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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

INVESTIGATION OF SOLUTIONS OF A MIXED PROBLEM FOR PARABOLIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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GENERAL CHARACTERISTICS OF THE DISSERTATION

The relevance of the topic and the degree of development.

The theory of nonlinear parabolic equations in non-smooth domains is important for research. Such equations are closely related to the problems of chemistry, biochemistry, physics, mechanics, biophysics, ecology and many other areas of science and applied issues.

The study of linear and nonlinear elliptic and parabolic equations in smooth domains has a long history. The behavior of solutions near boundary points, their smoothness properties, and the solvability of initial-boundary problems were studied. Research by Ladyzhenskaya, Uraltseva, Morrey, de Giorgi, Nash, Schauder, Serrin, Moser and other authors led to the solution of Hilbert's problems in the case of smooth domains. They also created new methods for solving various problems that play a fundamental role in various branches of mathematics. The presentation of the main results of these studies is contained in monographs.

In this direction, important results were obtained by E.Justi, M.Miranda, F.Browder, J.Lions, I.Neras, I.Skrypnik, O.Oleynik, V.Kondratiev, V.Mazya, E.Landis and others.

At present, a complete theory of boundary and initial-boundary problems for elliptic, parabolic and hyperbolic equations in domains with a smooth boundary has been constructed. The central result of this theory is that if the coefficient of the equation and boundary conditions, their right-hand sides, as well as the boundary of the region are sufficiently smooth, then the solution of the problem is a correspondingly smooth function. If the above conditions are violated, then this leads to the appearance of singularities in the solutions. Violations can be as follows: the coefficients of the equation are discontinuous , the boundary of the region is not smooth or the region is unbounded , there are degenerations, etc.

The dissertation work is devoted to the study of the regularity of solutions of non-linear degenerate divergent second-order parabolic equations in cylindrical domains and to the issues of removability for weak solutions. Precise conditions are set removability of a compact set for weak solutions.

Therefore, we believe that the topic of the dissertation work is relevant.

In the work, Harnack's inequality is obtained in various forms, the Hölder property of solutions, the behavior of solutions for nonlinear parabolic equations are studied. Mackenhaupt weights are considered that satisfy the duality condition with the support of the Poincaré inequality. Note that the case of nonlinear parabolic equations is very qualitatively different from the study of elliptic equations. We'll show you the details later.

We will consider nonlinear parabolic equations of the following form

$$u_t - div(\omega(x)|Du|^{p-2} Du) = 0 \quad (1)$$

$$u\big|_{\Gamma(Q_T)} = h \quad (2)$$

in a cylindrical domain, $Q_T = \Omega \times (0,T)$, where $\Omega \subset R^n, n \geq 2$ restricted area $T > 0, \Gamma(Q_T) = (\overline{\Omega} \times \{0\}) \bigcup (\partial\Omega \times [0,T])$ - parabolic boundary . $h : Q_T \to R$ is a continuous function, $\omega(x)$ - weights the Mackenhaupt function .

Our contribution in case $p \neq 2$. Note that the doubling condition and the Poincaré inequality are standard assumptions in analysis in metric spaces. It is known that Moser's method is based on a combination of Sobolev and Cacciopoli type inequalities. Let us show that the result is preserved in metric spaces, i.e. the doubling property and the Poincaré inequality imply a Sobolev-type inequality.

The object and subject of his research. The research object of the presented dissertation is the investigation of the solutions of the divergent second-order parabolic equations with nonlinear contraction in cylindrical domains and the problem of eliminating compactness for weak solutions.

The purpose and problems of the study. The purpose of the dissertation work is to solve the following main problems:

- study of Harnack's parabolic inequality
- Harnack's parabolic inequality
- estimates of super- and sub -solutions
- inverse Hölder inequality for subsolutions
- boundedness of subsolutions
- logarithmic estimate for supersolutions
- Harnack's inequality
- study of doubly non-linear parabolic equations
- regularity of solutions of equations

removability theorem for weak solutions
- estimates of weak solutions

removability theorem
- regularity of solutions of nonlinear parabolic problems with an obstacle

Research methods. The work uses methods of the theory of nonlinear partial differential equations, and methods of functional analysis .

Main conclusion presented for defense.
- Harnack's parabolic inequality
- estimates of super- and sub -solutions
- inverse Hölder inequality for subsolutions
- boundedness subsolutions
- logarithmic estimate for supersolutions
- Harnack's inequality
- doubly non-linear parabolic equations
- regularity of solutions of equations

removability theorem for weak solutions
- a priori estimates of weak solutions

removability theorem
- regularity of solutions of nonlinear parabolic problems with an obstacle

Scientific novelty of the research. The following new results were obtained in the dissertation work.
- the parabolic Harnack inequality is shown
- super- and sub -solutions were evaluated
- obtained the inverse Hölder inequality for subsolutions

- boundedness of subsolutions is shown
- obtained a logarithmic estimate for supersolutions
- Harnack's inequality is obtained for the solution
- the study of doubly nonlinear parabolic equations
- the regularity of these solutions is studied
removability theorem for weak solutions
- the existence of a solution is shown
- obtained a priori estimates of weak solutions
- the theorem on removability is proved

Theoretical and practical value of the study. New results on the qualitative theory of solutions of non-linear degenerate parabolic equations are obtained. They are used in many areas of natural science.

Approbation and application. The results obtained in the dissertation were reported and discussed at various international and republican conferences and seminars:

At scientific seminars at the IMM NAS of Azerbaijan in the departments "Differential Equations" (head - prof . A. Aliyev), "Functional Analysis" (head - prof. G.I. Aslanov), at the department "Differential Equations and Optimal Control "(Head - Prof. F. Feyziev) Sumgayit State University, as well as the XXIX International Conference "Problems of Decision Making under Uncertainty" (Ukraine - 2017), International Scientific Conference "Theoretical and Applied Problems of Mathematics" (Sumgayit - 2017), International Conference "Operators in Morrey Type Spaces and Applications" (Turkey - 2017), XXXI International Conference " Problems of Decision Making under Uncertainty " (Ukraine -2018), held at the Azerbaijan State Pedagogical University XXII Republican Scientific Conference of Doctoral Students and Young Researchers (Baku-2018) , International Scientific Conference "Information Systems and Technologies: Achievements and Prospects" (Sumgayit -2018), Inter international conference dedicated to the 90th anniversary of academician Azad Mirzajanzadeh (Baku-2018), Republican scientific conference “Fundamental problems of mathematics and the use of intelligent technologies in education”

(Sumgayit-2020), “Behavior of solutions of degenerate nonlinear parabolic equations” (Ukraine-2020) were presented at conferences.

Applicant’s personal contribution. All the main scientific results obtained in the dissertation are the result of the activity of the applicant personally as a result of application of the ideas of the supervisor in specific direction, formulation of the problems and development of the solution methods.

Publications of the author. The main results of the dissertation were published in 15 scientific works of the author. The list of these works is given at the end of the dissertation.

The name of the institution where the dissertation work was performed.

The dissertation work was carried out at the "Differential equations and optimal control" of the Sumgait State University.

The total volume of the dissertation in characters, indicating the volume of each structural unit separately.

The total volume of the dissertation is 199538 signs (title - 376 table of contents - 1931, introduction - 68363 signs, first chapter - 66000 signs, second chapter - 62000 signs, conclusion-868 signs). The dissertation consists of an introduction, two chapters, a conclusion and a 58-item bibliography.

BRIEF CONTENTS OF THE DISSERTATION

The dissertation consists of an introduction, two chapters, a conclusion and a list of references. In the introduction, an overview of the works related to the topic of the dissertation was given, and the brief content of the results obtained in the dissertation was explained.

The first chapter of the dissertation is devoted to the Hölder properties of the solutions of divergent nonlinear parabolic equations of the second order.

In section 1.1, we study the Hölder property of the solution of degenerate divergent nonlinear second-order parabolic equations.

Let $\omega(x)$ Mackenhaupt weight function $.\Omega$ bounded domain of $R^n, n \geq 2$. Weighted Sobolev space $W^1_{p,\omega(x)}(\Omega)$ define as the closure of functions from $C^\infty(\Omega)$ relative to the norm

$$\|u\|_{W^1_{p,\omega}(\Omega)} = (\int_\Omega |u|^p \omega(x)dx)^{1/p} + (\int_\Omega \omega(x)|Du|^p dx)^{1/p} \tag{3}$$

Functions belong to local space $W^1_{p,\omega(x),loc}(\Omega)$ if they belong $W^1_{p,\omega(x)}(\Omega')$ for any open subset $\Omega' \subset \Omega$ whose closure is a compact subset Ω. Weighted Sobolev space with zero boundary values $\overset{0}{W}{}^1_{p,\omega(x)}(\Omega)$, closure $C_0^\infty(\Omega)$ with respect to the norm (3).

Denote by $L_{p,loc}(t_1, t_2; W^1_{p,\omega(x),loc}(\Omega))$, $t_1 < t_2$ space of functions such that for all $t, t_1 < t < t_2$, the function $x \to u(x,t)$ belongs to $W^1_{p,\omega(x)}(\Omega)$ and $\int_{t_1}^{t_2}\int_\Omega (\omega(x)|u(x,t)|^p + \omega(x)|Du(x,t)|^p)dxdt < \infty$.

The time derivative denotes by u_t. Non-negative function u that belongs $L_{p,loc}(t_1, t_2; W^1_{p,\omega(x),loc}(\Omega))$ is a weak solution to (1) in $\Omega \times (t_1, t_2)$ if

$$\int_{t_1}^{t_2}\int_\Omega (|Du|^{p-2} Du\omega(x)D\eta - \frac{u\partial\eta}{\partial t})dxdt = 0 \tag{4}$$

for all $\eta \in C_0^\infty(\Omega \times (t_1, t_2))$.

Let $0 < \sigma \leq 1$, $\tau \in R$ and $B(z,R)$ ball in R^n.

Denote

$$U = B(z,R) \times (\tau - r^p, \tau + r^p),$$

$$\sigma U^+ = B(z,\sigma r) \times (\tau + (1/2)r^p - \frac{1}{2}(\sigma r)^p, \tau + \frac{1}{2}r^p + \frac{1}{2}(\sigma r)^p)$$

and

$$\sigma U^- = B(z,\sigma r) \times (\tau - \frac{1}{2}r^p - \frac{1}{2}(\sigma r)^p, \tau - \frac{1}{2}r^p + \frac{1}{2}(\sigma r)^p).$$

In section 1.2 we present the scale-invariant parabolic Harnack inequality.

Theorem 0.1. Let $1 < p < \infty$ and assume that the weight satisfies the duality condition and supports the weak $(1, p)$ is the Poincaré inequality.

Let $u \geq \rho > 0$ weak solution of equation (1) in U and $0 < \sigma < 1$. Then we have

$$\underset{\sigma U^-}{ess} \sup(u\omega) \leq C \underset{\sigma U^+}{ess} \inf(u\omega), \tag{5}$$

where is a constant C depends only on p, C_0, p_0 and σ.

Note that the constant C in (5) does not depend on ρ. A modification of the proof shows that the technical assumption $u \geq \rho$ can be eliminated and the result is true for all non-negative solutions.

For functions with zero boundary values, we have the following version of the Sobolev inequality.

Let's assume that $\upsilon \in \overset{0}{W}{}^{1}_{p,\omega(x)}(B(z,R))$.

Then

$$\Big(\int_{B(z,R)} |\upsilon|^k \omega(x)dx\Big)^{1/k} \leq CR\Big(\int_{B(z,R)} |D\upsilon|^p \omega(x)dx\Big)^{1/p}. \tag{6}$$

The following weighted Poincaré inequality follows from the duality property and $(1, p)$ Poincare's inequalities.

Theorem 0.2. Let's assume that $u \in W^1_{p,\omega}(B(z,R))$.

Let $\varphi(x) = \left(1 - \frac{|x-z|}{R}\right)_+^{\theta}$, where $\theta > 0$. Then there is a constant $C = C(p, C_0, P_0, \theta)$ such that for everyone $0 < r < 1$

$$\int_{B(z,r)} |u - u_\varphi|^p \varphi\omega(x)dx \leq Cr^p \int_{B(z,r)} |Du|^p \varphi\omega(x)dx$$

where

$$u_{\varphi} = \frac{\int\limits_{B(z,r)} u\varphi\omega dx}{\int\limits_{B(z,r)} \varphi\omega dx}.$$

We present some auxiliary lemmas.

We also apply the following modification of the Bombieri abstract lemma .

Lemma 0.1. Let U_{σ} bounded measurable set with $U_{\sigma'} \subset U_{\sigma}$ for $0<\delta\le\sigma'<\sigma\le 1. 0<\delta<1$ and $0<q\le\infty$. Moreover, if $q<\infty$, $q'<\infty$ suppose that the duality property holds as follows $\omega(U_1)\le C\upsilon(U_{\delta})$.

Let f positive measurable function on U_1, which satisfies the inverse Hölder inequality

$$(\int\limits_{U_{\sigma'}} f^{q}\omega(x)dx)^{1/q} \le \left(\frac{c}{(\sigma-\sigma')^{\theta}} \underset{U_{\sigma}}{f} f^{s}\omega(x)dx\right)^{1/s},$$

With $0<s<q$.

Let's f satisfies

$$\omega(\{x\in U_1 / \log f>\lambda\})\le \frac{c\cdot\omega(U_{\delta})}{\lambda^{\gamma}}$$

for everyone $\lambda>0$. Then

$$(\int\limits_{U_{\delta}} f^{q}\omega dx)^{1/q}\le c,$$

where c depends only on θ,δ,γ,q and c.

In Section 3 in Chapter 1, we estimate supersolutions and subsolutions .

The following Cacciopoli-type estimates follow with an appropriate choice of the tes function in (4).

Lemma 0.2. Let's $u\ge p>0$ supersolution in $\Omega\times(t_1,t_2)$. Then $\upsilon=u^{-1}$ sub-solution.

Lemma 0.3. Let $u \geq \rho > 0$ supersolution in $\Omega \times (t_1, t_2)$ and $\varepsilon > 0$ with $\varepsilon \neq p-1$. Then there is a constant $C(p,\varepsilon)$ such that

$$\int_{t_1}^{t_2}\int_{\Omega} |Du|^p u^{-\varepsilon-1} \varphi^p \omega(x)dxdt + ess\sup_{t_1<t<t_2} \int_{\Omega} u^{p-1-\varepsilon} \varphi^p \omega(x)dx \leq$$

$$\leq C\int_{t_1}^{t_2}\int_{\Omega} u^{p-1-\varepsilon} |D\varphi|^p \omega(x)dxdt + C\int_{t_1}^{t_2}\int_{\Omega} u^{p-1-\varepsilon} \varphi^{p-1} \left|\frac{\partial\varphi}{\partial t}\right| \omega(x)dxdt.$$

for any $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$, with $\varphi \geq 0$.

Next, we prove the corresponding result for the subsolution . Note that there may be quantities in the next lemma that are not a priori finite. But, we can make the necessary calculations using a truncated test function. Finally, we get the result by letting the truncation level go to infinity. This confirms the calculations made in the proof of the lemma, which will be given later.

Lemma 0.4. Let $u \geq \rho > 0$ sub solution in $\Omega \times (t_1, t_2)$ $\varepsilon > 0$. Then there is a constant $C(\varepsilon, p)$ such that

$$\int_{t_1}^{t_2}\int_{\Omega} |Du|^p u^{\varepsilon-1} \varphi^p \omega dxdt + ess\sup_{t_1<t<t_2} \int_{\Omega} u^{p-1+\varepsilon} \varphi^p \omega(x)dx \leq$$

$$\leq C\int_{t_1}^{t_2}\int_{\Omega} u^{p-1+\varepsilon} |D\varphi|^p \omega(x)dxdt + C\int_{t_1}^{t_2}\int_{\Omega} u^{p-1-\varepsilon} \varphi^{p-1} \left|\frac{\partial\varphi}{\partial t}\right| \omega dxdt$$

for all $\varphi \in C_0^\infty(\Omega \times (t_1, t_2))$, With $\varphi \geq 0$.

Finally, let us show Cacciopoli 's inequality for the logarithm of the supersolution .

Lemma 0.5. Let $u \geq \rho > 0$ super solution in $\Omega \times (t_1, t_2)$. Then there is a constant $C(p)$ such that

$$\int_{t_1}^{t_2}\int_{\Omega} |D(\log u)|^p \varphi^p \omega dxdt + ess\sup_{t_1<t<t_2} \left|\int_{\Omega} \log u \varphi^p \omega dx\right| \leq$$

$$\leq C\int_{t_1}^{t_2}\int_{\Omega} |D\varphi|^p \omega dxdt + C\int_{t_1}^{t_2}\int_{\Omega} |\log u| \varphi^{p-1} \left|\frac{\partial\varphi}{\partial t}\right| \omega dxdt,$$

for each $\varphi \in C_0^\infty(\Omega \times (t_1,t_2))$, with $\varphi \geq 0$.

Let $0 < \sigma \leq 1,\ \tau \in R,\ T > 0$ and $B(z,r)$ ball in R^n.

Denote

$$Q = B(z,r) \times (\tau - Tr^p, \tau + Tr^p)$$

$$\sigma Q = B(z,\sigma r) \times (\tau - T(\sigma r)^p, \tau + T(\sigma r)^p).$$

The parameter T will be chosen so that the time intervals in different lemmas are compatible. In the next lemma, we obtain a constant that does not depend on the parameter s.

Moser method , only a finite number of iterations is needed. In this case, it is not necessary to control the asymptotic behavior of the constants. In our case, the number of iterations is not limited and it divides the cylinder into geometrically converging parts in order to obtain a uniform estimate of the constant.

Lemmas 0.6. Let $u \geq \rho > 0$ sub-solution in Q and $0 < \delta < 1$. Then there are positive constants $C(p,q,C_0,P_0,T)$ and $\theta(p,C_0)$ such that

$$\left(\int_{\sigma' Q} u^q \omega dxdt\right)^{1/q} \leq \left(\frac{C}{(\sigma - \sigma')^\theta}\right)^{1/s} \left(\int_{\sigma Q} u^s \omega dxdt\right)^{1/s},$$

for all $0 < \delta \leq \sigma' < \sigma \leq 1$ and $0 < s < q < q_0$.

Here $q_0 = (p-1)(2 - p/k)$ and $k > p$.

In section 1.4 he proves the reverse Hölder 's inequality for super- and sub- solutions .

In section 1.5 , the proof of boundedness of subsolutions is based on the following lemma.

Lemma 0.7. Let $u \geq \rho > 0$ subsolution in Q and $0 < \delta < 1$. Then there are positive constants $C(p,C_0,P_0,T,\delta)$ and $\theta(p,C_0)$ such that

$$\operatorname*{ess\,sup}_{\sigma' Q} u \leq \left(\frac{C}{(\sigma - \sigma')^\theta}\right)^{1/s} \left(\int_{\sigma Q} u^s \omega dxdt\right)^{1/s}$$

for all $0 < \delta \leq \sigma' < \sigma \leq 1$ and $s > 0$.

In section 1.6 we show that the conditions for the logarithm under the assumptions of Lemma 0.1 are true.

Lemma 0.8. Let $u \geq \rho > 0$ super solution in Q and let it go

$$\varphi(x,t) = \varphi(x) = \left(1 - 2\frac{|x-z|}{(1+\sigma)r}\right)_+ ,$$

where

$$0 < \sigma < 1$$

and

$$(x,t) \in B(z,r) \times (\tau - (\sigma r)^p, \tau + (\sigma r)^p) .$$

Let

$$\beta = \int_{B(z,r)} \log u(x,\tau)\varphi^p(x)\omega dx.$$

Then there are constants $C(p, C_0, P_0, \sigma, T)$ and $C'(p, C_0, \sigma, T)$ such that

$$\nu(\{(x,t) \in \sigma Q^- \big| \log u(x,t) > \lambda + \beta + c'\}) \leq \frac{c}{\lambda^{p-1}} \nu(\sigma Q^-)$$

and

$$\nu(\{(x,t) \in \sigma Q^+ \big| \log u(x,t) < -\lambda + \beta - c'\}) \leq \frac{c}{\lambda^{p-1}} \nu(\sigma Q^+)$$

for any $\lambda > 0$.

In section 1.7 , we first give a weak Harnack inequality.

Theorem 0.3. Let $u \geq \rho > 0$ supersolution in U . Then there are constants $C(p, C_0, P_0, q, \delta)$ and

$q_0 = (p-1)(2 - p/k), k > p$ like in $k = \begin{cases} \dfrac{d_\mu p}{d_\mu - p}, & 1 < p < d_\mu \\ 2, & p \geq d_\mu \end{cases}$

such that

$$\left(\int_{\delta U^-} u^q \omega(x) dxdt\right)^{1/q} \leq C \operatorname*{ess\,inf}_{\delta U^+} u,$$

for $0 < \delta < 1$ and $0 < q < q_0$.

In section 1.8, Harnack's inequality will be obtained for solutions of doubly nonlinear parabolic equations of the following form

$$\frac{\partial(u^{p-1})}{\partial t} = div(\omega(x)|Du|^{p-2}Du), \ \ 1 < p < \infty . \tag{7}$$

Solution (7) can be scaled with non-negative factors, but due to the non-linear term (u^{p-1}) we cannot add a constant to the solution.

The derivation of the Harnack inequality for equation (7) is based on the Moser method and the parabolic version of the John-Nirenberg lemma. The above is also true for this equation. Moreover, the results are also valid for more general equations of the following form

$$\frac{\partial(u^{p-1})}{\partial t} = divA(x,t,u,Du) \tag{8}$$

where A are Carathéodore functions and satisfy the conditions

$$\begin{gathered} A(x,t,u,Du)Du \ge C_0\omega(x)|Du|^p \\ |A(x,t,u,Du)| \le C_1\omega(x)|Du|^{p-1}, \end{gathered} \tag{9}$$

where C_0 and are C_1 positive constants, the $\omega(x)$ Mackenhaupt -type function satisfies the duality condition.

Let $t_1 < t_2$ and $1 < p < \infty$. A non-negative function u belonging to the space $L_{p,loc}(t_1,t_2;W^1_{p,\omega(x),loc}(\Omega))$ is a generalized solution of equation (0.7) in $\Omega\times(t_1,t_2)$ if the integral identity is satisfied

$$\int_{t_1}^{t_2}\int_{\Omega}(|Du|^{p-2}DuD\eta - u^{p-1}\frac{\partial\eta}{\partial t})\omega(x)dxdt = 0 \tag{10}$$

for any $\eta \in C_0^\infty(\Omega\times(t_1,t_2))$.

Theorem 0.4. Let $1 < p < \infty$ the weight is dual and supports the weak $(1,p)$ Poincaré's inequality.

Let be $u \ge \rho > 0$ a weak solution of (7) in U and $0 < \sigma < 1$. Then

$$\underset{\sigma U^-}{ess\sup}\,\omega u \le C\,\underset{\sigma U^+}{ess\inf}\ \omega u, \tag{11}$$

where is a constant C depends only on p, C_0, P_0 and σ.

It is well known that at $p = 2$ the Hölder continuity of the weak solution follows from the Harnack inequality. However, thanks to the non-linear term $(u^{p-1})_+$, for $p \ne 2$, it is not clear how to change the same proof for a doubly non-linear equation.

Theorem 0.5. Let $u \ge \rho > 0$ supersolution in U. Then there are constants C and $q_0 = (p-1)(2 - p/k)$ such that

$$\Big(\int_{\delta U^-} u^q \omega(x)dxdt\Big)^{1/q} \le C\,\underset{\delta U^+}{ess\inf}\,(u\omega)$$

for $0 < \delta < 1$ and $0 < q < q_0$.

In chapter II, the exact regularity of solutions of nonlinear parabolic equations and removable sets are studied.

In this chapter, we will study the regularity of solutions of the Dirichlet problem for nonlinear parabolic equations and the removability of the compact set of solutions to this problem.

Let $Q = \Omega \times (0,T) \subset R^n \times R$ a cylindrical domain, where $\Omega \subset R^n$ is a smooth, bounded domain, $T > 0$, $n \ge 2$.

Denote ∂Q by $\partial Q = (\overline{\Omega} \times \{0\}) \bigcup (\partial\Omega \times [0,T])$ parabolic boundary Q.

Let a continuous function be given $h : \overline{Q} \to R$.

Consider the problem

$$u_t - div(\omega(x)\left|Du\right|^{p-2} Du) = 0 \text{ in } Q \tag{12}$$

$$u = h \quad \text{ in } \quad \partial Q \tag{13}$$

We can consider more general equations

$$u_t - div(\omega(x)a(Du)) = 0, \tag{14}$$

under certain conditions on the function $a : R^n \to R^n$. Weight $\omega(x)$ - as in the first chapter from Mackenhaupt 's class .

For this type of problems, the questions of the regularity of the solution are considered in the works of Di Benedetto . The apparent lack of isotropy and, as a consequence, the very problem of scaling properties becomes more complex compared to the linear case $p=2$. For this reason, the classical regularity analysis based on direct scaling and shrinking cylinder decay estimates do not apply in this case.

Overcoming this leads to the study of local regularity properties by analyzing the decay into truncated cylinders, the size of which depends on the solution itself. This is the basic idea behind Di Benedetto 's internal geometry .

We introduce the space of functions used below.

Let $A \subset R^{n+1}$ the function $f : A \to R^m, m \geq 1$

$$\underset{A}{oscf} = \sup_{(x_0,t_0),(x,t)\in A} \left| f(x_0,t_0) - f(x,t) \right|$$

means oscillation f on A. For given $(x_0,t_0) \in R^{n+1}$ and $r, \lambda > 0$ introduce cylinders

$$Q_r^\lambda(x_0,t_0) = \{(x,t) \in R^{n+1} : |x - x_0| < r, |t - t_0| < \lambda^{2-p} r^p\} .$$

For the function, j we introduce the following.

Let $W : R_+ \to R_+$ be a convex modulus of continuity, i.e. such a convex nondecreasing function that $W(1)=1$ and $W(0) = \lim_{r \to 0^-} W(r) = 0$. Then for the function $W(1) = 1$ we define on $Q = \Omega \times (0,T) \subset R^n \times R$

$$|f|_{\overline{C}^{w(\cdot)}_{\omega(x)}(Q)} = \inf\{\lambda > 0 / \sup_{Q_r^{\lambda w(r)} \subset R^n \times R} \left(\frac{1}{\lambda w(r)} \underset{Q_r^{\lambda w(r)} \cap Q}{osc} \omega f \right) \leq 1\}.$$

For time-independent functions, we define the resulting norm with respect to $W(\cdot)$, as

$$|f|_{C^{w(\cdot)}_{\omega(x)}(\Omega)} = \inf\{\lambda > 0 / \sup_{B(x,r) \subset R^n} \left(\frac{1}{\lambda w(r)} \underset{B(x,r) \cap \Omega}{osc} \omega f \right) \leq 1\}.$$

It is clear that the localized version of the above spaces is defined in the usual way and we write, for example, $f \in \overline{C}^{w(\cdot)}_{loc}(Q)$ if and only if $f \in \overline{C}^{w(\cdot)}_{\omega(x)}(Q')$, where $Q' \subseteq Q$. Moreover, $C^0(Q), C^0(\Omega)$ denotes sets of functions that are continuous in Q and Ω, respectively.

Note that in a special case the $W(r)=r^{\alpha}, \alpha \in (0,1]$ above spaces denote Hölder continuity:

$$W(r)=r^{\alpha}, \alpha \in (0,1], \ \left| f \right|_{\overline{C}^{w(\cdot)}_{\omega(x)}(Q)} < \infty \Leftrightarrow$$

$$\Leftrightarrow \sup_{z_1,z_2} \frac{\left| \omega(z_1) f(z_1) - \omega(z_2) f(z_2) \right|}{\left\| z_1 - z_2 \right\|_{\alpha}^{\alpha}} < \infty,$$

where is the parabolic metric as

$$\left\| (x_1,t_1)-(x_2,t_2) \right\|_{\alpha} = \max\{\left| x_1 - x_2 \right|, \left| t_1 - t_2 \right|^{1/[p-\alpha(p-2)]}\}.$$

In particular, the metric depends on the degree of regularity considered. Note also that when $p=2$ these spaces coincide with the spaces of functions that are Hölder continuous of order α with respect to the standard parabolic metric. Let us formulate a result on internal optimal regularity.

Theorem 0.6. Let u a solution to problem (12), (13) and $Q' \subset Q$ a bounded space-time cylinder such that $\overline{Q'} \cap \partial Q = \varnothing$. Then $u \in \overline{C}^{w(\cdot)}_{\omega(x)}(Q')$ and

$$\left| u \right|_{\overline{C}^{w(\cdot)}_{\omega(x)}(Q')} \le c(n, p, w(\cdot), \omega(x), Q, Q', \underset{Q}{osch}).$$

Note that from Theorem 0.6. the following result follows.

Theorem 0.7. Let be u a solution to (12), (13) with $h \in L_{\infty}(Q)$ .Then $Du \in L_{\infty,loc}(Q)$.

Applying Theorem 0.6, we establish exact conditions for the removability of sets for weak solutions. The problem will be considered in the cylinders introduced in the previous paragraph. Let us define the Hausdorff measure with the modulus of continuity $w(\cdot)$.

Let δ fixed, $0 < \delta < r_0$ and $E \subset R^{n+1}$.

$L(\delta, w(\cdot); E) = \{Q_{r_i}^{w(r_i)}(x_i,t_i)\}$ cylindric family such that

$$E \subseteq \bigcup Q_{r_i}^{w(r_i)}(x_i,t_i)$$

and

$$0 < r_i < \delta \text{ for } i = 1,2,\ldots.$$

Using these notations, we introduce the Hausdorff measure

$$H^{w(\cdot)}(E) = \lim_{\delta \to 0} \inf_{L(\delta, w(\cdot); E)} \{\sum r_i^n w(r_i) : E \subseteq \bigcup Q_{r_i}^{w(r_i)}(x_i, t_i)\},$$

where the infinimum is taken with respect to all possible covers of the $L(\delta, w(\cdot), E)$ set E.

Theorem 0.8 (removable singularities). Let a $Q \subset R^{n+1}$ cylindrical region and $E \subset Q$ a closed set.

We assume that the u weak solution

$$u_t - div(\omega(x)|Du|^{p-2} Du) = 0 \text{ in } Q \setminus E$$

and that $u \in \overline{C}^{w(\cdot)}_{\omega(x),loc}(Q)$.

Let's assume that $H^{w(\cdot)}(E) = 0$. Then the set E removable, i.e. u can be extended to a weak solution in Q.

Note that in the definition of the Hausdorff measure, different $w(\cdot)$ correspond to various Hausdorff measures (and dimensions) regarding the evaluation of functions and metrics for the foundations of the Hausdorff measure . The peculiarity, in our case, is that each time we define $H^{w(\cdot)}$, we consider the metric - or, equivalently, the cylinders used for coverings - and the measurable functions that are related to each other. For a closer comparison with the situation when the standard parabolic Hausdorff measure , we note that in the case $w(r) = r^{\alpha}, \alpha \in (0,1]$, we obtain

$$H_{\alpha}^{\sigma}(E) = \lim_{\delta \to 0} \inf_{L(\delta, r^{\alpha}, E)} \{\sum r_i^{\sigma} : E \subseteq \bigcup Q_{r_i}^{r_i^{\alpha}}(x_i, t_i)\}.$$

In this case, the Lebesgue measure of the cylinder $Q_{r_i}^{r_i^{\alpha}}(x_i, t_i)$ used in coating E essentially equals $r_i^{n+\alpha(2-p)+p}$ and by the assumption of Theorem 0.9. for a set E, $H_{\alpha}^{n+\alpha}(E) = 0$.

When $\alpha = 1$ the standard parabolic Hausdorff measure is considered .

$$H^{\sigma}(E) = \lim_{\delta \to 0} \inf_{L(\delta, r; E)} \{\sum r_i^{\sigma} : E \subseteq \bigcup B(x_i, r_i) \times (t_i - r_i^2, t_i + r_i^2)\}$$

In this case, the following theorem is true.

Theorem 0.9. Let Q, E are the same as in Theorem 0. 8 . We assume that the u weak solution

$$u_t - div(\omega(x)|Du|^{p-2} Du) = 0 \text{ in } Q \setminus E$$

and, what $u \in \overline{C}^{w(\cdot)}_{\omega(x),loc}(Q)$ with $w(r) = r, r \geq 0$.

Let $N = n+2$ and let $H^{N-1}(E) = 0$. Then the set E removable, i.e. u – can be extended to a weak solution in Q.

Note that the $N = n+2$ standard parabolic dimension. Also, Theorem 0.9 is the optimal parabolic analogue of the well-known results in the elliptic case.

Recall the result of Carleson, where he first obtained sufficient conditions on the set $E \subset R^n$ about removability with respect to harmonic functions in the form $H^{n-1}(E) = 0$.

Solution scaling

Let $(x_0, t_0) \in R^n \times R,\ R > 0$ and suppose that w solution (12), (13) in cylinders $Q^{\lambda w(R)}_{R,\pm}(x_0, t_0)$ entered above.

Consider $r \leq R,\ \lambda > 0$ and define

$$\tilde{w}(x,t) = \frac{w(x_0 + rx, t_0 + (\lambda w(r))^{2-p} r^p t))}{\lambda w(r)}. \qquad (15)$$

Then $\tilde{w}$ the solution of the equation

$$\tilde{w}_t - div(\omega(x)|D\tilde{w}|^{p-2} D\tilde{w}) = 0 \ \text{ in } \ Q^{\tilde{w}(R/r)}_{R/r,\pm},$$

where $\quad \tilde{w}(\gamma) = \dfrac{w(\gamma r)}{w(r)}$ for $\gamma > 0.$ (16)

In particular, when $r = R$, we have that $\tilde{w}$ solution in $Q^1_{1,\pm}$.

Gradient Estimates

Let us estimate the local supremum of the gradient of the solution in the form of the inverse Hölder inequality. In the non-degenerate case, the result is obtained. Our case is degenerate and more general equations $u_t - div(\omega(x) a(Du)) = 0$.

Let us present one more fundamental regularity result obtained by DiBenedetto and Friedmann for evolutionary parabolic equations.

Theorem 0.10. Let's pretend that w is a weak solution (12), (13) in the space-time cylinder Q. Then Dw Hölder continuous in Q. In addition, let $Q_{r,-}^{\lambda r} \subset Q$ for some $r, \lambda > 0$ such that

$$\sup_{Q_{r,-}^{\lambda r}} |Dw| \leq A\lambda$$

right, with some constant $A \geq 1$. Then there is $\tilde{\alpha}(n, p, L, A) \in (0,1]$ such that

$$\underset{Q_{\rho,-}^{\lambda\rho}}{osc}\, Dw \leq 4A\lambda \left(\frac{\rho}{r}\right)^{\tilde{\alpha}}, \tag{17}$$

true for everyone $\rho \in (0, r)$. Here $Q_{\rho,-}^{\lambda\rho} \subset Q_{r,-}^{\lambda r}$ for $0 < \rho \leq r$ an inner cylinder that shares its center with $Q_{r,-}^{\lambda r}$.

Later of removability theorem is proved.

Let u a weak solution

$$u_t - div(\omega(x)|Du|^{p-2} Du) = 0 \text{ in } Q \setminus E$$

and $H^{w(\cdot)}(E) = 0$. suppose that $u \in \overline{C}_{\omega(x),loc}^{w(\cdot)}(Q)$

Let $Q_2 \subseteq Q_1 \subseteq Q$ two arbitrary, not fixed, smooth space- time cylinders. To prove the theorem, it is necessary to show it in Q_1, since the solution, which is weak, has local properties. By assumption $u \in \overline{C}_{\omega(x),loc}^{w(\cdot)}(Q)$, there $M > 0$ is such that

$$\underset{Q_1}{osc}(\omega(x)u) \leq M \text{ and}$$

$$\underset{Q_r^{Mw(r)} \cap Q_1}{osc}(\omega(x)u) \leq Mw(r). \tag{18}$$

Using existence results, we obtain that there exists a unique continuous solution to υ problem (12),(13)

$$\upsilon_t - div(\omega(x)|D\upsilon|^{p-2} D(u-\upsilon)) = 0 \text{ in } Q_1$$

$$\upsilon = u \text{ on } \partial Q_1.$$

Let μ a non-negative Riesz measure of association with υ. existence μ follows from the fact that υ supersolution .

Let $F = \{(x,t) \in Q_1 \cdot \upsilon(x,t) = u(x,t)\}$. First of all, let us show that the support belongs to the following set

$$F \bigcap E \,. \tag{19}$$

Next, we study the problem with obstacles. Equation (12) will be denoted by Lu. Then for a continuous function $h: Q \to R$ and a continuous obstacle $\psi : Q \to R$ such that $h \geq \psi$ on ∂Q, consider the problem

$$\begin{aligned} &\max\{Lu, \psi - u\} = 0 \qquad в \quad Q \\ &u = h \qquad\qquad\qquad\quad на \quad \partial Q. \end{aligned} \tag{20}$$

We are interested in the question of the optimal regularity of the solution u subject to regularity h and ψ. The goal is to prove that the solutions (20) have the same degree of regularity as the data h,ψ. And here it is important that we do not assume differentiability of the obstacle ψ by time.

Other considerations regarding the internal geometry, cylinders, the designation of these cylinders, the definition of solutions in spaces, the definition of spaces remain valid.

Let us present a result on the optimal internal regularity of problem (20), assuming that the obstacle is from the space $\widetilde{C}^{w(\cdot)}_{\omega(x)}$.

Theorem 0.11. Let L operator from (20) and $\psi \in \widetilde{C}^{w(\cdot)}_{\omega(x)}(Q)$, u - solution (20). Suppose $Q' \subseteq Q$ a limited space-time cylinder such that $\overline{Q'} \bigcap \partial Q = \varnothing$. Then $u \in \widetilde{C}^{w(\cdot)}_{\omega(x)}(Q')$ and the estimate is hold

$$|u|_{\widetilde{C}^{w(\cdot)}_{\omega(x)}(Q')} \leq c(n, p, L, w(\cdot),\ Q, Q', \operatorname*{osc}_{Q} h, |\psi|_{\widetilde{C}^{w(\cdot)}_{\omega(x)}(Q)}).$$

CONCLUSION

The dissertation work is devoted to the study of the regularity of the solutions of the divergent second order parabolic equations with nonlinear cracking in cylindrical domains and the issues that can be overcome for weak solutions.

The following main results were obtained in the dissertation work.

- the parabolic Harnack inequality is shown
- super- and sub -solutions were estimates
- obtained the inverse Hölder inequality for subsolutions
- boundedness of subsolutions is shown
- obtained a logarithmic estimate for supersolutions
- Harnack's inequality is obtained for the solution
- the study of doubly nonlinear parabolic equations is considered
- the regularity of these solutions is studied
- removability theorem for weak solutions
- the existence of a solution is shown
- obtained a priori estimates of weak solutions
- the theorem on removability is proved.

The main results of the dissertation are published in the following scientific papers:

1. Gadjiev, T.S., *Yagnaliyeva, A.H.*, Zulfalieva, G. Regularity of solutions of degenerate parabolic non-linear equations and removability of solutions. // -USA: Journal of Applied Computational Mathematics, -2017. v.6, issue 3, -p.1-3.
2. Gadjiev, T.S., *Yagnaliyeva, A.H.*, Kerimova, M. The some property of solutions degenerate nonlinear parabolic equations. // International conference on "Operators in Morrey-type spaces and applications" dedicated to 60-th birthday of professor Guliyev V.S., -Turkey, Ahi Evran University, -10-13 July, -2017, -p. 176-177
3. Gadjiev, T., *Yagnaliyeva, A.H.*, Aliyev, Kh. Regularity of solution degenerates parabolic non-linear equation and removability theorem for solutions. // XXIX International conference "Problems of Decision Making Under Uncertainties". -Mukachevo, Ukraine, -10-13 May, -2017, -p.44-45
4. Gadjiev, T.S., *Yagnaliyeva, A.H.* Regularity of solution degenerates parabolic non-linear equations. // Sumqayıt Dövlət Universitetinin 55 illiyinə həsr olunan "Riyaziyyatın nəzəri və tətbiqi problemləri" beynəlxalq elmi konfrans, -Sumqayıt, -25-26 may -2017, -p. 116-117.
5. Gadjiev, T.S. Regularity of solution of degenerate parabolic nonlinear equations and removability of solutions./ T.S.Gadjiev, S.Y.Aliev, *A.H.Yagnaliyeva* [et all] // -Baku: The Reports of National Academy of Sciences of Azerbaijan, -2018. v. LXXIV, №1, -p. 6-11
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9. *Yagnaliyeva, A.H.* The removability of solution degenerates parabolic non-linear equations. // "Information systems and technologies: achievements and perspectives" International scientific conference, -Sumgayit, -15-16 november, -2018, -p. 234-235
10. Gadjiev, T.S., Aliyev, Kh., *Yagnaliyeva, A.H.* The behavior of solutions to degenerate nonlinear parabolic equations. // XXXV International conference "Problems of Decision Making Under Uncertainties", -Baku-Sheki, -11-15 may, -2020, -p.43-44
11. Gadjiev, T.S., Rustamov, Y.I., *Yagnaliyeva, A.H.* Harnacks inequality for degenerate double nonlinear parabolic equations. // "Fundamental problems of mathematics and application of intellectual technologies in education" Republican scientific conference. -Sumgayit State University, -03-04 July, -2020, №3, -p.88-89
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