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ABSTRACT

of the dissertation for the degree of Doctor of Science

STUDYING THE SOLUTIONS OF SOME CLASSES OF SURFACE INTEGRAL EQUATIONS BY PROJECTIVE-GRID METHODS

Speciality: 1202.01 -Analysis and functional analysis

Field of science: Mathematics

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GENERAL CHARACTERISTICS OF THE WORK

Rationale of the theme and development degree. İt is known that mathematical description of scattering of waves that harmonically depend on time leads to boundary value problem for the Helmholts equation $\Delta u + k^2 u = 0$, where Δ is a Laplace operator, k is a wave number and $\operatorname{Im} k \geq 0$. As in many cases it is impossible to find exact solution of boundary value problems for Helmholts equation, there arises an interest to develop approximate methods for solving boundary value problem for Helmholts equation with theoretical foundation. One of the widely used methods for studying approximation solutions of exterior boundary value problems for Helmholts equation is their reduction to an integral equation. The main advantage of using the method of boundary integral equations for studying exterior boundary value problems is that such a system allows to reduce the problem stated for an unbounded domain to the problem for a smaller dimension bounded domain.

Note that direct application of potential theory for deriving integral equations of exterior boundary value problem for Helmholts equation reduces to equations that have no a unique solution on eigen values of internal boundary value problems. However, searching the solutions of external boundary value problems in the form of combination of acoustic potentials of single and double layer and also using the Green formula for deriving integral equations of external boundary problems, the integral equations uniquely solvable for any value of the wave number and dependent on the operator

$$(T\rho)(x) = 2\frac{\partial}{\partial \vec{n}(x)} \int_S \frac{\partial \Phi_k(x,y)}{\partial \vec{n}(y)} \rho(y) dS_y, \quad x \in S,$$

were obtained, where S is a closed twice continuously differentiable surface in R^3, $\vec{n}(x)$ is a unit external normal at the point $x \in S$, while $\Phi_k(x,y)$ is a fundamental solution of the Helmholts equation, i.e.

$$\Phi_k(x,y)=\frac{\exp(ik|x-y|)}{4\pi|x-y|},\ x,y\in R^3,\ x\neq y.$$

A counterexample structured by Liapunov[1] shows that for a double layer potential with continuous density, a derivative, generally speaking, does not exist, i.e. the operator T was not determined in the space $C(S)$ of all continuous functions on S with the norm $\|f\|_\infty=\max_{x\in S}|f(x)|$. However, in D.Kolton and R.Kress book[2] it was proved that the operator T boundedly acts in the Holder space and a formula for calculating the derivative of a double layer acoustic potential was given by means of surface gradient. Furthermore, in this book it was showed that if in $\mathrm{Im}\,k>0$, the operator $T:\mathrm{N}(S)\to C(S)$ is invertible and the invertible operator T^{-1} is given by the relation

$$T^{-1}=-L\left(I-\widetilde{K}\right)^{-1}\left(I+\widetilde{K}\right)^{-1},$$

where

$$(L\rho)(x)=2\int_S\Phi_k(x,y)\,\rho(y)dS_y,\quad x\in S,$$

$$\left(\widetilde{K}\rho\right)(x)=2\int_S\frac{\partial\Phi_k(x,y)}{\partial\vec{n}(x)}\rho(y)dS_y,\quad x\in S,$$

$I-$is a operator in the space $C(S)$ and $\mathrm{N}(S)$ denotes a space of all continuous functions φ, whose double layer potential with density φ has continuous normal derivatives on both sides of the surface S. It should be indicated that the given formula in D.Kolton and R.Kress book for calculating the derivative of a double layer acoustic potential is not very practical. Furthermore, it should be noted that a practice cubic formula for calculating normal derivative of double

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[1] Гюнтер, Н.М. Теория потенциала и ее применение к основным задачам математической физики / Н.М. Гюнтер. – Москва: Гостехиздат, – 1953. – 415с.

[2] Колтон, Д. Методы интегральных уравнений в теории рассеяния / Д. Колтон, Р. Кресс. – Москва: Мир, – 1987. – 311с.

layer acoustic potential was not constructed yet. For example, A.Yu.Anfinogenov, I.K.Lifanov and P.L.Lifanov[3] a cubic formula was constructed for a normal derivative of a double layer acoustic potential in a sphere. However, the constructed cubic formula in this work is not practical in the sense that the coefficients of this cubic formula are singular integrals.

Furthermore, it is known that one of the methods for solving hypersingular integral equations of exterior boundary value problem for Helmholt equations is regularization of these equations by means of the inverse operator T^{-1}. As can be seen, in spite of inevitability of operators $\left(I+\widetilde{K}\right)^{-1}$ and $\left(I-\widetilde{K}\right)^{-1}$ obvious form of inverse operators $I+\widetilde{K}$ and $I-\widetilde{K}$, are not known, consequently the obvious form of the inverse operator T^{-1} is unknown. Because of these reasons, the approximate solution of some classes of integral equations of boundary value problem for Helmholts equation was not studied. Consequently, development of approximate methods for solving integral equations of boundary value problems for Helmholts equations for any value of the wave number and its theoretical foundation is very urgent.

Goal and tasks of the research. The main goal and tasks of the dissertation work is to derive a practical formula for calculating the derivative of a double layer acoustic potential, to construct a cubic formula for the normal derivative of a double layer acoustic potential and to study the approximate solution of some classes of integral equations of boundary value problems for Helmholts equation for any value of the wave number $\operatorname{Im} k \geq 0$.

Investigation methods. The methods of potential theory, theory of singular and hypersingular integral equations, theory of operators, functional analysis and general theory of approximate methods are

[3] Анфиногенов, А. Ю., Лифанов, И. К., Лифанов, П. И. О некоторых гипер-сингулярных одномерных и двумерных интегральных уравнениях // – Москва: Математический сборник,– 2001. т.192, №8, – с. 3–46.

used.

The basic aspects to be defended. The following main statements are defended:

1. To derive a practical formula for calculating the derivative of a double layer acoustic potential.

2. To study some properties of operators generated by the direct value of the derivative of a simple layer acoustic potential and the derivative of a double layer acoustic potential in generalized Holder spaces.

3. To construct cubic formulas for the direct value of the derivative of a simple layer acoustic potential and for the normal derivative of a double layer.

4. To study approximate solution of a class of weakly singular surface integral equations of exterior boundary value problems for Helmholts equation by projective methods.

5. To research approximate solution of a class of hypersingular surface integral equations of first and second kind by projective methods.

Scientific novelty of the research. In the dissertation work the following main results were obtained:

1. The boundedness of the operator generated by the direct value of the derivative of a simple layer acoustic potential in generalized Holder classes, was proved.

2. Practical formula for calculating the derivative of a double layer acoustic potential was given and boundedness of the operator generated by the derivative of a double layer acoustic potential in generalized Holder classes, was proved.

3. A cubic formula for a class of weakly singular surface integrals was constructed.

4. A method for constructing a cubic formula for a surface singular integral was given and based on this method, a cubic formula for the direct value of the derivative of a simple layer acoustic potential and for the normal derivative of a double layer acoustic layer is constructed.

5. Justification for the collocation method for a class of weakly singular surface integral equations of exterior boundary value problems for Helmholts equation is given.

6. Justification of the collocation method for the system of surface integral equation of a boundary conjugation problem for Helmholts equation is given.

7. The approximation method at the support points of the operator inverse to the operator generated by the normal derivative of a double layer acoustic potential is given. Based on this method, the approximate solution of a class of hypersingular surface integral equations of first and second kind, is studied.

Theoretical and practical value of the study. The work is mainly of theoretical character. But the results obtained in the work may be used for numerical solutions in many practical problems of natural science (for example, in theory of diffraction of electromagnetic and acoustic waves).

Approbation and application. The results of the dissertation work were reported at the seminar of the department "Applied mathematics" of Azerbaijan State Oil Academy (head: corr.-member of NASA, prof. K.R.Aida-zadeh), at the seminar of the department "General and applied mathematics" of Azerbaijan State University Oil and Industry (head: prof. A.R.Aliyev), at the seminar of the department "Mathematical analysis" of Baku State University (head: prof. S.S.Mirzoyev), at the seminar of the department "Mathematical physics equations" of Baku State University (head: acad. Y.A. Mamedov), at the seminar of the section "Mathematical analysis" of IMM of NASA (head: corr.-member of NASA, prof. V.S.Guliyev), at the seminar of the section "Nonharmonic analysis" of IMM of NASA (head: corr.-member of NASA, prof. B.T.Bilalov), at the seminar of the section "Functional analysis" of IMM of NASA (head: prof. H.I.Aslanov), at the seminar of the section "Function theory" of IMM of NASA (head: doct. math. eci. V.E.Ismailov), at the institute seminar of IMM of NASA (head: corr.-member of NASA, prof. M.J.Mardanov), and at the conference dedicated to 70-th anniversary of prof. Y.J.Mamedov (Baku, 2001), at the international conference "Actual problems of mathematics and mechanics" dedicated to 90-th anniversary H.A.Aliyev (Baku, 2013), at the international conference "Function spaces and function approximation theory" dedicated to the 100 years of acad. S.M. Nikolsky (Moscow, 2015), at the international conference "Mathematical analysis, differential equations and their applications" (MADEA-7, Baku, 2015), at the international scientific seminar "Nonharmonic analysis and differential operators" (Baku, 2016), at the conference "Functional analysis and their applications " dedicated to 100 years of prof. A.Sh.Habibzadeh (Baku, 2016), at the international conference "Weight estimates of differential and integral operators and their application" dedicated to 70-th jubilee of prof. R.Oynarov (Astana, 2017), at the international conference "Operators, functions, and systems of mathematical physics" dedicated to 70 years of prof. H.Isaxanli (Baku, 2018).

Personal contribution of the author is in formulation of the goal and choice of research direction. Furthermore, all conclusions and the obtained results and research methods belong personally to the author.

Publications of the author. Publications in editions recommended by HAC under President of the Republic of Azerbaijan – 23, conference materials – 1, abstracts of papers – 7.

Institution where the dissertation work was executed. The work was performed at the department of "General and Applied Mathematics" of Azerbaijan State Oil and Industry University.

Structure and volume of the dissertation (in signs, indicating the volume of each structural subsection separately) General volume of the dissertation work consists of – 409482 signs (title page – 326 signs, table of contents – 3656 signs, introduction – 69200 signs, chapter I – 137400 signs, chapter II – 45300 signs, chapter III– 102400 signs, chapter IV – 51200 signs). Then list of references 155 names.

THE MAIN CONTENT OF THE DISSERTATION

The dissertation consists of introduction, four chapters, list of references.

In the introduction rationale of the research work is justified, degree of its elaboration is shown, goal and takes of the research are formulated, scientific novelty is reduced, theoretical and practical value is noted, information on approbation of the work is given.

In chapter I a more practical formula is developed for calculating the derivative of a double layer acoustic potential and under weaker conditions the boundedness of operator generated by the direct value of the derivative of a simple layer acoustic layer and of the derivative of a double layer acoustic potential in the generalized Holder spaces are proved. The main results of this chapter were published in the author's works [5, 6, 7, 9, 11, 16, 24].

let us consider the direct value

$$V(x) = \int_S grad_x \Phi_k(x,y)\,\rho(y)dS_y, \quad x = \left(x_1, x_2, x_3\right) \in S, \tag{1}$$

of the derivative of a simple layer acoustic potential, where $S \subset R^3$ is Lyapunov, surface, while $\rho(y) -$ is a continuous function on S.

For a function $\varphi(x)$ continuous on the surface S we introduce a continuity modulus of the form

$$\omega(\varphi,\delta) = \delta \sup_{\tau \geq \delta} \frac{\overline{\omega}(\varphi,\tau)}{\tau}, \quad \delta > 0,$$

where $\overline{\omega}(\varphi,\tau) = \max\limits_{\substack{|x-y| \leq \tau \\ x,y \in S}} \left|\varphi(x) - \varphi(y)\right|.$

Theorem 1. *Let $S -$ be a Lyapunov surface and*

$$\int_0^{diam\,S} \frac{\omega(\rho,t)}{t}\,dt < +\infty.$$

Then integral (1) exists in the sense of the Cauchy principal value, and

$$\sup_{x\in S}|V(x)|\le M\left(\|\rho\|_\infty+\int_0^{diam\,S}\frac{\omega(\rho,t)}{t}dt\right).$$

Here and in sequel, by M we will denote positive constants different in various spaces.

Theorem 2. *Let S be a Lyapunov surface with the index $0<\alpha\le 1$ and*

$$\int_0^{diam\,S}\frac{\omega(\rho,t)}{t}dt<+\infty.$$

Then the following estimations are valid:

$$\omega(V,h)\le M_\rho\left(h^\alpha+\omega(\rho,h)+\int_0^h\frac{\omega(\rho,t)}{t}dt+h\int_h^{diamS}\frac{\omega(\rho,t)}{t^2}dt\right)$$

$$for\ 0<\alpha<1,$$

$$\omega(V,h)\le M_\rho\left(h|\ln h|+\omega(\rho,h)+\int_0^h\frac{\omega(\rho,t)}{t}dt+h\int_h^{diamS}\frac{\omega(\rho,t)}{t^2}dt\right)$$

$$for\ \alpha=1,$$

where M_ρ is a positive constant dependent only on S, k and ρ.

We introduce the following classes of functions determined on $(0,diamS]$:

$$\chi=\left\{\varphi:\ \varphi\uparrow,\lim_{\delta\to 0}\varphi(\delta)=0,\varphi(\delta)/\delta\downarrow\right\},$$

$$J_0(S)=\left\{\varphi\in\chi:\ \int_0^{diamS}\frac{\varphi(t)}{t}dt<+\infty\right\}$$

and consider the function

$$Z(\varphi)=\begin{cases}h^\alpha+\varphi(h)+\displaystyle\int_0^h\frac{\varphi(t)}{t}dt+h\int_h^{diamS}\frac{\varphi(t)}{t^2}dt, & if\ \ 0<\alpha<1,\\ h|\ln h|+\varphi(h)+\displaystyle\int_0^h\frac{\varphi(t)}{t}dt+h\int_h^{diamS}\frac{\varphi(t)}{t^2}dt, & if\ \ \alpha=1.\end{cases}$$

Let $\varphi\in\chi$ and by $H(\varphi)$ denote a linear space of all functions ρ continuous on S and satisfying the condition

$$\left|\rho(x)-\rho(y)\right|\le C_\rho \varphi(|x-y|), \quad x,y\in S,$$

where C_ρ is a positive constant dependent on S and ρ, but not on x and y. It is known that the space $H(\varphi)$ is a Banach space with the norm

$$\|\rho\|_{H(\varphi)}=\sup_{x\in S}|\rho(x)|+\sup_{\substack{x,y\in S\\ x\ne y}}\frac{|\rho(x)-\rho(y)|}{\varphi(|x-y|)}.$$

Theorem 3. *Let* $\varphi\in J_0(S)$, *then the operator* $(A\rho)(x)=$ $=V(x)$, $x\in S$, *boundedly acts from* $H(\varphi)$ *to* $H(Z(\varphi))$, *and*

$$\|V\|_{H(Z(\varphi))}\le M\,\|\rho\|_{H(\varphi)}.$$

Now we consider a double layer acoustic potential

$$W_{k,\rho}(x)=\int_S \frac{\partial \Phi_k(x,y)}{\partial \vec{n}(y)}\rho(y)dS_y, \quad x\in S,$$

where $S\subset R^3$ is Lyapunov's surface, while, a $\rho(y)-$is a continuous function on S.

Theorem 4. *Let* S *be a Lyapunov surface,* $\rho(x)$ *be a continuously differentiable function on* S *and*

$$\int_0^{diam\,S}\frac{\omega(grad\rho,t)}{t}dt<+\infty.$$

Then the double layer acoustic potential $W_{k,\rho}(x)$ *has a derivative on* S *and*

$$\begin{aligned} grad\,W_{k,\rho}(x)&=\int_S grad_x\left(\frac{\partial(\Phi_k(x,y)-\Phi_0(x,y))}{\partial\vec{n}(y)}\right)\rho(y)dS_y-\\ &-\frac{3}{4\pi}\int_S\frac{(y-x,\vec{n}(y))(y-x)}{|x-y|^5}(\rho(y)-\rho(x))dS_y+\\ &+\frac{1}{4\pi}\int_S\frac{\rho(y)-\rho(x)}{|x-y|^3}\vec{n}(y)dS_y, \quad x\in S, \qquad (2)\end{aligned}$$

and

$$\left|gradW_{k,\rho}(x)\right| \le M\left(\int_0^d \frac{\omega(grad\rho,t)}{t}dt + \|\rho\|_\infty + \|grad\rho\|_\infty\right), \quad \forall x \in S,$$

where $\Phi_0(x,y) = \Phi_k(x,y)|_{k=0}$ *and the last integral in the equality (2) exists in the sense of the Cauchy principal value.*

Theorem 5. *Let* S *be a Lyapunov surface with the index* $0 < \alpha \le 1$, $\rho(x)$ *be a continuously differentiable function on* S *and*

$$\int_0^{diam\,S} \frac{\omega(grad\rho,t)}{t}dt < +\infty.$$

Then for $0 < \alpha < 1$

$$\omega\left(gradW_{k,\rho},h\right) \le$$

$$\le M_\rho\left(h^\alpha + \omega(grad\rho,h) + \int_0^h \frac{\omega(grad\rho,t)}{t}dt + h\int_h^{diamS} \frac{\omega(grad\rho,t)}{t^2}dt\right),$$

while for $\alpha = 1$

$$\omega\left(gradW_{k,\rho},h\right) \le$$

$$\le M_\rho\left(h|\ln h| + \omega(grad\,\rho,h) + \int_0^h \frac{\omega(grad\,\rho,t)}{t}dt + h\int_h^{diamS} \frac{\omega(grad\,\rho,t)}{t^2}dt\right)$$

,

where M_ρ *is a positive constant dependent only on* S, k *and* ρ.

Let $\varphi \in \chi$. By $H_1(\varphi)$ we denote a linear space of all continuously differentiable functions ρ on the surface S and satisfying the condition

$$\left|grad\rho(x) - grad\rho(y)\right| \le C_\rho\varphi(|x-y|), \quad x,y \in S,$$

where C_ρ is a positive constants dependent on S and ρ, but not on x and y. It is known that the space $H_1(\varphi)$ is a Banach space with the norm

$$\|\rho\|_{H_1(\varphi)} = \sup_{x\in S}|\rho(x)| + \sup_{x\in S}|grad\rho(x)| + \sup_{\substack{x,y\in S\\ x\ne y}} \frac{|grad\rho(x) - grad\rho(y)|}{\varphi(|x-y|)}.$$

Theorem 6. *Let* $S-$ *be a Lyapunov surface, and* $\varphi \in J_0(S)$. *Then the operator* $A\rho = gradW_{k,\rho}(x)$, $x \in S$, *boundedly acts from* $H_1(\varphi)$ *to* $H(Z(\varphi))$,

$$\left\| gradW_{k,\rho} \right\|_{H(Z(\varphi))} \le M \left\| \rho \right\|_{H_1(\varphi)}.$$

Corollary 1. *Let* S *be a Lyapunov surface, and* $\varphi \in J_0(S)$. *Then the operator* T *boundedly acts from* $H_1(\varphi)$ *to* $H(Z(\varphi))$, *and*

$$\left\| (T\rho)(x) \right\|_{H(Z(\varphi))} \le M \left\| \rho \right\|_{H_1(\varphi)}.$$

In chapter II a method for constructing a cubic formula for a surface singular integral is given and based on this method cubic formulas for the direct value of the derivative of a simple layer acoustic potential and for the normal derivative of a double layer acoustic potential are constructed. Furthermore, a cubic formula for a class of weakly singular surface integrals is constructed in this chapter. The main results of this chapter are in the papers [3, 4, 8, 12, 13, 18].

Let S be a Lyapunov surface. We divide S into "regular" elementary parts $S = \bigcup_{l=1}^{N} S_l$. Under the "regular" elementary part we agree to understand the points set subjected to the following reguirements:

(1) for any $l \in \{1, 2, ..., N\}$ the elementary part of S_l is closed and its set of internal points $\overset{0}{S_l}$ with respect to S is not empty, and $mes \overset{0}{S_l} = mesS_l$ and for $j \in \{1, 2, ...N\}$, $j \ne l$, $\overset{0}{S_l} \cap \overset{0}{S_j} = \emptyset$;

(2) for any $l \in \{1, 2, ..., N\}$ the elementary part S_l is a connected piece of the surface S with a continuous boundary;

(3) for any $l \in \{1, 2, ..., N\}$ there exists the so-called support point $x(l) = (x_1(l), x_2(l), x_3(l)) \in S_l$ such that:

(3.1) $\quad r_l(N) \sim R_l(N) \qquad \left(r_l(N) \sim R_l(N) \Leftrightarrow C_1 \le \frac{r_l(N)}{R_l(N)} \le C_2 \right.$,

where C_1 and C_2 are positive constants independent of N), where $r_l(N) = \min\limits_{x \in \partial S_l} |x - x(l)|$ and $R_l(N) = \max\limits_{x \in \partial S_l} |x - x(l)|$;

(3.2) $R_l(N) \le d/2$, where $d-$ is a radius of a standard sphere;

(3.3) for any $j \in \{1,2,...,N\}$ $\quad r_j(N) \sim r_l(N)$.

Obviously, $r(N) \sim R(N)$ and $\lim\limits_{N \to \infty} r(N) = \lim\limits_{N \to \infty} R(N) = 0$, where $R(N) = \max\limits_{l = \overline{1,N}} R_l(N)$, $r(N) = \min\limits_{l = \overline{1,N}} r_l(N)$.

Let us consider a surface integral of the form

$$B(x) = \int_S \frac{K(x,y)}{|x-y|^n} \rho(y) dS_y, \ x \in S, \tag{3}$$

where $S \subset R^3$ is a Lyapunov surface with the index $\alpha \in (0,1]$, $n-$ is a natural number, $K(x,y)-$ is a continuous function on $S \times S$ and there exists a number $\lambda \in (0,2)$ such that $x, y \in S$

$$|K(x,y)| \le M |x-y|^{n-\lambda}, \tag{4}$$

$\rho(x)-$ is a continuous function on S. Let

$$b_{lj} = \begin{cases} 0 & \text{for} \quad l = j, \\ \frac{K(x(l), x(j))}{|x(l) - x(j)|^n} mes S_j & \text{for } l \ne j. \end{cases}$$

Theorem 7. *Let a function $S \times S$ continuous on $K(x,y)$ satisfy condition (4) and there exist a natural number m such that* $\forall x, y', y'' \in S$

$$|K(x,y') - K(x,y'')| \le M \sum_{j=1}^{m} |y' - y''|^{\alpha_j} |x - y'|^{\beta_j} |x - y''|^{\gamma_j}, \tag{5}$$

where $0<\alpha_j\le 1$, $\beta_j\ge 0$, $\gamma_j\ge 0$ *u* $\alpha_j+\beta_j+\gamma_j>n-2$, $j=\overline{1,m}$. *Then the expression*

$$B^N(x(l))=\sum_{\substack{j=1\\ j\neq l}}^{N} b_{lj}\,\rho(x(j)) \tag{6}$$

at the points $x(l)$, $l=\overline{1,N}$, *is a cubic formula for the integral (3) continuous with density* ρ *on* S *and the following estimation be valid:*

$$\max_{l=\overline{1,N}}\left|B(x(l))-B^N(x(l))\right|\le M\left[\|\rho\|_\infty (R(N))^\gamma\left|\ln R(N)\right|+\omega(\rho,R(N))\right],$$

where $\gamma=\min\{\eta\ ,\ 2-\lambda\ ,\ \eta+\beta+2-n\}$, $\beta=\min\limits_{j=\overline{1,m}}\{\alpha_j+\beta_j+\gamma_j\}-\eta$, $\eta=\min\limits_{j=\overline{1,m}}\alpha_j$.

Now we construct a cubic formula for the direct value of the derivative of a simple layer acoustic potential and for the normal derivative of a double layer acoustic potential. Let

$$P_l=\left\{j\,|\,1\le j\le N\ ,|x(l)-x(j)|\le (R(N))^{\frac{1}{1+\alpha}}\right\},$$

$$Q_l=\left\{j\,|\,1\le j\le N\ ,|x(l)-x(j)|>(R(N))^{\frac{1}{1+\alpha}}\right\}$$

and $V(x)=(V_1(x),V_2(x),V_3(x))$, where

$$V_m(x)=\int_S\frac{\partial\Phi_k(x,y)}{\partial x^{(m)}}\rho(y)\,dS_y,\quad x\in S\quad (m=1,2,3).$$

Theorem 8. *Let* S *be a Lyapunov surface with the index* $0<\alpha\le 1$ *and* $\rho\in H_\beta$, $0<\beta\le 1$. *Then the expression*

$$V_m^N(x(l))=$$

$$=\sum_{\substack{j=1\\ j\neq l}}^{N}\frac{\left(ik\,|x(l)-x(j)|\exp(ik|x(l)-x(j)|)+(1-\exp(ik|x(l)-x(j)|))\right)(x_m(l)-x_m(j))}{4\pi|x(l)-x(j)|^3}\times$$

$$\times\rho(x(j))mesS_j+\sum_{j\in Q_l}\frac{x_m(j)-x_m(l)}{4\pi|x(l)-x(j)|^3}\rho(x_j)\,mesS_j$$

at the points $x(l),\ l=\overline{1,N}$*, is a cubic formula for* $V_m(x(l))$*, and*

$$\max_{l=\overline{1,N}}\left|V_m(x(l))-V_m^N(x(l))\right|\le M_\rho\left[(R(N))^{\frac{\alpha}{1+\alpha}}+(R(N))^{\frac{\beta}{1+\alpha}}\right],\quad m=\overline{1,3},$$

where M_ρ *is a positive constant dependent only on* S, k *and* ρ.

Note that by the method for constructing a cubic formula for the direct value of the derivative of a simple layer acoustic potential one can construct a cubic formula for other singular integrals along the Lyapunov surface as well.

Theorem 9. *Let* S *be a Lyapunov surface with the index* $0<\alpha\le 1$*,* $\rho(x)-$ *be a continuously differentiable function on* S *and*

$$\int_0^{diam\,S}\frac{\omega(grad\rho,t)}{t}dt<+\infty.$$

Then the expression

$$(T\rho)^N(x(l))=2\sum_{\substack{j=1\\ j\ne l}}^{N}\frac{\partial}{\partial\vec n(x(l))}\left(\frac{\partial(\Phi_k(x(l),x(j))-\Phi_0(x(l),x(j)))}{\partial\vec n(x(j))}\right)\rho(x(j))mesS_j-$$

$$-\frac{3}{2\pi}\sum_{\substack{j=1\\ j\ne l}}^{N}\frac{(x(j)-x(l),\vec n(x(j)))(x(j)-x(l),\vec n(x(l)))}{|x(l)-x(j)|^5}(\rho(x(j))-\rho(x(l)))mesS_j+$$

$$+\frac{1}{2\pi}\sum_{j\in Q_l}\frac{(\vec n(x(l)),\vec n(x(j)))}{|x(l)-x(j)|^3}(\rho(x(j))-\rho(x(l)))mesS_j$$

at the points $x(l),\ l=\overline{1,N}$*, is a cubic formula for* $(T\rho)(x)$*, and the following estimations are valid:*

$$\max_{l=\overline{1,N}}\left|(T\rho)(x(l))-(T\rho)^N(x(l))\right|\le$$

$$\le M\left[\|\rho\|_\infty(R(N))^\alpha+\|grad\,\rho\|_\infty(R(N))^{\frac{\alpha}{1+\alpha}}+\int_0^{(R(N))^{\frac{1}{1+\alpha}}}\frac{\omega(grad\,\rho,t)}{t}dt\right]$$

for $0<\alpha<1$,

$$\max_{l=\overline{1,N}}\left|(T\rho)(x(l))-(T\rho)^N(x(l))\right|\le$$

$$\leq M\left[\|\rho\|_\infty R(N)|\ln(R(N))|+\|grad\,\rho\|_\infty\sqrt{R(N)}+\int_0^{\sqrt{R(N)}}\frac{\omega(grad\,\rho,t)}{t}dt\right]$$

$$for\ \alpha=1.$$

Chapter III deals with justification of the collocation method for a class of weakly singular surface integral equations of exterior boundary value problems for Helmholts equation. Furthermore, a sequence convergent to the exact solution of initial boundary value problem is constructed and error estimation is given. The main results of this chapter are in the author's papers [1, 2, 3, 10, 14, 17, 19, 20, 22, 25, 26, 29].

let us consider the integral equation

$$\rho+B\rho=f, \tag{7}$$

where

$$(B\rho)(x)=\int_S\frac{K(x,y)}{|x-y|^n}\rho(y)dS_y,\ \ x\in S,$$

$S\subset R^3$ is a Lyapunov surface, n is a natural numbers, $K(x,y)$ is a continuous function on $S\times S$ and satisfies the condition (4), f is a given continuous function on the surface S, and $\rho(x)$ is a desired continuous function on S.

As earlier, we divide S into “regular” elementary parts $S=\bigcup_{l=1}^N S_l$ and consider the matrix with the elements $B^N=\left(b_{lj}\right)_{l,j=1}^N$

$$b_{lj}=0\ \text{ for } l=j;$$

$$b_{lj}=\frac{K(x(l),x(j))}{|x(l)-x(j)|^n}mesS_j\ \text{ for } l\neq j.$$

Let C^N be a space of N–dimensional vectors $z^N=\left(z_1^N,z_2^N,\ldots,z_N^N\right)^{\mathrm{T}}$, $z_l^N\in C,\ l=\overline{1,N}$, with the norm $\left\|z^N\right\|=\max_{l=\overline{1,N}}\left|z_l^N\right|$, where the notation “$a^{\mathrm{T}}$” means transportation of the vector a. Using cubic formula (6), we substitute integral equation (7) by the system of algebraic equations z_l^N with respect to approximate values $\rho(x(l)),\ l=\overline{1,N}$, that will be written in the form

$$\left(I^N + B^N\right)z^N = f^N, \tag{8}$$

where I^N – is a unit operator on the space C^N, $f^N = p^N f$, while $p^N : C(S) \to C^N$ – is a linear bounded operator determined by the formula $p^N f = \left(f(x(1)), f(x(2)),..., f(x(N))\right)^{\mathrm{T}}$ and called a simple drift operator.

Theorem 10. *Let* $Ker(I+B)=\{0\}$*, the function* $K(x,y)$ *satisfy conditions (4) and (5) there exist a natural number* ℓ *such that*

$$\left|K(x',y)-K(x'',y)\right| \le M\sum_{j=1}^{\ell}\left|x'-x''\right|^{a_j}\left|x'-y\right|^{b_j}\left|x''-y\right|^{c_j},\ \forall x',x'',y\in S,$$

where $0<a_j\le 1,\ b_j\ge 0,\ c_j\ge 0$ *and* $a_j+b_j+c_j>n-2,\ j=\overline{1,\ell}$*. Then equations (7) and (8) have unique solutions* $\rho_*\in C(S)$ *and* $z_*^N\in C^N$*, respectively and* $\left\| z_*^N - p^N\rho_*\right\|\to 0$ *for* $N\to\infty$ *with the estimation*

$$\left\| z_*^N - p^N\rho_*\right\| \le M\ \left[\|f\|_\infty \left(R(N)\right)^\eta \left|\ln R(N)\right| + \omega\left(f,R(N)\right)\right],$$

where $\eta=\min\{\gamma,c\}$, $c=\min\{a\ ,\ 2-\lambda\ ,\ a+b+2-n\}$, $b=\min\limits_{j=\overline{1,\ell}}\{a_j+b_j+c_j\}-a$, $a=\min\limits_{j=\overline{1,\ell}} a_j$.

Let $D\subset R^3$ be a bounded domain with a twice differentiable boundary S. In the mentioned monograph of D.Kolton and R.Kress, it was proved that if the function $u(x)$ has a normal derivative in the sense of uniform convergence, i.e. the limit

$$\frac{\partial u(x)}{\partial\vec{n}(x)} = \lim_{\substack{h\to 0\\ h>0}}\left(\vec{n}(x), grad\, u\left(x+h\vec{n}(x)\right)\right),\quad x\in S,$$

exists uniformly on S, then the solution of Helmholts equation u, satisfying the ray equations, may be represented as follows:

$$u(x)=\int_S\left\{u(y)\frac{\partial\Phi_k(x,y)}{\partial\vec{n}(y)}-\frac{\partial u(y)}{\partial\vec{n}(y)}\Phi_k(x,y)\right\}dS_y\,,\, x\in R^3\setminus\overline{D}\,. \tag{9}$$

Using this representation in the work of Burton and Miller[4] the Dirichlet exterior boundary value problem for Helmholts equation is reduced to the following second order integral equation uniquely solvable in the space $C(S)$ for any value of the wave number $\operatorname{Im}k\geq 0$

$$\rho+\widetilde{K}\rho-i\eta L\rho=Tf-i\eta(Kf-f), \tag{10}$$

where

$$(Kf)(x)=2\int_S\frac{\partial\Phi_k(x,y)}{\partial\vec{n}(y)}f(y)dS_y\,,\quad x\in S\,,$$

$f\in N(S)$ is a given function, while $\eta\neq 0$ is an arbitrary real number and $\eta\operatorname{Re}k\geq 0$. Note that the solution of equation (10) is a normal derivative in the sense of uniform convergence of the solution of the Dirichlet exterior boundary value problem for Helmholts equation on the surface S. This time the function

$$u(x)=\int_S\left\{f(y)\frac{\partial\Phi_k(x,y)}{\partial\vec{n}(y)}-\rho(y)\Phi_k(x,y)\right\}dS_y\,,\, x\in R^3\setminus\overline{D}\,,$$

is the solution of the Dirichlet exterior boundary value problem for Helmholts equation. Furthermore. equation (10) has the advantage that its solution is the solution of the moments equation obtained first by Waterman[5] for electromagnetic waves scattering. We write equation (10) in the form

$$\rho(x)+(A\rho)(x)=(Bf)(x), \tag{11}$$

where

$$(A\rho)(x)=\left(\widetilde{K}\rho\right)(x)-i\eta(L\rho)(x),\quad x\in S\,,$$

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[4] Burton, A.J., Miller, G.F. The application of integral equation methods to the numerical solution of some exterior boundary–value problems // Proceedings of the Royal Society London, – 1971. v. A323, – p. 201–220.

[5] Waterman, P.C. Matrix formulation of electromagnetic scattering // Proceedings of the IEEE, – 1965. v.53, – p.805–812.

$$(Bf)(x) = (Tf)(x) - i\eta\,((Kf)(x) - f(x)),\ x \in S.$$

Again we divide S into "regular" elementary parts $S = \bigcup_{l=1}^{N} S_l$. Then the expression

$$(A\rho)^N(x(l)) = \sum_{j=1}^{N} a_{l\,j}\,\rho(x(j)) \tag{12}$$

at the points $x(l), l = \overline{1,N}$, is a cubic formula for the integral $(A\rho)(x)$, where

$$a_{l\,j} = 0\ ,\ \text{if } l = j\ ,$$

$$a_{l\,j} = 2\left[\frac{\partial\Phi_k(x(l),x(j))}{\partial\vec{n}(x(l))} - i\eta\,\Phi_k(x(l),x(j))\right] mesS_j,\ \text{if}\ \ l \neq j,$$

and the following estimation is valid

$$\max_{l=\overline{1,N}}\left|(A\rho)(x(l)) - (A\rho)^N(x(l))\right| \le M\left[\|\rho\|_\infty R(N)|\ln R(N)| + \omega(\rho, R(N))\right].$$

Furthermore, if the function f is continuously differentiable on S and

$$\int_0^d \frac{\omega(grad\, f, t)}{t}\,dt < \infty,$$

then the expression

$$(Bf)^N(x(l)) = \sum_{j=1}^{N} b_{l\,j}\, f(x(j)) \tag{13}$$

at the points $x(l), l = \overline{1,N}$, is a cubic formula for the integral $(Bf)(x)$, where

$$b_{l\,l} = \frac{3}{2\pi}\sum_{\substack{j=1\\ j\neq l}}^{N} \frac{(x(j)-x(l),\vec{n}(x(j)))\ (x(j)-x(l),\vec{n}(x(l)))}{|x(l)-x(j)|^5}\, mesS_j -$$

$$-\frac{1}{2\pi}\sum_{j\in Q_l}\frac{(\vec{n}(x(l)),\vec{n}(x(j)))}{|x(l)-x(j)|^3}\, mesS_j + i\eta \quad \text{for}\ \ l = \overline{1,N}\ ,$$

$$b_{lj}=2\Bigg[\frac{\partial}{\partial\vec{n}(x(l))}\left(\frac{\partial(\Phi_k(x(l),x(j))-\Phi_0(x(l),x(j)))}{\partial\vec{n}(x(j))}\right)-$$

$$-\frac{3}{4\pi}\frac{(x(j)-x(l),\vec{n}(x(j)))\,(x(j)-x(l),\vec{n}(x(l)))}{|x(l)-x(j)|^5}-$$

$$-i\eta\,\frac{\partial\Phi_k(x(l),x(j))}{\partial n(x(j))}\Bigg]mesS_j \quad \text{for } j\in P_l \quad \text{and} \quad j\neq l\,,$$

$$b_{lj}=2\Bigg[\frac{\partial}{\partial\vec{n}(x(l))}\left(\frac{\partial(\Phi_k(x(l),x(j))-\Phi_0(x(l),x(j)))}{\partial\vec{n}(x(j))}\right)-$$

$$-\frac{3}{4\pi}\frac{(x(j)-x(l),\vec{n}(x(j)))\,(x(j)-x(l),\vec{n}(x(l)))}{|x(l)-x(j)|^5}+$$

$$+\frac{1}{4\pi}\frac{(\vec{n}(x(l)),\vec{n}(x(j)))}{|x(l)-x(j)|^3}-i\eta\,\frac{\partial\Phi_k(x(l),x(j))}{\partial n(x(j))}\Bigg]mesS_j \quad \text{for} \quad j\in Q_l\,,$$

and

$$\max_{l=\overline{1,N}}\left|(Bf)(x(l))-(Bf)^N(x(l))\right|\leq$$

$$\leq M\left[\|f\|_\infty R(N)|\ln R(N)|+\|grad\,f\|_\infty\sqrt{R(N)}+\int_0^{\sqrt{R(N)}}\frac{\omega(grad\,f,t)}{t}dt\right].$$

Using cubic formulas (12) and (13), we substitute the integral equation (11) by the system of algebraic equations with respect to z_l^N – approximate values $\rho(x(l))$, $l=\overline{1,N}$, that is written in the form

$$\left(I^N+A^N\right)z^N=B^Nf^N\,, \tag{14}$$

where $f^N=p^Nf$, $A^N=\left(a_{lj}\right)_{l,j=1}^N$ and $B^N=\left(b_{lj}\right)_{l,j=1}^N$.

Theorem 11. *Let f be a continuously differentiable function on S and*

$$\int_0^d\frac{\omega(gradf,\,t)}{t}dt<\infty.$$

Then equations (11) and (14) have unique solutions $\rho_* \in C(S)$ *and* $z_*^N \in C^N$ $(N \geq N_0)$, *as* $\left\| z_*^N - p^N \rho_* \right\| \to 0$ *for* $N \to \infty$ *with the estimation of convergence rate*

$$\left\| z_*^N - p^N \rho_* \right\| \leq M\left(\sqrt{R(N)} + \omega\left(grad\, f, \sqrt{R(N)}\right)\right).$$

Corollary 2. *Let* f *be a continuously differentiable function on* S *and*

$$\int_0^d \frac{\omega(gradf, t)}{t} dt < \infty,$$

$z_*^N = \left(z_1^*, z_2^*, \ldots, z_N^*\right)^{\mathrm{T}}$ *be a solution of the system of algebraic equations (14) and* $x_0 \in R^3 \setminus \overline{D}$. *then the sequence*

$$u_N(x_0) = \sum_{j=1}^{N} \frac{\partial \Phi_k(x_0, x(j))}{\partial \vec{n}(x(j))} f(x(j)) mesS_j - \sum_{j=1}^{N} \Phi_k(x_0, x(j)) z_j^* \, mesS_j$$

converges to the value $u(x_0)$ *of the solution* $u(x)$ *of the Dirichlet exterior value problem for Helmholts equation at the point* x_0, *and*

$$\left| u_N(x_0) - u(x_0) \right| \leq M\left(\sqrt{R(N)} + \omega\left(grad\, f, \sqrt{R(N)}\right)\right).$$

We give justification of the collocation method for a boundary integral equation of the mixed boundary value problem for Helmholts equation. Let $D \subset R^3$ be a bounded domain with twice continuously differentiable boundary S, f be a given continuous function on S, λ be a given value and $\operatorname{Im}\left(\overline{k}\lambda\right) \geq 0$,

$$\overline{\Phi}_k(x, y) = 2 \frac{\partial}{\partial \vec{n}(y)} \left(\Phi_k(x, y) - \Phi_0(x, y)\right), \; x, y \in R^3, x \neq y,$$

$$\upsilon_1(x, \rho) = 2 \int_S \frac{\partial \Phi_k(x, y)}{\partial \vec{n}(y)} \rho(y) dS_y, \quad \upsilon_2(x, \rho) = 2 \int_S \Phi_k(x, y) \rho(y) dS_y,$$

and $\psi(x) = \upsilon_{20}(x, \rho) -$ is a simple layer potential with density $\rho \in C(S)$ for the Laplace equation, i.e.

$$\upsilon_{20}(x, \rho) = 2 \int_S \Phi_0(x, y) \rho(y) dS_y.$$

In the paper of O.I.Panich[6] it was shown that the function

$$u(x)=\upsilon_2(x,\rho)-\mu\upsilon_1(x,\psi),\ \ x\in R^3\setminus\overline{D},$$

where μ is a complex number, and if $\operatorname{Im}k=0$, then $\operatorname{Im}\mu\neq 0$, and if $\operatorname{Im}k>0$, then $\mu=0$, is the solution of the mixed problem for Helmholts equation if the density ρ is the solution of uniquely solvable integral equation

$$\rho+A\rho=\varphi, \tag{15}$$

where

$$\varphi=(\mu-1)^{-1}f,$$

$$A=(\mu-1)^{-1}\left(\widetilde{K}-2\mu(G+2R)+\lambda\left(L-\mu\widetilde{L}-4\mu Q\right)\right),$$

$$\left(\widetilde{L}\rho\right)(x)=(L\rho)(x)|_{k=0}=2\int_S\Phi_0(x,y)\rho(y)dS_y,\ x\in S,$$

$$(G\rho)(x)=\int_S\frac{\partial\overline{\Phi}_k(x,y)}{\partial\vec{n}(x)}\left(\int_S\Phi_0(y,t)\rho(t)dS_t\right)dS_y,\ x\in S,$$

$$(R\rho)(x)=\int_S\frac{\partial\Phi_0(x,y)}{\partial\vec{n}(x)}\left(\int_S\frac{\partial\Phi_0(y,t)}{\partial\vec{n}(y)}\rho(t)dS_t\right)dS_y,\ x\in S,$$

$$(Q\rho)(x)=\int_S\frac{\partial\Phi_k(x,y)}{\partial\vec{n}(y)}\left(\int_S\Phi_0(y,t)\rho(t)dS_t\right)dS_y,\ \ x\in S.$$

Dividing S into "regular" elementary parts $S=\bigcup_{l=1}^{N}S_l$, we assume

$$a_{lj}=(\mu-1)^{-1}\left(2b_{lj}^{(k)}-2\mu\left(\sum_{m=1}^{N}g_{lm}^{(k)}c_{mj}^{(0)}+2\sum_{m=1}^{N}b_{lm}^{(0)}b_{mj}^{(0)}\right)+\right.$$
$$\left.+\lambda\left(2c_{lj}^{(k)}-2\mu\,c_{lj}^{(0)}-4\mu\sum_{m=1}^{N}e_{lm}^{(k)}c_{mj}^{(0)}\right)\right),$$

where

[6] Панич, О.И. К вопросу о разрешимости внешних краевых задач для волнового уравнения и для системы уравнений Максвелла // – Москва: Успехи математических наук, – 1965. т. 20, №1, – с. 221–226.

$$b_{ll}^{(k)} = g_{ll}^{(k)} = c_{ll}^{(k)} = e_{ll}^{(k)} = 0 \quad \text{for} \quad l = \overline{1,N} \ ,$$

$$b_{lj}^{(k)} = \frac{\partial \Phi_k\left(x(l), x(j)\right)}{\partial \vec{n}\left(x(l)\right)} mesS_j \quad \text{for} \quad l, j = \overline{1,N} \ \text{ и } \ l \neq j \ ,$$

$$g_{lj}^{(k)} = \frac{\partial \overline{\Phi}_k\left(x(l), x(j)\right)}{\partial \vec{n}\left(x(l)\right)} mesS_j \ \text{for} \quad l, j = \overline{1,N} \ \text{ и } \ l \neq j \ ,$$

$$c_{lj}^{(k)} = \Phi_k\left(x(l), x(j)\right) mesS_j \quad \text{for} \quad l, j = \overline{1,N} \ \text{ и } \ l \neq j \ ,$$

$$e_{lj}^{(k)} = \frac{\partial \Phi_k\left(x(l), x(j)\right)}{\partial \vec{n}\left(x(j)\right)} mesS_j \ \text{for} \quad l, j = \overline{1,N} \ \text{ и } \ l \neq j \ ,$$

$$b_{lj}^{(0)} = b_{lj}^{(k)}\Big|_{k=0}, \ \ c_{lj}^{(0)} = c_{lj}^{(k)}\Big|_{k=0}, \ \ e_{lj}^{(0)} = e_{lj}^{(k)}\Big|_{k=0} .$$

Theorem 12. *The expression*

$$\left(A\rho\right)^N \left(x(l)\right) = \sum_{j=1}^{N} a_{lj}\, \rho\left(x(j)\right) \tag{16}$$

at the points $x(l), l = \overline{1,N}$, *is a cubic formula for* $(A\rho)(x)$, *and the following estimation is valid:*

$$\max_{l=\overline{1,N}} \left| (A\rho)(x(l)) - (A\rho)^N \left(x(l)\right) \right| \leq M \left[\|\rho\|_\infty R(N) \left|\ln R(N)\right| + \omega\left(\rho, R(N)\right) \right].$$

Using cubic formula (16), we substitute integral equation (15) by the system of algebraic equations with respect to z_l^N −approximate values $\rho(x(l))$, $l = \overline{1,N}$, that are written in the form

$$\left(I^N + A^N\right) z^N = \varphi^N , \tag{17}$$

where $A^N = \left(a_{lj}\right)_{l,j=1}^{N}$ и $\varphi^N = \left(\mu - 1\right)^{-1} p^N f$.

Theorem 13. *Equations (15) and (17) have unique solutions* $\rho_* \in C(S)$ *and* $z_*^N \in C^N$, *respectively and* $\left\| z_*^N - p^N \rho_* \right\| \to 0$ *as* $N \to \infty$ *with the estimation*

$$\left\| z_*^N - p^N \rho_* \right\| \leq M \left[R(N) \left|\ln R(N)\right| + \omega\left(f, R(N)\right) \right].$$

Corollary 3. *Let* $z_*^N = \left(z_1^*, z_2^*, \ldots, z_N^*\right)^{\mathrm{T}}$ *be the solution of the system of algebraic equations (17) and* $x_0 \in R^3 \setminus \overline{D}$. *Then the sequence*

$$u_N(x_0) = 2\sum_{j=1}^{N} \Phi_k\left(x_0, x(j)\right) z_j^* mesS_j -$$

$$-4\mu \sum_{j=1}^{N} \frac{\partial \Phi_k\left(x_0, x(j)\right)}{\partial \vec{n}\left(x(j)\right)} \left(\sum_{\substack{m=1 \\ m \neq j}}^{N} \Phi_0\left(x(j), x(m)\right) z_m^* \, mesS_m \right) mesS_j$$

convergences to the value $u(x_0)$ *of the solution* $u(x)$ *of the mixed boundary value problem for Helmholts equation at the point e* x_0, *and*

$$\left|u_N(x_0) - u(x_0)\right| \leq M\left[\omega\left(f, R(N)\right) + R(N)\left|\ln R(N)\right|\right].$$

Now let us justify the collocation method for the system of integral equations of a conjugation boundary value problem for Helmholts equation. Let $D \subset R^3$ be a bounded domain with twice continuously differentiable boundary S, f and g be the given continuous function on S, while k, k_0, μ and μ_0 at the given complex numbers $\mathrm{Im}k \geq 0$, $\mathrm{Im}k_0 \geq 0$ and $\mu + \mu_0 \neq 0$. Kress and Roach[7] proved that combination of simple and double layer potentials

$$u(x) = \int_S \left\{ \frac{\partial \Phi_k(x,y)}{\partial \vec{n}(y)} \psi(y) + \mu \Phi_k(x,y) \varphi(y) \right\} dS_y, \; x \in R^3 \setminus \overline{D},$$

$$u_0(x) = \int_S \left\{ \frac{\partial \Phi_{k_0}(x,y)}{\partial \vec{n}(y)} \psi(y) + \mu_0 \Phi_{k_0}(x,y) \varphi(y) \right\} dS_y, \; x \in D,$$

[7] Kress, R., Roach, G.F. Transmission problems Helmholtz equation // Journal of Mathematical Physics, – 1978. v.19, – p.1433–1437.

with continuous densities ψ and φ, is the solution of the conjugation problem if ψ and φ are the solutions of uniquely solvable system of integral equations

$$\begin{aligned}&(\mu+\mu_0)\psi+(\mu K-\mu_0 K_0)\psi+\left(\mu^2 L-\mu_0^2 L_0\right)\varphi=2f,\\&(\mu+\mu_0)\varphi-(T-T_0)\psi-\left(\mu\widetilde{K}-\mu_0\widetilde{K}_0\right)\varphi=-2g,\end{aligned}\tag{18}$$

where

$$L_0=L\,|_{k=k_0},\quad K_0=K\,|_{k=k_0},\quad \widetilde{K}_0=\widetilde{K}\,|_{k=k_0},$$

$$((T-T_0)\psi)(x)=2\int_S\frac{\partial}{\partial\vec{n}(x)}\left(\frac{\partial\left(\Phi_k(x,y)-\Phi_{k_0}(x,y)\right)}{\partial\vec{n}(y)}\right)\psi(y)\,dS_y,\ x\in S.$$

On the space $C(S)\times C(S)$ introduce the operator

$$A=\frac{1}{\mu+\mu_0}\begin{pmatrix}\mu K-\mu_0K_0 & \mu^2L-\mu_0^2L_0\\ T_0-T & \mu_0\widetilde{K}_0-\mu\widetilde{K}\end{pmatrix}.$$

Then we can rewrite system (18) in the form

$$(I+A)\rho=h,\tag{19}$$

where I is a unit operator on $C(S)\times C(S)$,

$$\rho=\begin{pmatrix}\psi\\ \varphi\end{pmatrix},\quad h=\frac{2}{\mu+\mu_0}\begin{pmatrix}f\\ -g\end{pmatrix}.$$

It should be indicated that $C(S)\times C(S)$ is a Banach space with the norm $\|\rho\|_1=\max\{\|\psi\|_\infty,\|\varphi\|_\infty\}$.

We again divide S into "regular" elementary parts $S=\bigcup_{l=1}^{N}S_l$ and let $\widetilde{p}^{2N}:C(S)\times C(S)\to C^{2N}$ be a linear bounded operator determined by the following formula

$$\widetilde{p}^{2N}\rho=\widetilde{p}^{2N}\begin{pmatrix}\psi\\ \varphi\end{pmatrix}=$$

$$=\left(\psi(x(1)),\psi(x(2)),\ldots,\psi(x(N)),\varphi(x(1)),\varphi(x(2)),\ldots,\varphi(x(N))\right)^{\mathrm{T}}.$$

Consider the $2N-$dimensional matrix $A^{2N}=\left(a_{lj}\right)_{l,j=1}^{2N}$ with the elements

$$a_{lj}=0 \quad \text{for } l=\overline{1,N},\ j=\overline{1,N} \text{ and } l=j;$$

$$a_{lj}=\frac{mes\,S_j}{\mu+\mu_0}\left(\mu\frac{\partial\Phi_k\left(x(l),x(j)\right)}{\partial\vec{n}\left(x(j)\right)}-\mu_0\frac{\partial\Phi_{k_0}\left(x(l),x(j)\right)}{\partial\vec{n}\left(x(j)\right)}\right) \text{ for } l=\overline{1,N},$$
$$j=\overline{1,N} \text{ and } l\neq j;$$

$$a_{lj}=0 \text{ for } l=\overline{1,N},\ j=\overline{N+1,2N} \text{ and } l=j-N;$$

$$a_{lj}=\frac{mes\,S_{j-N}}{\mu+\mu_0}\left(\mu^2\Phi_k\left(x(l),x(j-N)\right)-\mu_0^2\Phi_{k_0}\left(x(l),x(j-N)\right)\right)$$
$$\text{for } l=\overline{1,N},\ j=\overline{N+1,2N} \text{ and } l\neq j-N;$$

$$a_{lj}=0 \text{ for } l=\overline{N+1,2N},\ j=\overline{1,N} \text{ and } l=j+N;$$

$$a_{lj}=\frac{mes\,S_j}{\mu+\mu_0}\frac{\partial}{\partial\vec{n}\left(x(l-N)\right)}\left(\frac{\partial\left(\Phi_{k_0}\left(x(l-N),x(j)\right)-\Phi_k\left(x(l-N),x(j)\right)\right)}{\partial\vec{n}\left(x(j)\right)}\right)$$
$$\text{for } l=\overline{N+1,2N},\ j=\overline{1,N} \text{ and } l\neq j+N;$$

$$a_{lj}=0 \text{ for } l=\overline{N+1,2N},\ j=\overline{N+1,2N} \text{ and } l=j;$$

$$a_{lj}=\frac{mes\,S_{j-N}}{\mu+\mu_0}\left(\mu_0\frac{\partial\Phi_{k_0}\left(x(l-N),x(j-N)\right)}{\partial\vec{n}\left(x(l-N)\right)}-\mu\frac{\partial\Phi_k\left(x(l-N),x(j-N)\right)}{\partial\vec{n}\left(x(l-N)\right)}\right)$$
$$\text{for } l=\overline{N+1,2N},\ j=\overline{N+1,2N} \text{ and } l\neq j.$$

Theorem 14. *Let* $\rho=\begin{pmatrix}\psi\\ \varphi\end{pmatrix}\in C(S)\times C(S)$. *Then the expression*

$$\begin{pmatrix}\sum_{j=1}^{N}a_{lj}\psi\left(x(j)\right)+\sum_{j=1}^{N}a_{l,N+j}\,\varphi\left(x(j)\right)\\ \sum_{j=1}^{N}a_{N+l,j}\,\psi\left(x(j)\right)+\sum_{j=1}^{N}a_{N+l,N+j}\,\varphi\left(x(j)\right)\end{pmatrix} \tag{20}$$

at the points $x(l), l=\overline{1,N}$, *is a cubic formula for* $(A\rho)(x)$, *and the following estimation is valid*

$$\left\|\widetilde{p}^{2N}(A\rho)-A^{2N}\left(\widetilde{p}^{2N}\rho\right)\right\|\leq M\left[\|\rho\|_1 R(N)|\ln R(N)|+\omega(\rho,R(N))\right].$$

Using cubic formula (20), we substitute the system of integral equations (19) by the system of algebraic equations with respect for to $z^{2N}=\left(z_1^{2N},z_2^{2N},...,z_{2N}^{2N}\right)\in C^{2N}$, being the approximate value of $\widetilde{p}^{2N}\rho$ (here z_l^{2N}, $l=\overline{1,N}$, is the approximate value of $\psi(x(l))$, while z_{N+l}^{2N}, $l=\overline{1,N}$ is the approximate value is $\varphi(x(l))$). In its turn, we write this system in the form

$$\left(I^{2N}+A^{2N}\right)z^{2N}=h^{2N}, \tag{21}$$

where $h^{2N}=\widetilde{p}^{2N}h$ and $I^{2N}-$is a unit operator on C^{2N}.

Theorem 15. *Let* $h\in C(S)\times C(S)$. *Then equations (19) and (21) have unique solutions* $\rho_*\in C(S)\times C(S)$ *and* $z_*^{2N}\in C^{2N}$, *respectively and* $\lim\limits_{N\to\infty}\left\|z_*^{2N}-\widetilde{p}^{2N}\rho_*\right\|=0$ *with the estimation of convergence rate*

$$\left\|z_*^{2N}-\widetilde{p}^{2N}\rho_*\right\|\leq M\left[R(N)|\ln R(N)|+\omega(h,R(N))\right].$$

Corollary 4. *Let* $z_*^{2N}=\left(z_1^*,z_2^*,\ldots,z_{2N}^*\right)^{\mathrm{T}}$ *be the solution of the system of algebraic equations (21). Then the sequence*

$$u^N(x^*)=\sum_{j=1}^{N}\left(\frac{\partial\Phi_k\left(x^*,x(j)\right)}{\partial\vec{n}(x(j))}z_j^*+\mu\Phi_k\left(x^*,x(j)\right)z_{N+j}^*\right)mesS_j,\ x^*\in R^3/\overline{D},$$

converges to $u\left(x^*\right)$, *while the sequence*

$$u_0^N(x_*)=\sum_{j=1}^{N}\left(\frac{\partial\Phi_{k_0}\left(x_*,x(j)\right)}{\partial\vec{n}(x(j))}z_j^*+\mu_0\Phi_{k_0}\left(x_*,x(j)\right)z_{N+j}^*\right)mesS_j\ ,\ x_*\in D,$$

converges to $u_0(x_*)$, *and*

$$\left|u^N\left(x^*\right)-u\left(x^*\right)\right|\leq M\left[R(N)|\ln R(N)|+\omega(h,R(N))\right],$$

$$\left|u_0^N(x_*)-u_0(x_*)\right|\leq M\left[R(N)|\ln R(N)|+\omega(h,R(N))\right].$$

Method of approximation at the support points of the operator inverse to the operator generated by the normal derivative of a double layer acoustic potential is given in chapter IV. Based on this method, the approximate solution of a class of surface integral equations of first kind and hypersingular integral equation of second kind of boundary value problem for Helmholts equation is studied by the projective methods. Furthermore, the sequences convergent to exact solution of the consider boundary value problems are constructed, the error estimations are given. The main results of this chapter are in the author's papers [15, 21, 23, 27, 28, 30, 31].

Let $D \subset R^3$ be a bounded domain with twice continuously differentiable boundary S, while g is a given function on S. In the mentioned book of D.Colton and R.Kress it is proved that the double layer potential

$$u(x) = \int_S \frac{\partial \Phi_k(x,y)}{\partial \vec{n}(y)} \varphi(y) dS_y , \quad x \in R^3 \setminus S ,$$

with the density $\varphi \in \mathrm{N}(S)$ is the solution of Neumann interior and exterior boundary value problem for Helmholts equation if φ is the solution of the first order hypersingular integral equation

$$T\varphi = 2g . \tag{22}$$

Note that the operator T is unbounded in the space $\mathrm{N}(S)$. However, in this work it is shown that if $\operatorname{Im} k > 0$, then for any right hand side $g \in C(S)$ the hypersingular integral equation (22) is uniquely solvable in the space $\mathrm{N}(S)$, and the solution of integral equation (22) has the form

$$\varphi = -2L\left(I - \widetilde{K}\right)^{-1}\left(I + \widetilde{K}\right)^{-1} g .$$

Consequently, the operator T^{-1}, inverse to the operator T, is given by the relation

$$T^{-1} = -L\left(I - \widetilde{K}\right)^{-1}\left(I + \widetilde{K}\right)^{-1} .$$

As earlier we divide S into "regular" elementary parts $S = \bigcup_{l=1}^{N} S_l$.

Let I^N be N dimensional unit matrix and $\widetilde{K}^N = \left(\widetilde{k}_{lj}\right)_{l,j=1}^{N}$, where

$$\widetilde{k}_{lj} = \begin{cases} 0 & \text{for} \quad l = j, \\ 2\dfrac{\partial \Phi_k(x(l), x(j))}{\partial \vec{n}(x(l))} \, mesS_j & \text{for} \quad l \neq j. \end{cases}$$

Lemma 1. *If* $\operatorname{Im} k > 0$, *then there exists the inverse operator* $\left(I^N + \widetilde{K}^N\right)^{-1}$, *and*

$$M_1 = \sup_N \left\| \left(I^N + \widetilde{K}^N\right)^{-1} \right\| < +\infty$$

u

$$\max_{l=\overline{1,N}} \left| \left(\left(I + \widetilde{K}\right)^{-1} g\right)(x(l)) - \sum_{j=1}^{N} \widetilde{k}_{lj}^{+} \, g(x(j)) \right| \leq$$
$$\leq M \left[\|g\|_\infty R(N) |\ln R(N)| + \omega(g, R(N)) \right],$$

where $\widetilde{k}_{lj}^{+}$ *is an element of the* $l-th$ *row and* $j-th$ *column of the matrix* $\left(I^N + \widetilde{K}^N\right)^{-1}$.

Lemma 2. *If* $\operatorname{Im} k > 0$, *then there exists the inverse matrix* $\left(I^N - \widetilde{K}^N\right)^{-1}$, *and*

$$M_2 = \sup_N \left\| \left(I^N - \widetilde{K}^N\right)^{-1} \right\| < +\infty$$

and

$$\max_{l=\overline{1,N}} \left| \left(\left(I - \widetilde{K}\right)^{-1} g\right)(x(l)) - \sum_{j=1}^{N} \widetilde{k}_{lj}^{-} \, g(x(j)) \right| \leq$$
$$\leq M \left[\|g\|_\infty R(N) |\ln R(N)| + \omega(g, R(N)) \right],$$

where $\widetilde{k}_{lj}^{-}$ *– is the element of the* $l-th$ *row and* $j-th$ *column of the matrix* $\left(I^N - \widetilde{K}^N\right)^{-1}$.

Let

$$f_{lj} = \begin{cases} 0 & \text{for} \quad l = j, \\ 2\,\Phi_k(x(l), x(j)) mesS_j & \text{for} \quad l \neq j. \end{cases}$$

Theorem 16. *If* $\operatorname{Im} k > 0$, *then the expression*

$$\varphi^N(x(l)) = -2\sum_{j=1}^{N} f_{lj}\left(\sum_{n=1}^{N}\tilde{k}_{jn}^{-}\left(\sum_{m=1}^{N}\tilde{k}_{nm}^{+}\, g(x(m))\right)\right)$$

is the approximate value of the solution $\varphi(x)$ of equation (22) at the points $x(l), l=\overline{1,N}$, and

$$\max_{l=\overline{1,N}}\left|\varphi(x(l)) - \varphi^N(x(l))\right| \le M\left[\|g\|_\infty R(N)|\ln R(N)| + \omega(g, R(N))\right].$$

Corollary 5. *Let* $\operatorname{Im} k > 0$*,*

$$\varphi^N(x(l)) = -2\sum_{j=1}^{N} f_{lj}\left(\sum_{n=1}^{N}\tilde{k}_{jn}^{-}\left(\sum_{m=1}^{N}\tilde{k}_{nm}^{+}\, g(x(m))\right)\right)$$

and $x_0 \in D$ ($x_0 \in R^3 \setminus \overline{D}$). Then the sequence

$$u_N(x_0) = \sum_{l=1}^{N}\frac{\partial\Phi_k(x_0, x(l))}{\partial\vec{n}(x(l))}\varphi^N(x(l))\, mesS_l$$

converges to the value $u(x_0)$ of the solution $u(x)$ of the Neumann interior (exterior) boundary value problem for Helmholts equation at the point x_0, and

$$|u_N(x_0) - u(x_0)| \le M\left[\|g\|_\infty R(N)|\ln R(N)| + \omega(g, R(N))\right].$$

Now we study approximate solution of the first kind boundary integral equation of Dirichlet interior and exterior boundary value problems for Helmholts equation. Let $D \subset R^3$ be a bounded domain with twice continuously differentiable boundary S, while f be a given continuous function on S. In the mentioned book of D.Kolton and R.Kress it was shown that the simple layer potential

$$u(x) = \int_S \Phi_k(x, y)\varphi(y)\, dS_y\,,\ x \in R^3 \setminus S,$$

with continuous density φ is the solution of Dirichlet interior and exterior boundary value problems if φ is the solution of the integral equation

$$L\varphi = 2f. \tag{23}$$

It should be indicated that the operator L^{-1}, inverse to the compact operator L, is unbounded in the space $\mathrm{N}(S)$. However, in this book it was shown that if $\operatorname{Im} k > 0$, then for any right hand side $f \in \mathrm{N}(S)$ the equation (23) has a unique solution and the solution of the integral equation (23) is of the form

$$\varphi = -2T(I-K)^{-1}(I+K)^{-1} f . \tag{24}$$

However, theorem 4 shows that if $g \in J_1(S)$, then a double layer potential with density has a continuous derivative, where $J_1(S)$ denotes a space of continuously differentiable functions on g for S, which

$$\int_0^{diam\, S} \frac{\omega(grad\, g\, , t)}{t} dt < +\infty .$$

As can be seen, the use of representation (24) for studying approximate solution of equation (23) is not convenient in the sense that additionally we have to verify fulfillment of the condition

$$(I-K)^{-1}(I+K)^{-1} f \in J_1(S).$$

Therefore, it is necessary to obtain another representation for solving equation (23). If $\operatorname{Im} k > 0$, then the operator

$$T^{-1} = -L\left(I-\widetilde{K}\right)^{-1}\left(I+\widetilde{K}\right)^{-1}$$

is an inverse operator to T, consequently the inverse operator L^{-1} is determined by the relation

$$L^{-1} = -\left(I-\widetilde{K}\right)^{-1}\left(I+\widetilde{K}\right)^{-1} T .$$

Then the solution of equation (23) has the form

$$\varphi = -2\left(I-\widetilde{K}\right)^{-1}\left(I+\widetilde{K}\right)^{-1} Tf .$$

Dividing S into "regular" elementary parts $S = \bigcup_{l=1}^{N} S_l$, we assume

$$t_{ll} = \frac{3}{2\pi} \sum_{\substack{j=1 \\ j \neq l}}^{N} \frac{\left(x(j)-x(l), \vec{n}(x(j))\right)\left(x(j)-x(l), \vec{n}(x(l))\right)}{|x(l)-x(j)|^5} mesS_j -$$

$$-\frac{1}{2\pi} \sum_{j \in Q_l} \frac{\left(\vec{n}(x(l)), \vec{n}(x(j))\right)}{|x(l)-x(j)|^3} mesS_j \quad \text{for} \quad l = \overline{1, N} ;$$

$$t_{lj}=\left[2\frac{\partial}{\partial\vec{n}(x(l))}\left(\frac{\partial(\Phi_k(x(l),x(j))-\Phi_0(x(l),x(j)))}{\partial\vec{n}(x(j))}\right)-\right.$$

$$\left.-\frac{3}{2\pi}\frac{(x(j)-x(l),\vec{n}(x(j)))(x(j)-x(l),\vec{n}(x(l)))}{|x(l)-x(j)|^5}\right]mesS_j \quad \text{for } j\in P_l,\, j\neq l;$$

$$t_{lj}=\left[2\frac{\partial}{\partial\vec{n}(x(l))}\left(\frac{\partial(\Phi_k(x(l),x(j))-\Phi_0(x(l),x(j)))}{\partial\vec{n}(x(j))}\right)+\right.$$

$$+\frac{1}{2\pi}\frac{(\vec{n}(x(l)),\vec{n}(x(j)))}{|x(l)-x(j)|^3}-$$

$$\left.-\frac{3}{2\pi}\frac{(x(j)-x(l),\vec{n}(x(j)))(x(j)-x(l),\vec{n}(x(l)))}{|x(l)-x(j)|^5}\right]mesS_j \quad \text{for } \quad j\in Q_l.$$

Theorem 17. *Let* $\mathrm{Im}k>0$ *and* $f\in J_1(S)$. *Then the expression*

$$\varphi^N(x(l))=-2\sum_{j=1}^{N}\widetilde{k}_{lj}^{-}\left(\sum_{n=1}^{N}\widetilde{k}_{jn}^{+}\left(\sum_{m=1}^{N}t_{nm}f(x(m))\right)\right)$$

at the points $x(l), l=\overline{1,N}$, *is the approximate value of the solution* $\varphi(x)$ *of equation (23), and*

$$\max_{l=\overline{1,N}}\left|\varphi(x(l))-\varphi^N(x(l))\right|\leq M\left[\sqrt{R(N)}+\omega(grad\, f,R(N))+\right.$$

$$\left.+\int_0^{\sqrt{R(N)}}\frac{\omega(grad\, f,t)}{t}dt+R(N)\int_{R(N)}^{diam\, S}\frac{\omega(grad\, f,t)}{t^2}dt\right].$$

Corollary 6. *Let* $\mathrm{Im}k>0$, $f\in J_1(S)$,

$$\varphi^N(x(l))=-2\sum_{j=1}^{N}\widetilde{k}_{lj}^{-}\left(\sum_{n=1}^{N}\widetilde{k}_{jn}^{+}\left(\sum_{m=1}^{N}t_{nm}f(x(m))\right)\right)$$

and $x_0\in D$ *(* $x_0\in R^3\setminus\overline{D}$ *). Then the sequence*

$$u_N(x_0)=\sum_{l=1}^{N}\Phi_k(x_0,x(l))\varphi^N(x(l))mesS_l$$

converges to the value $u(x_0)$ of the solution $u(x)$ of Dirichlet's interior (exterior) boundary value problem for Helmholts equation at the point x_0, and

$$|u_N(x_0)-u(x_0)| \le M\Bigg[\sqrt{R(N)}+\omega(grad\, f, R(N))+$$
$$+\int_0^{\sqrt{R(N)}} \frac{\omega(grad\, f,t)}{t}dt + R(N)\int_{R(N)}^{diam\, S}\frac{\omega(grad\, f,t)}{t^2}dt\Bigg].$$

Now we justify the collocation method for second kind hypersingular integral equations for Neumann's exterior boundary value problem and for a boundary value problem of Helmholts equation with impedance condition.

Let $D \subset R^3$ be a bounded domain with twice continuously differentiable boundary S, while $g-$be a given continuous function on S. Using representation (9), in the mentioned book of D.Kolton and R.Kress, the Neumann exterior boundary value problem is reduced to uniquely solvable in the space $\mathrm{N}(S)$ a hypersingular integral equation of second kind

$$\psi - K\psi - i\eta T\psi = -Lg - i\eta\left(g+\widetilde{K}g\right), \tag{25}$$

where $\eta \neq 0$ is an arbitrary real number and $\eta\,\mathrm{Re}\,k \ge 0$. Note that the solution of equation (25) is a boundary value of the solution of the Neumann exterior boundary value problem for Helmholts equation on S. This time the function

$$u(x)=\int_S\left\{\psi(y)\frac{\partial\Phi_k(x,y)}{\partial\vec{n}(y)} - g(y)\Phi_k(x,y)\right\}dS_y\,,\ x\in R^3\setminus\overline{D}\,,$$

is the solution of the Neumann exterior boundary value problem if $\psi \in \mathrm{N}(S)$ is the solution of hypersingular integral equation (25). Furthermore, the solution of equation (25) is the solution of the equation of zero field method obtained by Waterman[8] for acoustic waves scattering.

[8] Waterman, P.C. New formulation of acoustic scattering // The journal of the acoustical society of America, – 1969. v.45, – p.1417–1429.

Let the wave number k_0 do not coincide with the eigen-value of Dirichlet or Neumann interior problems (for that it suffices to choose any value k_0 with $\operatorname{Im} k_0 > 0$). Further, we denote by zero index the circumstance that the parameter k, that enters into the operator $\widetilde{K}$, L and T, equals the value k_0. Since the operator

$$A_0 = -L_0\left(I - \widetilde{K}_0\right)^{-1}\left(I + \widetilde{K}_0\right)^{-1} : C(S) \to \mathrm{N}(S)$$

is an operator inverse to $T_0 : \mathrm{N}(S) \to C(S)$, then conducting regularization, we can transform (25) to the equivalent form

$$\psi + A\psi = Bg\,, \tag{26}$$

and this obtained equality is considered in the space $C(S)$, where

$$A\psi = \frac{1}{i\eta} A_0\left(K + i\eta\left(T - T_0\right) - I\right)\psi\,,\quad Bg = \frac{1}{i\eta} A_0\left(L + i\eta\left(I + \widetilde{K}\right)\right)g\,.$$

Let S be divided into "regular" elementary parts $S = \bigcup_{l=1}^{N} S_l$ and

$$f_{lj}^{0} = f_{lj}\,|_{k=k_0}\,,\ l, j = \overline{1, N}\,,$$

$$c_{ll} = -1 \quad \text{for} \quad l = \overline{1, N}\,,$$

$$c_{lj} = 2i\eta \frac{\partial}{\partial \vec{n}(x(l))}\left(\frac{\partial\left(\Phi_k(x(l), x(j)) - \Phi_{k_0}(x(l), x(j))\right)}{\partial \vec{n}(x(j))}\right) mesS_j +$$

$$+ 2\frac{\partial \Phi_k(x(l), x(j))}{\partial \vec{n}(x(j))} mesS_j \quad \text{for} \quad l, j = \overline{1, N}\,,\ l \neq j\,,$$

$$g_{ll} = i\eta \quad \text{for} \quad l = \overline{1, N}\,,$$

$$g_{lj} = 2\left[\Phi_k(x(l), x(j)) + i\eta \frac{\partial \Phi_k(x(l), x(j))}{\partial \vec{n}(x(l))}\right] mesS_j$$

$$\text{for} \quad l, j = \overline{1, N}\,,\ l \neq j\,.$$

Theorem 18. *The expression*

$$(A\psi)^N(x(l)) = \sum_{j=1}^{N} a_{lj}\, \psi(x(j)) \tag{27}$$

at the points $x(l), l=\overline{1,N}$*, is a cubic formula for* $(A\psi)(x)$*, and*

$$\max_{l=\overline{1,N}}\left|(A\psi)x(l)-(A\psi)^N x(l)\right| \le M\,\left[\|\psi\|_\infty R(N)\left|\ln R(N)\right|+\omega(\psi,R(N))\right],$$

where

$$a_{lj}=-\frac{1}{i\eta}\sum_{n=1}^{N}\left(f_{ln}^{0}\left(\sum_{m=1}^{N}\widetilde{k}_{nm}^{-}\left(\sum_{t=1}^{N}\widetilde{k}_{mt}^{+}c_{tj}\right)\right)\right),\ \ l,j=\overline{1,N}.$$

Theorem 19. *The expression*

$$(Bg)^N(x(l))=\sum_{j=1}^{N}b_{lj}\,g(x(j)) \tag{28}$$

at the points $x(l), l=\overline{1,N}$*, is a cubic formula for* $(Bg)(x)$*, and*

$$\max_{l=\overline{1,N}}\left|(Bg)x(l)-(Bg)^N x(l)\right| \le M\,\left[\|g\|_\infty R(N)\left|\ln R(N)\right|+\omega(g,R(N))\right],$$

where

$$b_{lj}=-\frac{1}{i\eta}\sum_{n=1}^{N}\left(f_{ln}^{0}\left(\sum_{m=1}^{N}\widetilde{k}_{nm}^{-}\left(\sum_{t=1}^{N}\widetilde{k}_{mt}^{+}g_{tj}\right)\right)\right),\ \ l,j=\overline{1,N}.$$

Using cubic formulas (27) and (28), we substitute equation (26) by the system of algebraic equations respect to z_l^N $-$approximate values $\psi(x(l))$, $l=\overline{1,N}$, and write in the form

$$\left(I^N+A^N\right)z^N=B^N g^N, \tag{29}$$

where $A^N=\left(a_{lj}\right)_{l,j=1}^{N}$, $B^N=\left(b_{lj}\right)_{l,j=1}^{N}$ и $g^N=p^N g$.

Theorem 20. *Equations (26) and (29) have unique solutions* $\psi_*\in C(S)$ *and* $z_*^N\in C^N$*, and respectively, moreover* $\left\|z_*^N-p^N\psi_*\right\|\to 0$ *as* $N\to\infty$ *with the estimation of convergence ratio*

$$\left\|z_*^N-p^N\psi_*\right\|\le M\,\left[\|g\|_\infty R(N)\left|\ln R(N)\right|+\omega(g,R(N))\right].$$

Corollary 7. *Let* $z_*^N=\left(z_1^*,z_2^*,\dots,z_N^*\right)^{\mathrm{T}}$ *be the solution of the system of algebraic equations (29) and* $x_0\in R^3\setminus\overline{D}$*. Then the sequence*

$$u_N(x_0)=\sum_{j=1}^{N}\frac{\partial\Phi_k(x_0,x(j))}{\partial\vec{n}(x(j))}z_j^*\,mesS_j-\sum_{j=1}^{N}\Phi_k(x_0,x(j))g(x(j))mesS_j$$

converges to the value $u(x_0)$ *of the solution* $u(x)$ *of the Neumann exterior boundary value problem for Helmholts equation at the point* x_0, *moreover*

$$|u_N(x_0)-u(x_0)|\le M\,\left[\|g\|_\infty R(N)|\ln R(N)|+\omega(g,R(N))\right].$$

Let $D\subset R^3$ be a domain with twice continuously differentiable boundary S, while f and g be the given continuous function on S. In the mentioned book of D.Kolton and R.Kress it was shown that combination of simple and double layer potentials

$$u(x)=\int_S\left\{\Phi_k(x,y)+i\eta\frac{\partial\Phi_k(x,y)}{\partial\vec{n}(y)}\right\}\varphi(y)dS_y\,,\ x\in R^3\setminus\overline{D}\,,$$

where $\eta\neq 0-$ is an arbitrary real number, moreover $\eta\ \mathrm{Re}\,k\ge 0$, is the solution of a boundary value problem for Helmholts equation with impedance condition if the density φ is the solution of the following hypersingular integral equation

$$(1-i\eta f)\varphi-\left(\widetilde{K}+i\eta T+i\eta f K+f L\right)\varphi=-2g\,. \tag{30}$$

Let $\mathrm{Im}k_0>0$. Then we can transform equation (30) into the equivalent form

$$\varphi+\widetilde{A}\varphi=\widetilde{B}g\,, \tag{31}$$

and the obtained equation is considered in the space $C(S)$, where

$$\widetilde{A}\varphi=-\frac{1}{i\eta}A_0\left[(1-i\eta f)I-\left(\widetilde{K}+i\eta(T-T_0)+i\eta f K+f L\right)\right]\varphi,$$

$$\widetilde{B}g=\frac{2}{i\eta}A_0 g\,.$$

We divide S into "regular" elementary parts $S=\bigcup_{l=1}^{N}S_l$ and let

$$\widetilde{c}_{ll}=1-i\eta f(x(l))\quad\text{for}\quad l=\overline{1,N}\,;$$

$$\widetilde{c}_{lj} = -2i\eta \frac{\partial}{\partial \vec{n}(x(l))}\left(\frac{\partial\left(\Phi_k(x(l),x(j)) - \Phi_{k_0}(x(l),x(j))\right)}{\partial \vec{n}(x(j))}\right) mesS_j -$$
$$-2\frac{\partial \Phi_k(x(l),x(j))}{\partial \vec{n}(x(l))} mesS_j - 2i\eta f(x(l)) \frac{\partial \Phi_k(x(l),x(j))}{\partial \vec{n}(x(j))} mesS_j -$$
$$-2f(x(l))\Phi_k(x(l),x(j))\, mesS_j \quad \text{for} \quad l,j=\overline{1,N}, \quad l \neq j;$$
$$\widetilde{a}_{lj} = \frac{1}{i\eta}\sum_{n=1}^{N}\left(f_{ln}^0\left(\sum_{m=1}^{N}\widetilde{k}_{nm}^-\left(\sum_{t=1}^{N}\widetilde{k}_{mt}^+\widetilde{c}_{tj}\right)\right)\right), \quad l,j=\overline{1,N};$$
$$\widetilde{b}_{lj} = -\frac{2}{i\eta}\sum_{n=1}^{N}\left(f_{ln}^0\left(\sum_{m=1}^{N}\widetilde{k}_{nm}^-\widetilde{k}_{mj}^+\right)\right), \quad l,j=\overline{1,N}.$$

Then the expressions

$$\left(\widetilde{B}g\right)^N(x(l)) = \sum_{j=1}^{N}\widetilde{b}_{lj}\, g(x(j)), \tag{32}$$

$$\left(\widetilde{A}\varphi\right)^N(x(l)) = \sum_{j=1}^{N}\widetilde{a}_{lj}\, \varphi(x(j)) \tag{33}$$

at the points $x(l), l=\overline{1,N}$, are cubic formulas for $\left(\widetilde{B}g\right)(x)$ and $\left(\widetilde{A}\varphi\right)(x)$, respectively, moreover

$$\max_{l=\overline{1,N}}\left|\left(\widetilde{B}g\right)^N(x(l)) - \left(\widetilde{B}g\right)(x(l))\right| \le M\left[\omega(g,R(N)) + \|g\|_\infty R(N)\left|\ln R(N)\right|\right],$$

$$\max_{l=\overline{1,N}}\left|\left(\widetilde{A}\varphi\right)^N(x(l)) - \left(\widetilde{A}\varphi\right)(x(l))\right| \le$$
$$\le M\left[\omega(\varphi,R(N)) + \|\varphi\|_\infty \omega(f,R(N)) + \|\varphi\|_\infty R(N)\left|\ln R(N)\right|\right].$$

Using cubic formulas (32) and (33), we substitute equation (31) by the system of algebraic equations with respect to z_l^N – approximate values $\varphi(x(l)),\ l=\overline{1,N}$, and write in the form

$$\left(I^N + \widetilde{A}^N\right)z^N = \widetilde{B}^N g^N, \tag{34}$$

where $\widetilde{A}^N = \left(\widetilde{a}_{lj}\right)_{l,j=1}^N$, $\widetilde{B}^N = \left(\widetilde{b}_{lj}\right)_{l,j=1}^N$ and $g^N = p^N g$.

Theorem 21. *Equations (31) and (34) have unique solutions* $\varphi_* \in C(S)$ *and* $z_*^N \in C^N$*, respectively, this time* $\left\| z_*^N - p^N \varphi_* \right\| \to 0$ *as* $N \to \infty$ *with the estimation of the convergence rate*

$$\left\| z_*^N - p^N \varphi_* \right\| \le M \left[\omega(g, R(N)) + \omega(f, R(N)) + R(N) \left| \ln R(N) \right| \right].$$

Corollary 8. *Let* $z_*^N = \left(z_1^*, z_2^*, ..., z_N^* \right)^{\mathrm{T}}$ *be the solution of the system of algebraic equations (34) and* $x_0 \in R^3 \setminus \overline{D}$ *. Then the sequence*

$$u^N(x_0) = \sum_{j=1}^{N} \left(\Phi_k(x_0, x(j)) + i\,\eta \frac{\partial \Phi_k(x_0, x(j))}{\partial \vec{n}(x(j))} \right) z_j^* \, mesS_j$$

converges to the value $u(x_0)$ *of the solution* $u(x)$ *of the boundary value problem for Helmholts equation with impedance condition at the point* x_0*, moreover*

$$\left| u^N(x_0) - u(x_0) \right| \le M \left[\omega(g, R(N)) + \omega(f, R(N)) + R(N) \left| \ln R(N) \right| \right].$$

Conclusions

The dissertation work is devoted to study of approximate solutions of surface integral equations of boundary value problems for Helmholts equation by projective – grid methods.

The main results of the dissertation work are the followings:

1. Boundedness of the operator generated by the direct value of the derivative of a simple layer acoustic potential in generalized Holder classes, is proved.
2. A practical formula for estimating the derivative of a double layer acoustic potential is given, boundedness of the operator generated by the derivative of a double layer acoustic potentials in generalized Holder spaces, was proved.
3. A cubic formula for a class of weakly singular surface integrals was constructed.
4. A method for constructing a cubic formula for a surface singular integral is given and based on this method, a cubic formula for the direct value of the derivative of a simple layer acoustic potential and for normal derivative of a double layer acoustic potential, was constructed.
5. Justification of the collocation method for a class of weakly singular surface integral equation of exterior boundary value problem for Helmholts equation, was given.
6. Justification of the collocation method for the system of surface integral equations of a boundary conjugation value problem for Helmholts equation, is given.
7. Method of approximation at support points of the operator inverse to the operator generated by the normal derivative of a double layer acoustic potential was given. Based on this method, approximate solution of a class of hypersingular surface integral equations of first and second kind was studied.

The main results of the dissertation work were published in the following works:

1. Халилов, Э.Г. О разрешимости одного класса линейных сингулярных интегральных уравнений // Тезисы научной конференции, посвяшенной 70–летию проф. Я. Дж. Ма- медова, – Баку: – 2001, – с. 133–134.
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