering of particles in a potential field is fully determined by the asymptotics of wave functions at infinity. The inverse problem of the scattering theory means finding the potential of the field based on the asymptotics of the wave functions at infinity. In mathematical terms, the inverse problem of scattering is understood as the recovery of the operator due to the asymptotics of the normalized eigenfunctions of the linear operator (eigenfunctions corresponding to the continuous and discrete spectrum) at infinity. The collection of quantities that determine the asymptotics of normalized eigenfunctions at infinity is called scattering data. So, the inverse problem of scattering consists in recovering the linear operator from the scattering data.

V. Bargman showed for the first time that the potential $q(x)$ of the one-dimensional Schrödinger equation $-y'' + q(x)y = \lambda y$ given on the semi-axis is not defined as single-valued due to the scattering function and eigenvalues. N. Levinson proved that if there are no specific numbers, the potential is defined as single-valued. Later, V.A. Marchenko showed that the scattering function, eigenvalues and corresponding normalizing values define the potential as single-valued.

Note that the inverse problem of scattering is mainly studied by the method of transformation operators (Gelfand-Levitan-Marchenko formalism) or the method of the Riemann-Hilbert problem. It is with the help of the first method that the inverse problem of scattering along the entire axis for the one-dimensional Schrödinger equation was studied in the works of L.D. Faddeyev. The results obtained in this direction were further clarified later in the works of V.A. Marchenko, B.M. Levitan, H.M. Huseynov and others. In the aforementioned works on inverse scattering problems for one-dimensional Schrödinger equations, the potential usually falls into the class of functions with finite moments and decaying rapidly. On the other hand, when the potential is an unbounded function, the study of the scattering problem is important both from a mathematical and physical point of view. The first results in this direction were obtained in the work of P.P. Kulish. Of particular interest are the one-dimensional Schrödinger equations with an additional increasing potential in the form of unbounded potential equations. Such equations appear as quantum-mechanical operators of electron energy in crystals located in an external uniform electric field. Examples of such models include the perturbed oscillator and the one-dimensional Stark operator.

M.G.Gasimov and B.A.Mustafayev's work can be mentioned as one of the first works on the study of the inverse spectral problem for the Schrödinger equation with an additional increasing potential. In these cases, with the method of the conversion operator the inverse problem of scattering has been studied for the problem

$$-y'' - x^2 y + p(x)y = \lambda y, 0 < x < +\infty , \qquad (*)$$

$$y(0) = 0$$

In the mentioned works, a transformation operator satisfying the condition at infinity was established by the Riemann function method. In the work of F. Calogero and A. Degasperis, a formal solution of the inverse spectral problem is given for the equation

$$-y'' + xy + q(x)y = \lambda y, -\infty < x < +\infty \qquad (**)$$

Later, this issue was studied more precisely in the works of L. Yishen, A.P. Kachalov and Ya.V. Kurilev. In the works of H.P. McKean and E. Trubovitsin, D. Chelkak, P. Kargaev, E. Korotyaev, the inverse spectral problem was studied by the method of spectral analysis for the perturbed oscillator

$$-y'' + x^2 y + q(x)y = \lambda y, -\infty < x < +\infty \qquad (***)$$

A number of interesting results related to this issue were also obtained by B.M. Levitan. It should be noted that the inverse spectral problem in another setting for the last equation was investigated by the transformation operator method in the works of A.K. Khanmamedov and S.M. Baghirova, A.Kh. Khanmamedov and N.F. Gafarova, A.Kh. Khanmamedov and M.F. Muradov.

It should be noted that in the above-mentioned works, when constructing the transformation operators for the equations (*) and (***), the question of the solution of the integral equations obtained for the kernels of these operators satisfying the hyperbolic equations for the kernels, that is, the expressions obtained using the transformation operators solving the equations (*) and (***) remains open. There are serious flaws in the proof of construction of transformation operators for equation (**).

From a physical point of view, it is of particular importance that the potential behaves differently on the right and left sides, that is, it has a stepped type. Inverse problems of scattering for one-dimensional Schrödinger equations with step-type potentials were studied in the work of V.S. Buslayev and V. Fomin, I. A. Anders and V. P. Kotlyarov, and others. The works of H. Sh. Huseynov, H. M.

Huseynov, A. Kh. Khanmammadov and others were devoted to the inverse problems of scattering for the discrete analogues of these equations. Such a problem for infinitely increasing potentials was studied in the work of H.M. Huseynov and A.Kh. Khanmammadov, the inverse problem of scattering when the potential is infinitely small on the left side, and when it increases on the right side. In this work, the solution of the inverse problem is based on the separation formula for Airy functions of the first kind.

Of particular interest is the case where the potential is infinitesimally small on the left side and increases on the right side as x^2, in other words, the case where there is additional potential in the form $\theta(x)x^2$. Here, $\theta(x)$ is Heavyside function. In this case, the analogue of the Airy function of the first type is the parabolic cylinder function (Weber function). However, there is no suitable derivation formula for parabolic cylinder functions. This creates certain difficulties in the investigation of the issue of scattering and requires a different approach. It should be noted that when $q(x)=0$, the equation describes the motion of a particle in a parabolic potential hole, that is, a quantum harmonic oscillator. The movement of a particle in a semi-open parabolic potential hole, that is, in the area bounded by a curve $\theta(x)x^2$ is also of great interest from a physical point of view.

Subject and object of the research. For the one-dimensional Schrödinger equations with an additional increasing potential, it is relevant to construct transformation operators that satisfy the condition at infinity and apply them to direct and inverse problems of spectral analysis. When the potential of the one-dimensional Schrödinger equation is infinitesimally small on one side and infinitely increasing on the other side, the eigenfunctions corresponding to the continuous spectrum are related by only one relation. In addition, the modulus of the reflection coefficient is equal to unity, as in the case of semi-axial scattering. For these reasons, there is a need to seriously modify many classical judgments specific to the problem of all-axis scattering. Therefore, it is of particular interest to study the direct and inverse problems of scattering when the potential of the one-dimensional Schrödinger equation is infinitesimally small on the left-hand side, and increases as x^2 on the right-hand side.

Aims and objectives of the research. Overcoming defects in the proof of construction of transformation operators satisfying the condition at infinity for one-dimensional Schrödinger equations with additional quadratic or linear potentials and construction of transformation operators in wider classes of excitation potentials. Investigation of direct and inverse problems of scattering for the one-dimensional Schrödinger equation when the potential increases as an infinitely decreasing function on the left-hand side and as a quadratic function on the right-hand side.

Research methods. The methods of spectral theory of differential operators, functional analysis and the theory of functions with complex variables are used in the dissertation work.

Basic theses for defence. The following main propositions are defended: the construction of transformation operators satisfying the condition at infinity for the one-dimensional Schrödinger equation when the potential is an infinitely decreasing function on the left side and increases as a quadratic function on the right side; construction of transformation operators in wider classes of excitation potential for one-dimensional Schrödinger equations with additional linear potential; construction of transformation operators satisfying the condition at infinity for one-dimensional Schrödinger equations with complex-valued additional periodic potential; study of the scattering problem for the one-dimensional Schrödinger equation when the potential increases as an infinitely decreasing function on the left-hand side and as a quadratic function on the right-hand side.

Scientific novelty of the research. The following main results were obtained in the work:

- a transformation operator satisfying the condition at infinity is constructed for the one-dimensional Schrödinger equation with the first moment of the additional linear potential being the finite excitation potential. The properties of the kernels of transformation operators have been studied;

- for the one-dimensional Schrödinger equation with a complex-valued additive periodic potential and a finite excitation potential with the first moment, transformation operators satisfying the condition at infinity were constructed. The properties of the kernels of transformation operators have been studied;

- for the one-dimensional Schrödinger equation with an additional potential in the form $\theta(x)x^2$ the direct problem of scattering along the entire axis was studied, the separation formulas for the eigenfunctions and the basic Marchenko-type integral equations were derived;

- for the one-dimensional Schrödinger equation with an additional potential in the form $\theta(x)x^2$ the inverse problem of scattering along the entire axis is studied, and the algorithm for solving the inverse problem is given.

Theoretical and practical significance of the research. The results obtained in the dissertation are theoretical in nature. The obtained results can be used in the spectral theory of differential operators, integration of nonlinear equations.

Approbation and implementation. The main results of the dissertation were presented at the scientific seminars of Baku Engineering University department of "Mathematics" (head professor R.F.Efendiyev), Baku State University department of "Applied Mathematics" (head professor E.H.Eyvazov), Azerbaijan State Oil and Industrial University department of "General and Applied Mathematics" (head professor A.R.Aliyev), South Korean "INHA" univeristy department of "Mathematics" (head professor Hyundai Lee), scientific seminar of department of "Functional Analysis" (head professor H.I.Aslanov) of the Institute of Mathematics and Mechanics of ANAS, at the national scientific conference dedicated to 100th anniversary of the birth of national leader Heydar Aliyev "VII International Scientific Conference of Young Researchers" (Baku, 2023), at the international scientific conference "6th International HYBRID Conference on Mathematical Advances and Applications" (İstanbul, 2023), at the national scientific conference "NASCO XXV", at the international scientific conference "Modern problems of applied mathematics, informatics, mechanics" (Russia, Nalchik, 2023).

Applicant's personal contribution. All results obtained in the dissertation belong to the applicant.

Applicant's publications. 7 scientific articles of the author have been published in scientific publications recommended by the EAC under the President of the Republic of Azerbaijan (2 of them in journals indexed by Web of Science and SCOPUS, 1 of them with an Impact factor of 0.719), and 3 works in various conference materials (2 of them was an international conference, both were held abroad) were published.

The name of the organization where the dissertation was conducted. Dissertation work was performed at the "Mathematics" department of Baku Engineering University.

The volume of the dissertation's structural sections separately and the general volume. The total volume of the dissertation is ~ 271549 characters (title – 387 characters, table of contents ~ 2044 characters, introduction ~ 42128 characters, chapter I ~ 138119 characters, chapter II ~ 72000 characters, results – 1463 characters). The bibliography consists of 103 names.

BASIC CONTENT OF THE DISSERTATION

The dissertation consists of an Introduction, two chapters and a bibliography.

In the introduction, the relevance of the dissertation work is justified, a brief summary of the results related to the content of the work is given, and the main results are explained.

In Chapter I, triangular notations for transformation operators preserving the asymptotics of solutions at infinity for one-dimensional Schrödinger equations with additional potentials were found.

In paragraph 1.1, we look at the equation

$$-y'' + \theta(x)x^2 y = \lambda y, \ -\infty < x < \infty, \quad \lambda \in C, \qquad (1)$$

here, $\theta(x)$ is Heavyside function, meaning

$$\theta(x)=\begin{cases}1, x\ge 0,\\ 0, x<0.\end{cases}$$

Let us denote by G the complex λ-plane whose positive semi-axis is cut. Let us denote by ∂G the border of plane G, that is, the lower and upper “shores” of the cut along the $(0,+\infty)$ semi-axis. Let us consider on the plane G the function $\sqrt{\lambda}$, such that, the analytic branch of the radical is selected so that, for $\lambda>0$, $\sqrt{\lambda+i0}>0$. It is clear that, for $\lambda>0$, $\sqrt{\lambda-i0}=-\sqrt{|\lambda|}<0$.

Theorem 1. *At each complex value of the parameter* λ*, equation (1) has solutions* $\psi_{\pm}(x,\lambda)$ *that can be expressed as follows:*

$$\psi_{+}(x,\lambda)=\begin{cases} D_{\frac{\lambda-1}{2}}\left(\sqrt{2}x\right), x\ge 0,\\ \frac{1}{2}\left[D_{\frac{\lambda-1}{2}}(0)-i\sqrt{\frac{2}{\lambda}}D'_{\frac{\lambda-1}{2}}(0)\right]e^{i\sqrt{\lambda}x}+\\ +\frac{1}{2}\left[D_{\frac{\lambda-1}{2}}(0)+i\sqrt{\frac{2}{\lambda}}D'_{\frac{\lambda-1}{2}}(0)\right]e^{-i\sqrt{\lambda}x}, x<0,\end{cases} \tag{2}$$

$$\psi_{-}(x,\lambda)=\begin{cases} \frac{1}{2}\left[D^{-1}_{\frac{\lambda-1}{2}}(0)-i\sqrt{\frac{\lambda}{2}}\left(D'_{\frac{\lambda-1}{2}}(0)\right)^{-1}\right]D_{\frac{\lambda-1}{2}}\left(\sqrt{2}x\right)+\\ +\frac{1}{2}\left[D^{-1}_{\frac{\lambda-1}{2}}(0)+i\sqrt{\frac{\lambda}{2}}\left(D'_{\frac{\lambda-1}{2}}(0)\right)^{-1}\right]D_{\frac{\lambda-1}{2}}\left(-\sqrt{2}x\right), x\ge 0,\\ e^{-i\sqrt{\lambda}x}, x<0.\end{cases} \tag{3}$$

Here, $D_{\nu}(x)$ is parabolic cylinder function. In this paragraph, the properties of solutions $\psi_{+}(x,\lambda)$ and $\psi_{-}(x,\lambda)$ have been studied.

In the second paragraph of the first chapter, the closure L_0 of the symmetric operator defined by the left side of the differential equation (1) on finite functions that are continuously differentiable twice is considered. The operator L_0 is a self-adjoint operator defined on $L_2(-\infty,+\infty)$ space. In this paragraph, a description of the domain of the operator L_0 is given, its resolvent and spectrum are studied.

Theorem 2. *The spectrum of operator L_0 is continuous and fills the half-axis $[0,+\infty)$.*

In paragraph 1.3, the functions

$$a_0(\lambda)=\frac{1}{2}D_{\frac{\lambda-1}{2}}(0)-i\frac{1}{\sqrt{2\lambda}}D'_{\frac{\lambda-1}{2}}(0)\ ,\ r_0(\lambda)=\frac{\overline{a_0(\lambda)}}{a_0(\lambda)} \qquad (4)$$

are introduced, the scattering problem for operator L_0 is studied and separation formulas for its eigenfunctions are derived.

In fourth paragraph of this chapter, the existence of Jost type solutions for the equation

$$-y''+\theta(x)x^2y+q(x)y=\lambda y, \qquad -\infty<x<+\infty\ , \qquad (5)$$

when the potential $q(x)$ satisfies the condition $\int_{-\infty}^{+\infty}|q(x)|dx<\infty$ is proven.

In paragraph 1.5, transformation operators satisfying the condition at infinity are established for equation (5). More precisely, it is proven that the potential $q(x)$ is a real-valued function and when it satisfies the condition

$$\int_{-\infty}^{0}(1+|x|)|q(x)|dx+\int_{0}^{+\infty}(1+x^2)|q(x)|dx<\infty \tag{6}$$

the equation (5) has solutions which satisfy the conditions

$$f_{\pm}(x,\lambda)=\psi_{\pm}(x,\lambda)[1+o(1)],\, x\to\pm\infty .$$

Let

$$\sigma_{+}(x)=\int_{x}^{\infty}\left|\theta(t)t^2-t^2+q(t)\right|dt\,,\sigma_{-}(x)=\int_{-\infty}^{x}\left|\theta(t)t^2+q(t)\right|dt\ .$$

Theorem 3. *Let the potential $q(x)$ satisfies the condition (6). Then, for all values of the parameter $\lambda\in G$, the equation (5) has solutions which can be written as follows:*

$$f_{\pm}(x,\lambda)=\psi_{\pm}(x,\lambda)\pm\int_{x}^{\pm\infty}K^{\pm}(x,t)\,\psi_{\pm}(t,\lambda)dt\,, \tag{7}$$

here, the kernels $K^{\pm}(x,t)$ are continuous functions and satisfy the following relations:

$$K^{\pm}(x,t)=O\left(\sigma_{\pm}\left(\frac{x+t}{2}\right)\right),x+t\to\pm\infty\,, \tag{8}$$

$$K^{\pm}(x,x)=\pm\frac{1}{2}\int_{x}^{\pm\infty}\left[\theta(t)t^2-\frac{1\pm1}{2}t^2+q(t)\right]dt\ . \tag{9}$$

In this paragraph, it is also proven that the kernels $K^{\pm}(x,t)$ are absolutely continuous with respect to both of the variables and evaluations for their first order partial derivatives is obtained.

In sixth paragraph of first chapter, we consider the perturbed Airy equation

$$-y''+xy+q(x)y=\lambda y,\qquad -\infty<x<+\infty \tag{10}$$

here, the potential $q(x)$ has real values and satisfies the condition

$$\int_{-\infty}^{+\infty}(1+|x|)|q(x)|dx<\infty \tag{11}$$

Let us note that, in works of L.Yishen, A.P.Kachalov və Ya.V.Kurilev the transformation operator which satisfies a condition at inifinity is constructed for equation (10) when the potential $q(x)$ satisfies the conditions

$$q(x)\in C^{(1)}(-\infty,+\infty), \qquad \int_{-\infty}^{+\infty}(1+|x|)|q(x)|dx<\infty .$$

In this paragraph, it is shown that, the proofs in those works have flaws. Those flaws have been eliminated and the transformation operator has been constructed when potential $q(x)$ satisfies only the condition (11).

In the final paragraph of this chapter, that is, in paragraph seven, we consider the equation

$$-y''+e^{ix}y+p(x)y=\lambda^2 y, -\infty<x<+\infty , \qquad (12)$$

with complex valued additional periodic coefficient, here, function $p(x)$ satisfies the condition

$$\int_{-\infty}^{+\infty}(1+|x|)|p(x)|dx<\infty \qquad (13)$$

As is known from works of M.G.Gasımov, L.A.Pastur and V.A.Tkachenko, when $p(x)=0$, the equation (12) has solutions which can be written as $g_0(x,\lambda)=e^{i\lambda x}\left(1+\sum_{n=1}^{\infty}\frac{1}{n+2\lambda}\sum_{\alpha=n}^{\infty}V_{n\alpha}e^{i\alpha x}\right)$.

Let $\sigma_0^{\pm}(x)=\pm\int_x^{\pm\infty}|q(t)|dt, \sigma_1^{\pm}(x)=\pm\int_x^{\pm\infty}\sigma^{\pm}(t)dt$.

Theorem 4. *If potential $q(x)$ satisfies the condition (13), then, the equation (12) has solutions $g_{\pm}(x,\lambda)$ which can be written as follows:*

$$g_{\pm}(x,\lambda)=g_0(x,\pm\lambda)\pm\int_x^{\pm\infty}A^{\pm}(x,t)\, g_0(t,\pm\lambda)dt ,$$

here, the kernels $A^{\pm}(x,t)$ are continuous functions and satisfy the following relations:

$$\left|A^{\pm}(x,t)\right| \le \frac{1}{2}\sigma_0^{\pm}\left(\frac{x+t}{2}\right)e^{\sigma_1^{\pm}(x)}, \quad A^{\pm}(x,x) = \pm\frac{1}{2}\int_x^{\pm\infty} p(t)\,dt\,.$$

Chapter II is dedicated to studying the direct and inverse problems for equation (5) when the potential $q(x)$ satisfies the condition

$$\int_{-\infty}^{0}(1+|x|)|q(x)|dx + \int_{0}^{+\infty}(1+x^5)|q(x)|dx < \infty \qquad (14)$$

In paragraph 2.1, the direct scattering problem is studied. It is proven that the following connection formula exists between the solutions $f_+(x,\lambda)$ and $f_-(x,\lambda)$ defined by formula (7):

$$f_+(x,\lambda) = a(\lambda)\overline{f_-(x,\lambda)} + \overline{a(\lambda)}f_-(x,\lambda), \lambda \in \partial G, \lambda \ne 0\,. \qquad (15)$$

The functions $t(\lambda) = a^{-1}(\lambda)$ and $r(\lambda) = \dfrac{\overline{a(\lambda)}}{a(\lambda)}$ are respectively called the transition and reflection coefficients for the scattering problem. In this paragraph, the expression for the function $a(\lambda)$ is obtained using the solutions (7), it is proven that, the function $a(\lambda)$ is analytic on the plane G and is continuous up to the border ∂G of this plane, except maybe for the point $\lambda = 0$. It is studied how this function behaves around the point $\lambda = 0$. The following theorems are proven.

Theorem 5. *The function $a(\lambda)$ can only have a finite number of zeros.*

Theorem 6. *The zeros of function $a(\lambda)$ are simple.*

Additionally, it is shown that the zeros of the function $a(\lambda)$ can only be from the negative semi-axis. Let us denote by $\lambda_j, \lambda_j < 0, j = 1.2,..,N$ the zeros of the function $a(\lambda)$. Let us introduce the following notations:

$$\left(m_j^{\pm}\right)^2 \overset{def}{=} \int_{-\infty}^{\infty} f_{\pm}^2\left(x,\lambda_j\right)dx,\ j=1,\ldots,N. \qquad (16)$$

Then, the following equalities are true:

$$m_j^{+} m_j^{-} = 2i\sqrt{\lambda_j}\,\dot{a}\left(\lambda_j\right),\ j=1,2,\ldots,N\ , \qquad (17)$$

here, the point on the function denotes the derivative with respect to λ .

Moreover, in paragraph 2.1, the asymptotics at infinity for the function $a(\lambda)$ are defined:

$$a(\lambda)=a_0(\lambda)\left[1+O\left(\frac{1}{\sqrt{\lambda}}\right)\right],\lambda\to\infty\ , \qquad (18)$$

here, the function $a_0(\lambda)$ is defined by formula (4).

In second paragraph of second chapter, the operator L defined in $L_2(-\infty,+\infty)$ space with the domain which is the set

$$D_L = \left\{y\in L_2(-\infty,+\infty): y\in W_{2,loc}^2, \ell(y)\in L_2(-\infty,+\infty)\right\}$$

and which is made by the differential expression

$$\ell(y) = -y'' + \theta(x)x^2 y + q(x)y$$

is studied.

Theorem 7. *For* $\lambda \notin [0,+\infty), \lambda \neq \lambda_j, j=1,\ldots,N$, *the integral operator* R_λ *defined in space* $L_2(-\infty,+\infty)$ *by the formulas* $(R_\lambda f)(x) = \int_{-\infty}^{+\infty} R(x,t,\lambda) f(t) dt$ *and*

$$R(x,t,\lambda) = -\frac{1}{2i\sqrt{\lambda}a(\lambda)}\begin{cases} f_+(x,\lambda)f_-(t,\lambda), t \le x, \\ f_-(x,\lambda)f_+(t,\lambda), t > x. \end{cases}$$ *is called the resolvent of the operator* L.

Theorem 8. *The continuous spectrum of the operator* L *fills the semi-axis* $[0,+\infty)$. *In addition to this, the operator* L *can have a finite number of simple negative valued eigenvalues* $\lambda_j, \lambda_j < 0, j=1,\ldots,N$.

At the same time, it is proven that, the eigenfunctions of the operator L coincide with the zeros of the function $a(\lambda)$.

In this paragraph, the separation formulas for the eigenfunctions of the operator L are derived.

$\{t(\lambda) = a^{-1}(\lambda), \lambda \in \partial G; \lambda_j, \lambda_j < 0; m_j^+ > 0, j=1,\ldots,N\}$ collection of quantities is called the scattering data for the equation (8). When we say the inverse scattering problem for the equation (5) or the operator L, we understand the problem of recovering the potential $q(x)$ from the class (14) when the scattering data are given. In solving the inverse problem, the main integral equations of the Marchenko type play an important role. Section 2.3 is devoted to the derivation of the main equations. When the scattering data are known, let us introduce the following functions:

$$\xi(\lambda) = \begin{cases} \pi \sum_{\lambda_j < \lambda} \left(m_j^+\right)^{-2}, \lambda \le 0, \\ \pi \sum_{j=1}^{N} \left(m_j^+\right)^{-2} + \int_0^\lambda \frac{|a(u)|^{-2} - |a_0(u)|^{-2}}{4\sqrt{u}} d\lambda, \lambda > 0. \end{cases} \quad (19)$$

$$F^{+}(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\psi_{+}(x,\lambda)\psi_{+}(y,\lambda)d\xi(\lambda), \tag{20}$$

$$F^{-}(x,y)=\sum_{j=1}^{N}\left(m_j^{-}\right)^{-2}\psi_{-}(x,\lambda_j)\psi_{-}(y,\lambda_j)+ \\ +\frac{1}{4\pi}\int_{\partial\Gamma}\frac{r(\lambda)-r_0(\lambda)}{\sqrt{\lambda}}\psi_{-}(x,\lambda)\psi_{-}(y,\lambda)d\lambda \tag{21}$$

Here, the quantities $r(\lambda)=\frac{\overline{a(\lambda)}}{a(\lambda)}$ and m_j^{-}, $j=1,2,\dots,N$ are defined by the formula (16).

Theorem 9. *For each given* x, *the functions* $K^{\pm}(x,y)$ *included in the expression (7) satisfy the following integral equations:*

$$F^{\pm}(x,y)+K^{\pm}(x,y)\pm\int_{x}^{\pm\infty}K^{\pm}(x,t)F^{\pm}(t,y)dt=0, \pm y>\pm x. \tag{22}$$

The equations (22) are called the main integral equations.

In paragraph 2.4, the properties of the scattering functions $F^{+}(x,y)$ and $F^{-}(x,y)$ are studied. It follows from the results of paragraphs 1 and 4 of this chapter that the following properties of scattering data are true:

I. *The transition coefficient* $t(\lambda)=a^{-1}(\lambda)$ *is a continuous function on the cut* $\lambda\in\partial G$, *that is,* $t(\lambda-i0)=\overline{t(\lambda+i0)}$, $\lambda>0$. *The function* $t(\lambda)$ *is analytical on the plane* G, *except for the simple poles* $\lambda_j<0, j=1,2,\dots,N$ *and satisfies the following relations:*

$$t^{-1}(\lambda)=\frac{C}{2i\sqrt{\lambda}}+O(1),\lambda\to 0\,,$$

$$t^{-1}(\lambda)=\left[\frac{1}{2}D_{\frac{\lambda-1}{2}}(0)-i\frac{1}{\sqrt{2\lambda}}D'_{\frac{\lambda-1}{2}}(0)\right]\cdot\left[1+O\left(\frac{1}{\sqrt{\lambda}}\right)\right],\ \lambda\to\infty\,.$$

II$_{\pm}$. *The function* $F^{\pm}(x,y)$ *is a continuous function on the entire plane, and is absolutely continuous with respect to both variables. The function* $F^{\pm}(x,x)$ *is absolutely continuous. For each finite number* a*, the relations*

$$\left|F^{\pm}(x,y)\right|\le C,\pm x>\pm a,\pm y>\pm a,\ \int_a^{\pm\infty}\left(1+y^{2\pm 2}\right)\sup_{\pm x\ge\pm a}\left|F^{\pm}(x,y)\right|dy<\infty\,,$$

$$\sup_{\pm x\ge\pm a}\int_a^{\infty}\left(1+y^{1\pm 1}\right)\left[\left|\frac{\partial F^{\pm}(x,y)}{\partial x}\right|+\left|\frac{\partial F^{\pm}(x,y)}{\partial y}\right|\right]dy<\infty\,,$$

$$\int_a^{\pm\infty}\left(1+|x|^{3\pm 2}\right)\left|\frac{d}{dx}F^{\pm}(x,x)\right|dx<\infty$$

are true.

The last paragraph of the second chapter is dedicated to the solution of the inverse problem of scattering. In this paragraph, the uniqueness of the solution of the main integral equations is proven.

Theorem 10. *If the conditions I and II*$_{\pm}$ *are satisfied, then, for each given* x *the integral equation (22) has a unique solution* $K^{\pm}(x.\cdot)\in L_1(x,\pm\infty)$.

The uniqueness of solution of the main integral equations gives us the following algorithm to find the solution of the inverse scaterring problem.

Algorithm 1. *Let the collection of quantities* $\{t(\lambda)=a^{-1}(\lambda),\lambda\in\partial G;\lambda_j,\lambda_j<0;m_j^{+}>0,\ j=1,\dots,N\}$ *is given. Then:*

1) *We consturct the functions* $F^{+}(x,y)$ *and* $F^{-}(x,y)$ *using the formulas (19)-(21);*

2) *We solve the main integral equations (22) and find the functions* $K^{+}(x,y)$ *and* $K^{-}(x,y)$*;*

3) *Using the formulas (9), we define the potential* $q(x)$.

Furthermore, the necessary and sufficient conditions for the solution of the inverse scattering problem are found in this paragraph.

Theorem 11. *The necessary and sufficient condition for the collection of quantities* $\{t(\lambda)=a^{-1}(\lambda), \lambda\in\partial G; \lambda_j, \lambda_j<0; m_j^{+}>0, j=1,\ldots,N\}$ *to be called the scattering data for the equation* $-y''+\theta(x)x^2y+q(x)y=\lambda y, -\infty<x<+\infty$ *with the real valued potential* $q(x)$ *from the class*

$$\int_{-\infty}^{0}(1+|x|)|q(x)|dx+\int_{0}^{+\infty}(1+x^5)|q(x)|dx<\infty$$

is for the properties I, $II_{\pm}$ *to be satisfied.*

Finally, the author expresses his deep gratitude to the scientific supervisor, Professor A.K.Khanmamedov, for always keeping an eye on the setting of the issue and the completion of the work.

CONCLUSION

The dissertation work is devoted to the construction of transformation operators for one-dimensional Schrödinger equations with additional potentials and the study of direct and inverse scattering problems for one-dimensional Schrödinger equations with additional increasing potentials by the method of transformation operators. The following main results were obtained:

- a transformation operator satisfying the condition at infinity is constructed for the one-dimensional Schrödinger equation with the first moment of the additional linear potential being the finite excitation potential. The properties of the kernels of transformation operators have been studied;

- for the one-dimensional Schrödinger equation with a complex-valued additive periodic potential and a finite excitation potential with the first moment, transformation operators satisfying the condition at infinity were constructed. The properties of the kernels of transformation operators were studied;

- for the one-dimensional Schrödinger equation with an additional potential in the form $\theta(x)x^2$, the direct problem of scattering along the entire axis was studied, the separation formulas for the eigenfunctions and the basic Marchenko-type integral equations were derived;

- for the one-dimensional Schrödinger equation with an additional potential in the form $\theta(x)x^2$, the inverse problem of scattering along the entire axis is studied, and the algorithm for solving the inverse problem is given.

The main results of the dissertation were published in the following works:

1. Оруджев, Д.Г. Спектральный анализ одномерного оператора Шредингера с растущим потенциалом // – Баку: Вестник Бакинского Университета: серия физика-математических наук, – 2021. №1, – с. 39-46.
2. Оруджев, Д.Г. Разложения по собственным функциям одномерного оператора Шредингера с дополнительным растущим потенциалом // – Хырдалан: Журнал Бакинского Инженерного Университета, – 2022. – т. 6, №1, – с. 19-24.
3. Khanmamedov A.Kh., Orujov D.H. Eigenfunction expansions for the Schrodinger operator with a square potential // The Reports of National Academy of Sciences of Azerbaijan, – Baku: – 2022. – v. 78, №1-2, – p. 14-16.
4. Orujov, D.H. On the transformation operator for the Schrodinger equation with an additional increasing potential // – Baku: Proceedings of the Institute of Mathematics and Mechanics, – 2022. – v. 48, №2, – p. 311-322.
5. Ханмамедов А.Х., Оруджев Д.Г., Обратная задача рассеяния для уравнения Шредингера с дополнительным растущим потенциалом на всей оси // Теоретическая и математическая физика, Москва:-

2023.- т. 216, № 1,-с.117-132.
6. Ханмамедов А.Х., Оруджев Д.Г. О решении характеристической задачи Коши для гиперболического уравнения второго порядка // Сборник Трудов Международной Научной Конференции “Современные Проблемы Прикладной Математики, Информатики и Механики”, - Нальчик : 22.06.23г.-26.06.23г., Изд.-во Кабардино-Балкарского государственного университета им.Х.М. Бербекова, - 2023, 181-182.
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9. Orujov, D.H. Triangular representations of solutions of the Schrodinger equation with an additional quadratic potential // NASCO XXV, – Baku: – 23-24 November, – 2023, – v. 3, – p. 181-183.
10. Orujov, D.H., Khanmamedov, A.Kh. Inverse scattering problem for the Schrodinger equation with an additional growing potential on the entire axis // 6th International HYBRID Conference on Mathematical Advances and Applications, – Istanbul: – 10-13 May, – 2023, – p. 37.

The defense will be held on 19.03.2024 at 14:00 at the meeting of the Dissertation council ED 2.17 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at Baku State University.

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Dissertation is accessible at the Baku State University Library

Electronic versions of dissertation and its abstract are available on the official website of the Baku State University

Abstract was sent to required addresses on 15.02.2024

Signed for print: 09.02.2024

Paper format: A5

Volume: 39149

Number of hard copies: 20