REPUBLIC OF AZERBAIJAN

As a manuscript

OPTIMALITY CONDITIONS IN MULTİ-STAGE OPTİMAL CONTROL PROBLEMS DESCRİBED BY DİFFERENTİAL AND İNTEGRAL EQUATİONS

Specialty: 1211.01 - Differential equations

Field of science: Mathematics

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DISSERTATION ABSTRACT

for the degree of Doctor of Philosophy in Mathematics

Baku - 2025

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GENERAL CHARACTERISTICS OF THE WORK

The relevance and the background of the work. By the middle of the 1940s,when space navigation, rocket dynamics, chemical technologies and robotics were rapidly developing,new problems of calculus of variation arose – optimal control problems. L.S.Pontryagin made a great contribution to the mathematical theory of optimal control by introducing the concept of optimality condition (maximum principle). As is known, the qualitative theory of optimal control problems described by the systems of ordinary differential equations has been first studied by V.G.Boltyanski, R.V.Gamkrelidze and L.S.Pontryagin to be further developed by L.T.Ashepkov, V.F.Demyanov, A.I.Kalinin, A.A.Milyutin and A.Y.Dubovitski, T.G.Melikov, M.J.Mardanov, R.G.Gabasov and F.M.Kirillova, V.V.Gorokhovik, S.Y.Gorokhovik, L.I.Rozonoer, V.A.Srochko, etc.

Pontryagin's maximum principle has different proofs by the above authors. The one proposed by Rozonoer (so called growth method) was the most widely used. Inspired by Rozonoer’s works, F.P.Vasilyev, G.T.Ahmedov and S.S.Hakhiyev, T.G.Melikov, M.J.Mardanov, R.G.Gabasov, A.I.Yegorov, etc made their own contributions to this growth method.

Over time, many scientists have done considerable research in the field of the theory of necessary conditions for optimality in optimal control problems described by various types of partial differential equations, integral equations and integro-differential equations.

Many Azeri and Russian scientists such as A.A.Abdullayev, V.M.Abdullayev, K.R.Ayda-zadeh, A.I.Yegorov, S.S.Hakhiyev, K.G.Hasanov, E.N.Mahmudov, T.G.Melikov, K.B.Mansimov, M.J.Mardanov, R.O.Mastaliyev, H.F.Guliyev, F.P.Vasilyev, O.V.Vasilyev, A.B.Rahimov, M.A.Sadigov, V.A.Srochko, R.G.Taghiyev, M.H.Yagubov, Y.A.Sharifov have publications related directly to this field.

In practice, many complex processes are multi-stage. Such multi-stage processes are called stepwise processes or sometimes variable structure processes. Described by different equations in different time periods or in different domains, multi-stage processes are controlled by controllers which are independent of each other. N.M.Avalishvili, A.A.Lempert, K.B.Mansimov, Sh.F.Muharremov, M.S.Nikolski, V.N.Rozova, T.A.Tadumadze, G.L.Kharatishvili, G.K.Zakharov, etc obtained various optimality conditions for multi-stage processes described by the ordinary differential equations or the system of difference equations.

However, optimal control problems for multi-stage processes described by the system of integral and differential equations have never been considered before. That’s why the topic of this dissertation dedicated to the optimality conditions for the optimal control problems described by the systems of Volterra type integral equations and ordinary differential equations in different periods of time can be considered as relevant.

The object and the subject of research.

The object of this research are the (stepwise or variable structure) multi-stage optimal control problems described by the system of ordinary differential equations and integral equations.

And the subject of this research is to obtain optimality conditions for the considered optimal control problems.

The purpose and the objectives of research. The purposes of this research areto construct new growth formulas by means ofthe improved version of growth method,to apply these formulas to obtain necessary and sufficient conditions for optimality in the form of Pontryagin's maximum principle for the variable structure (two-stage, stepwise) optimal control problems described by the system of differential and Volterra type integral equations in linear case, to obtain first order necessary conditions for optimality in the forms of maximum principle, linearized maximum principle and analogue of Euler’s equation in case where the quality criterion is nonlinear and then to consider the cases where they are degenerate.

We plan to treat the similar problems in nonlinear case.

Research methods. We usethe methods of qualitative theory of added parameters optimal control, the methods of classical calculus of variation and the methods of qualitative theory of ordinary differential equations and Volterra type integral equations.

Main points to be defended in this dissertation.

- Various necessary conditions for optimality have been obtained for some optimal control problems described by the system of differential and Volterra type integral equations;
- Under some restrictions, the cases where the first order necessary conditions for optimality are degenerate have been considered;
- Constructively verifiable second order necessary conditions for optimality have been obtained;
- Necessary conditions for optimality in the forms of Pontryagin's maximum principle, linearized maximum principle and Euler's equation have been obtained for the considered optimal control problems;
- Pointwise and integral-type necessary conditions for quasi-eigen optimality have been obtained in case where the linearized maximum principle is degenerate (quasi-eigen case).

Scientific novelty of research. The statementsof the main considered optimal control problems are new. The main results are also new.

Theoretical and practical significance of research. Results obtained in this work are new and have theoretical importance. The obtained optimality conditions can be used to construct the numerical algorithms for solving the considered problems. To do so, the method of successive approximations, their improved versions, and the gradient method can be used.

Approbation and applications. The main results of this work have been presented in the seminars held at the Department of Mathematics and Computer Science in Lankaran State University,at the Department of Mathematical Cybernetics in Baku State University, and in many international conferences such as International Conference on Dynamic Systems: Stability, Control, Optimization (Minsk, Belarus, 2018), International Conference dedicated to the 100th Anniversary of Mathematical Education in Eastern Siberia (Irkutsk, Russia, 2019), International Conference on

Dynamic Systems and Computer Sciences: Theory and Practice (Irkutsk, Russia, 2021).

Author's personal contribution and publications. All the obtained results belong to the author.

Publications.

Author's publications include 7(seven) papers in scientific journals recommended by the Higher Certification Commission under the President of the republic of Azerbaijan and 3(three) international conference materials.

The name of the institution where the dissertation was completed. This work was performed at the Department of Mathematics and Computer Science in Lankaran State University, Lankaran, Azerbaijan.

Volume and structure of the dissertation (in characters, with the volume of each structural unit shown separately).The dissertation includes title page, contents, introduction, three chapters, conclusion and reference list of 60 items. It consists of 238668 characters (title page – 394, contents – 4929, Introduction – 37345, Chapter 1 – 42000, Chapter 2 – 96000, Chapter 3 – 58000).

THE CONTENT OF THE DISSERTATION

This dissertation consists of Introduction, three chapters, Conclusion and Reference list.

Introduction presents the topic of research, provides the background, briefly summarizes the existing results and informs about the results obtained in the dissertation.

Chapter 1 consists of three paragraphs.

In Paragraph 1 of Chatper 1, two-stage (variable structure) optimal control problem described by the system of linear integral and differential equations is considered.

Let the controlled continuous process be described in the fixed time interval $T=[t_0,t_1]\cup[t_1,t_2]$ $\left(T_1=[t_0,t_1],\ T_2=[t_1,t_2]\right)$ by the following systems of ordinary differential and Volterra type integral equations:

$$\dot{x}=A(t)x(t)+f(t,u_1),\quad t\in T_1, \tag{1}$$

$$x(t_0)=x_0, \tag{2}$$

$$y(t)=\int_{t_1}^{t}\left(B(t,\tau)y(\tau)+g(t,\tau,u_2(\tau))\right)d\tau+Gx(t_1),\ \ t\in T_2 \tag{3}$$

where x_0 is a given n-dimensional constant vector, G is a given $(n\times m)$-dimensional constant matrix, $A(t)$ is a given $(n\times n)$-dimensional continuous matrix function, $f(t,u_1)$ a given n-dimensional vector function continuous with respect to all of its variables, $B(t,\tau)$ is a given $(m\times m)$ dimensional matrix function continuous with respect to all of its variables, $g(t,\tau,u_2)$ is a given m dimensional vector function continuous with respect to all of its variables, U_1 and U_2 are the given bounded non-empty sets, $u_1(t)$ and $u_2(t)$ are the r and q dimensional, respectively, control vector functions with a finite number of discontinuities of the first kind which take values on the given bounded non-empty sets U_1 and U_2, i.e.

$$\begin{aligned} &u_1(t)\in U_1\subset R^r, \qquad t\in T_1,\\ &u_2(t)\in U_2\subset R^q, \qquad t\in T_2. \end{aligned} \tag{4}$$

Every pair $(u_1(t),u_2(t))$ which satisfies these conditions is called a possible control.

It is assumed that every given possible control $(u_1(t),u_2(t))$ is corresponded by the unique piecewise smooth solution $x(t)$ of the Cauchy problem (1)-(2) and the unique continuous solution $y(t)$ of the system of integral equations (3).

Now let’s define a terminal-type functional

$$J(u_1,u_2)=c'x(t_1)+d'y(t_2) \tag{5}$$

over the solutions of the problem (1)-(3) corresponding to all possible controls, where c and d are the given n and m -dimensional constant vectors, respectively, and ' means transposition.

Let’s consider the problem of finding the minimum of the functional(5) under conditions (1)-(4).

A possible control $(u_1(t), u_2(t))$ which provides the minimum of the functional (5) under conditions (1)-(4)is called an optimal control.

Necessary and sufficient condition for optimality has been obtainedfor the considered problem.

Let $(u_1(t), u_2(t))$ be a fixed possible control, and $\psi(t)$ and $p(t)$ be the n and m -dimensional vector functions, respectively, which are the solutions of the following problems:

$$\dot{\psi}(t) = -A'(t)\psi(t), \quad t \in T_1,$$

$$\psi(t_1) = -c + \int_{t_1}^{t_2} G'p(t)dt - G'd, \quad t \in T_2,$$

$$p(t) = \int_{t}^{t_2} B'(\tau, t)p(\tau)d\tau - B'(t_2, t)d .$$

Introduce the Hamilton-Pontryagin functions of the form

$$H(t, u_1(t), \psi(t)) = \psi'(t) f(t, u_1(t)),$$

$$M(t, u_2(t), p(t)) = -d'g(t_2, t, u_2(t)) + \int_{t}^{t_2} p'(\tau)g(\tau, t, u_2(t))d\tau .$$

The following theorem has been proved:

Theorem 1. For the possible control $(u_1(t), u_2(t))$ to be an optimal control in the problem (1)-(5), it is necessary and sufficient that the relations

$$\max_{\upsilon_1 \in U_1} H(\theta, \upsilon_1, \psi(\theta)) = H(\theta, u_1(\theta), \psi(\theta)), \ \theta \in [t_0, t_1),$$

$$\max_{\upsilon_2 \in U_2} M(\theta, \upsilon_2, p(\theta)) = M(\theta, u_2(\theta), p(\theta)), \ \theta \in [t_1, t_2)$$

hold, where $\theta \in [t_0, t_1)$ and $\theta \in [t_1, t_2)$ are the continuity points of the possible controls $u_1(t)$ and $u_2(t)$, rerspectively.

In the second part of the paragraph, we consider the case where the quality criterion has the form

$$J(u_1,u_2)=\varphi_1(x(t_1))+\varphi_2(y(t_2)), \tag{6}$$

with the given continuous differentiable and convex functions $\varphi_1(x)$ and $\varphi_2(y)$.

Using growth method, sufficient condition for optimality in the form of Pontryagin's maximum principle for the problem (1)-(4), (6) has been obtained.

In Paragraph 2 of Chapter 1, the problem of finding the minimum of the multipoint functional

$$J(u_1,u_2)=\sum_{i=1}^{k}\left[c_i' x(\theta_i)+d_i' y(\xi_i)\right] \tag{7}$$

is considered under conditions (1)-(4), where $\theta_i \in [t_0,t_1)$, $i=\overline{1,k}$ $(t_0<\theta_1<\ldots<\theta_k\le t_1)$, $\xi_i\in[t_1,t_2)$, $i=\overline{1,k}$ $(t_1<\xi_1<\ldots<\xi_k\le t_2)$ are the given points.

In the problem (1)-(4), (7) the growth formula of the functional was constructed, necassary and sufficient conditions for optimality in the form of maxsimum principle was proved.

In Paragraph 3 of Chapter 1, the problem

$$\dot{x}=A(t)x(t)+f(t,u(t)),\ t\in T_1,$$

$$x(t_0)=x_0,$$

$$y(t)=\int_{t_1}^{t}\left(B(t,\tau)y(\tau)+g(t,\tau,\upsilon(\tau))\right)d\tau+Gx(t_1),\ t\in T_2,$$

$$u(t)\in U\subset R^r,\ t\in T_1,$$

$$\upsilon(t)\in V\subset R^q,\ t\in T_2,$$

$$\Phi(u,\upsilon)=\varphi_1(x(t_1))+\varphi_2(y(t_2))\to\min$$

is considered.

It is assumed that $\varphi_1(x)$ and $\varphi_2(y)$ are the given twice continuously differentiable scalar functions.

In the problem under consideredat first the analogue of the maximum principle is provedand then the case where it degenerate is stadied.

Assume that $\left(u^0(t), \upsilon^0(t), x^0(t), y^0(t)\right)$ is a fixed possible process in the optimal control problem under considered

Introduce the analog of the Hamilton-Pontryagin function for this optimal control problem as follows:

$$H\left(t, u(t), \psi^0(t)\right) = \psi^{0'}(t) f\left(t, u(t)\right),$$

$$M\left(t, \upsilon(t), p^0(t)\right) = -\frac{\partial \varphi_2'\left(y^0(t_2)\right)}{\partial y} g\left(t_2, t, \upsilon(t)\right) + \int_t^{t_2} p^{0'}(\tau) g\left(\tau, t, \upsilon(t)\right) d\tau$$

where the n and m - dimensional vector functions $\psi^0(t)$ and $p^0(t)$ are the solutions of the following equations, respectively:

$$\dot{\psi}^0(t) = -A'(t)\psi^0(t),$$

$$\psi^0(t_1) = -\frac{\partial \varphi_1\left(x^0(t_1)\right)}{\partial x} - G' \frac{\partial \varphi_2\left(y^0(t_2)\right)}{\partial y} + \int_{t_1}^{t_2} G' p^0(t) dt,$$

$$p^0(t) = -\frac{\partial \varphi_2'\left(y^0(t_2)\right)}{\partial y} B(t_2, t) + \int_t^{t_2} B'(\tau, t) p^0(\tau) d\tau .$$

Theorem 2. For the possible control $\left(u^0(t), \upsilon^0(t)\right)$ to be an optimal control in the considered problem, it is necessary that the relations

$$\max_{u \in U} H\left(\theta, u, \psi^0(\theta)\right) = H\left(\theta, u^0(\theta), \psi^0(\theta)\right),$$

$$\max_{\upsilon \in V} M\left(\xi, \upsilon, p^0(\xi)\right) = M\left(\xi, \upsilon^0(\xi), p^0(\xi)\right)$$

hold for arbitrary $\theta \in [t_0, t_1)$ and $\xi \in [t_1, t_2)$,respectively.

Definition 1. If the equalities

$$H\left(\theta, u, \psi^0(\theta)\right) = H\left(\theta, u^0(\theta), \psi^0(\theta)\right),$$

$$M\left(\xi, \upsilon, p^0(\xi)\right) = M\left(\xi, \upsilon^0(\xi), p^0(\xi)\right)$$

hold for every $\theta \in [t_0, t_1)$, $u \in U$ and $\xi \in [t_1, t_2)$, $\upsilon \in V$,then the possible control $\left(u^0(t), \upsilon^0(t)\right)$is called a singular casein the considered optimal control problem in the sense of Pontryagin'smaximum principle.

Then, a condition for the singular control to be optimal in the sence of the maximum principle, was found.

Chapter 2 consists of three paragraphs.

In Paragraph 1 of Chapter 2, it is assumed that the controlled process is described in the fixed time interval $T = T_1 \cup T_2$ $(T_1 = [t_0, t_1]$ $T_2 = [t_1, t_2])$by the following system of equations:

$$\dot{x} = f(t, x, u), \quad t \in T_1, \tag{8}$$

$$x(t_0) = x_0, \tag{9}$$

$$y(t) = \int_{t_1}^{t} g(t, s, y(s), \upsilon(s))ds + G(x(t_1)), \; t \in T_2 \tag{10}$$

where $f(t, x, u)(g(t, s, y, \upsilon))$ is a given $n(m)$ -dimensional vector function continuous with respect to all of its variables together with its second order derivative with respect to$(x, u)((y, \upsilon))$are the given numbers, x_0 is a given constant vector, $G(x)$ is a given twice continuously differentiable m -dimensional vector function, $u(t)(\upsilon(t))$ is an $r(q)$-dimensional piecewise continuous control vector function with a finite number of discontinuities of the first kind which takes values on the bounded, non-empty open set $U(V)$,i.e. the relations

$$u(t) \in U \subset R^r, \; t \in T_1,$$

$$\upsilon(t) \in V \subset R^q, \; t \in T_2 \tag{11}$$

hold.

It is assumed that every given possible control $(u(t), \upsilon(t))$ in the problem (8)-(10) correspondsto the unique solution$(x(t), y(t))$.

We consider in this paragraph the problem of finding the minimum of the nonlinear functional

$$J(u, \upsilon) = \varphi_1(x(t_1)) + \varphi_2(y(t_2)) \tag{12}$$

under conditions (8)-(11), where $\varphi_1(x)$ and $\varphi_2(y)$ are the given twice continuously differentiable scalar functions.

Let $\left(u^0(t), \upsilon^0(t), x^0(t), y^0(t)\right)$ be a fixed possible control in the considered problem. Introduce the analogues of the Hamilton-Pontryagin function:

$$H\left(t, x(t), u(t), \psi^0(t)\right) = \psi^{0'} f(t, x(t), u(t)),$$

$$M\left(t, y(t), \upsilon(t), p^0(t)\right) = -\frac{\partial \varphi_2'\left(y^0(t_2)\right)}{\partial y} g(t_2, t, y(t), \upsilon(t)) +$$

$$+\int_t^{t_2} p^{0'}(s) g(s, t, y(t), \upsilon(t)) ds$$

wherethe n and m -dimensional vector functions $\psi^0(t)$ and $p^0(t)$ are the solutions of the system of equations

$$\dot{\psi}^0(t) = -H_x\left(t, x^0(t), u^0(t), \psi^0(t)\right),$$

$$\psi^0(t_1) = -\frac{\partial \varphi_1\left(x^0(t_1)\right)}{\partial x} + G_x'\left(x^0(t_1)\right)\frac{\partial \varphi_2\left(y^0(t_2)\right)}{\partial y} + \int_{t_1}^{t_2} G_x'\left(x^0(t_1)\right) p^0(t) dt,$$

$$p^0(t) = M_y\left(t, y^0(t), \upsilon^0(t), p^0(t)\right)$$

By means of growth formula, second order growth formula for the minimized functionalis constructed in the considered problem,and, using special variation of the possible control $\left(u^0(t), \upsilon^0(t)\right)$the first and second variations of quality criterion are calculated.By means of these variations,first and second order necessary conditions are obtained for implicit optimality.Then, first order necessary condition in the form of the analogue of Euler's equation (in other words, necessary condition satisfied by the classical extremal) is derived.And, finally, using the condition that the second variation is not negative during optimal process, second order necessary conditions for optimality are obtained.

Theorem 3. For the possible control $\left(u^0(t), \upsilon^0(t)\right)$ to be an optimal control in the optimal control problem (8)-(12), it is necessary that the equalities

$$H_u\left(\theta, x^0(\theta), u^0(\theta), \psi^0(\theta)\right) = 0,$$
$$M_\upsilon\left(\theta, y^0(\theta), \upsilon^0(\theta), p^0(\theta)\right) = 0$$

hold for every $\theta \in [t_0, t_1)$ and $\theta \in [t_1, t_2)$.

In Paragraph 2, the problem considered in Paragraph 1 is treated under condition that the sets U and V are convex.

First, a necessary condition in the form of linearized maximum principle is obtained.

Theorem 4. Let the sets U and V in the optimal control problem (8)-(12) be convex.Then, for the possible control $\left(u^0(t), \upsilon^0(t)\right)$ to be an optimal control, it is necessary that the inequalities

$$\int_{t_0}^{t_1} H_u'\left(t, x^0(t), u^0(t), \psi^0(t)\right)\left(u(t) - u^0(t)\right) dt \le 0$$

$$\int_{t_1}^{t_2} M_\upsilon'\left(t, y^0(t), \upsilon^0(t), p^0(t)\right)\left(\upsilon(t) - \upsilon^0(t)\right) dt \le 0$$

hold for every $u(t) \in R^r$, $t \in T_1$ və $\upsilon(t) \in R^q$, $t \in T_2$.

From these integral-type necessary conditions, we obtain a pointwise necessary condition.

At the end of Paragraph 2, we consider the case where the linearized maximum condition degenerates.

Definition 2. If the equalities

$$\int_{t_0}^{t_1} H_u'\left(t, x^0(t), u^0(t), \psi^0(t)\right)\left(u(t) - u^0(t)\right) dt = 0,$$

$$\int_{t_1}^{t_2} M_\upsilon'\left(t, y^0(t), \upsilon^0(t), p^0(t)\right)\left(\upsilon(t) - \upsilon^0(t)\right) dt = 0$$

then the possible control $\left(u^0(t), \upsilon^0(t)\right)$ is called a quasi-eigen control in the optimal control problem (8)-(12).

Then in the problem considered, a necessary and sufficient conditiopn for quasi-eigen control to be optimal was proved.

In Paragraph 3 of Chapter 2, the problem

$$\dot{x} = f(t,x,u), \quad t \in T_1, \tag{13}$$

$$x(t_0) = x_0, \tag{14}$$

$$y(t) = \int_{t_1}^{t} A(t,s)g(s,y(s),\upsilon(s))ds + G(x(t_1)), \quad t \in T_2, \tag{15}$$

$$\begin{aligned} &u(t) \in U \subset R^r,\ t \in T_1, \\ &\upsilon(t) \in V \subset R^q,\ t \in T_2 \end{aligned} \tag{16}$$

$$J(u,\upsilon) = \varphi_1(x(t_1)) + \varphi_2(y(t_2)) \to \min \tag{17}$$

is considered, where $f(t,x,u)(g(s,y,\upsilon))$ is a given $n(m)$-dimensional vector function continuous with respect to all of its variables together with its second order derivative with respect to $(x)((y))$, $G(x)$ is a given twice continuously differentiable m -dimensional vector function, $A(t,s)$ is a given $(m \times m)$ -dimensionalcontinuous matrix function,$x_0$is a given constant vector,$u(t)(\upsilon(t))$is a $r(q)$-dimensional piecewise continuous control vector function with discontinuities of the first kind, $\varphi_1(x)$ and $\varphi_2(y)$ re the given twice continuously differentiable scalar functions.

Let $(u^0(t),\upsilon^0(t))$be a fixed possible control, and $(x^0(t), y^0(t))$ be the solutions of the problem(13)-(16) corresponding to this possible control.

Introduce the Hamilton-Pontryagin functions of the following form:

$$H(t,x,u,\psi^0) = \psi^{0\prime} f(t,x,u),$$

$$M(t,y,\upsilon,p^0) = -\frac{\partial\varphi_2(y^0(t_2))}{\partial y} A(t_2,t)g(t,y,\upsilon) + \int_t^{t_2} p^{0\prime}(s)A(s,t)g(t,y,\upsilon)ds$$

where the n and m-dimensional vector functions $\psi^0(t)$ and $p^0(t)$are the solutions of the system of equations

$$\dot{\psi}^0(t) = -H_x(t,x^0(t),u^0(t),\psi^0(t)), \tag{18}$$

$$\psi^0(t_1) = -\frac{\partial\varphi_1\left(x^0(t_1)\right)}{\partial x} + \frac{\partial N\left(p^0, x^0(t_1)\right)}{\partial x}, \tag{19}$$

$$p^0(t) = M_y\left(t, y^0(t), \upsilon^0(t)\right) \tag{20}$$

and

$$N\left(p^0, x\right) = -\left[-\frac{\partial\varphi_2'\left(y^0(t_2)\right)}{\partial y} + \int_{t_1}^{t_2} {p^0}'(t)dt\right]G(x).$$

In the considered problem, second order growth formula for a functionalis constructed and, using this formula, necessary condition for optimality in the form of Pontryagin'smaximum principle is obtained.

Theorem 5. For the possible control $\left(u^0(t), \upsilon^0(t)\right)$ to be an optimal control in the problem (13)-(17), it is necessary that the inequalities

$$H\left(\theta, x^0(\theta), u, \psi^0(\theta)\right) - H\left(\theta, x^0(\theta), u^0(\theta), \psi^0(\theta)\right) \le 0,$$

$$M\left(\theta, y^0(\theta), \upsilon, p^0(\theta)\right) - M\left(\theta, y^0(\theta), \upsilon^0(\theta), p^0(\theta)\right) \le 0$$

hold for every $\theta \in [t_0, t_1)$, $u(t) \in R^r$ və $\theta \in [t_1, t_2)$, $\upsilon(t) \in R^q$, respectively.

Further, the case where the maximum principle degenerates (i.e. special case) is considered and necessary condition for optimality of singular case is obtained.

Chapter 3 consists of three paragraphs. Variable structure optimal control problem described by the systems of Volterra type integral equations and ordinary differential equations is studied in this chapter.

In Paragraph 1, we consider the optimal control problem

$$x(t) = \int_{t_0}^{t} \left[A(t,\tau)x(\tau) + f\left(t,\tau,u(\tau)\right)\right]d\tau, \quad t \in T_1 = [t_0, t_1], \tag{21}$$

$$\dot{y}(t) = B(t)y(t) + g\left(t, \upsilon(t)\right), \quad t \in T_2 = [t_1, t_2], \tag{22}$$

$$y(t_1) = G \cdot x(t_1), \tag{23}$$

$$\begin{gathered} u(t) \in U \subset R^r,\ t \in T_1, \\ \upsilon(t) \in V \subset R^q,\ t \in T_2 \end{gathered} \tag{24}$$

$$\Phi(u,\upsilon)=c'x(t_1)+d'y(t_2)\to \min \tag{25}$$

where $A(t,\tau)$, $B(t)$ are the given $(n\times n)$ and $(m\times m)$ -dimensional continuous matrix functions, respectively, G is an $(n\times m)$ dimensional constant matrix, $f(t,\tau,u)(g(t,\upsilon))$ is a given $n(m)$ -dimensional vector function continuous with respect to all of its variables, t_0,t_1,t_2 $(t_0<t_1<t_2)$ are the given numbers, U and V are the given bounded non-empty sets, $u(t)(\upsilon(t))$ is an $r(q)$ dimensional piecewise continuous control vector function with a finite number of discontinuities of the first kind, c and d are the n və m -dimensional constant vectors, respectively.

Every pair $(u(t),\upsilon(t))$ which satisfies the above smoothness conditions is called a possible control.

Let $(u^0(t),\upsilon^0(t))$ be a fixed possible control. Denote

$$H(t,u(t),\psi^0(t),p^0(t))=-c'f(t_1,t,u(t))+$$

$$+(G'p^0(t))'f(t_1,t,u(t))+\int_t^{t_1}\psi^{0'}(\tau)f(\tau,t,u(t))d\tau,$$

$$M(t,\upsilon(t),p^0(t))=p^{0'}(t)g(t,\upsilon(t))$$

where the n and m -dimensional vector functions $\psi^0(t)$ and $p^0(t)$, respectively, are the solutions of the conjugate system

$$\psi^0(t)=-A'(t_1,t)c+\int_t^{t_1}A'(\tau,t)\psi^0(\tau)d\tau+A'(t_1,t)G'p^0(t_1),$$

$$\dot p^0(t)=-B'(t)p^0(t),\quad p^0(t_2)=-d .$$

The following statement has been proved for the considered problem:

Theorem 6. For the possible control $(u^0(t),\upsilon^0(t))$ to be an optimal control in the optimal control problem (21)-(25), it is necessary and sufficient that the relations

$$\max_{u\in U} H(\theta,u,\psi^0(\theta),p^0(\theta))=H(\theta,u^0(\theta),\psi^0(\theta),p^0(\theta)),$$

$$\max_{\upsilon\in V} M\left(\theta,\upsilon, p^0(\theta)\right)= M\left(\theta,\upsilon^0(\theta), p^0(\theta)\right)$$

hold for every $\theta \in [t_0, t_1)$ and $\theta \in [t_1, t_2)$, respectively.

In Paragraph 2 of Chapter 3, a variablestructure optimal control problem with linear system of equations and linear connectedness condition, but nonlinear quality criterion, is considered.

In other words, a problem of finding the minimum of the functional

$$\Phi(u,\upsilon)=\varphi_1(x(t_1))+\varphi_2(y(t_2)) \tag{26}$$

is considered under conditions (21)-(24), where $\varphi_1(x)$ and $\varphi_2(y)$ are the given twice continuously differentiable scalar functions.

Let $\left(u^0(t),\upsilon^0(t), x^0(t), y^0(t)\right)$ be a fixed possible control, and the vector functions $\psi^0(t)$ and $p^0(t)$ be the solutions of the equations

$$\psi^0(t)=-A'(t_1,t)\frac{\partial\varphi_1\left(x^0(t_1)\right)}{\partial x}+A'(t_1,t)G'p^0(t_1)+\int_t^{t_1} A'(\tau,t)\psi^0(\tau)d\tau,$$

$$\dot{p}^0(t)=-B'(t)p^0(t), \qquad p^0(t_2)=-\frac{\partial\varphi_2'\left(y^0(t_2)\right)}{\partial y}$$

respectively.

Introduce the analogue of the Hamilton-Pontryagin functions as follows:

$$H\left(t, x(t), u(t), \psi^0(t), p^0(t)\right)=\int_t^{t_1}\psi^{0'}(\tau)f(\tau,t,u(t))d\tau-$$

$$-\left(\frac{\partial\varphi_1\left(x^0(t_1)\right)}{\partial x}-G'p^0(t_1)\right)' f(t_1,t,u(t)),$$

$$M\left(t, y(t),\upsilon(t), p^0(t)\right)= p^{0'}(t)g(t,\upsilon(t))$$

Let the matrix function $R(t,\tau)$ be the solution of the Volterra type matrix integral equation

$$R(t,\tau)=\int_{\tau}^{t} R(t,s)A(s,\tau)ds+A(t,\tau),$$

and the matrix function $F(t,\tau)$ be the solution ofthe problem

$$F_{\tau}(t,\tau)=-F(t,\tau)B(\tau),\ t\in T_1,\ \ F(t,t)=E$$

where E is an $(m\times m)$-dimensional unit matrix.

For this problem, we construct a second order growth formula for the minimized functional and then we use it to obtain a necessary condition for optimality in the form of Pontryagin's maximum principle. We also consider the case where this necessary condition is degenerate (a special case).

Theorem7. For the possible control $\left(u^0(t),\upsilon^0(t)\right)$ to be an optimal control in the problem (21)-(24), (26), it is necessary that the relations

$$\max_{u\in U} H\left(\theta,u,\psi^0(\theta),p^0(t_1)\right)=H\left(\theta,u^0(\theta),\psi^0(\theta),p^0(t_1)\right),$$

$$\max_{\upsilon\in V} M\left(\xi,\upsilon,p^0(\xi)\right)=M\left(\xi,\upsilon^0(\xi),p^0(\xi)\right)$$

hold for every $\theta\in[t_0,t_1)$ and $\xi\in[t_1,t_2)$respectively.

Definition3. If the equalities

$$H\left(\theta,u,\psi^0(\theta),p^0(t_1)\right)=H\left(\theta,u^0(\theta),\psi^0(\theta),p^0(t_1)\right),$$

$$M\left(\xi,\upsilon,p^0(\xi)\right)=M\left(\xi,\upsilon^0(\xi),p^0(\xi)\right)$$

hold for $\theta\in[t_0,t_1),\ u\in U$ and $\xi\in[t_1,t_2),\ \upsilon\in V$ respectively,then the possible control $\left(u^0(t),\upsilon^0(t)\right)$is called a singular case in the sense of Pontryagin's maximum principle.

It is clear from the definition that in the singular case the maximum condition loses the essence of its content. There fore, in the problem considered it is necessary to find new necessary conditions for effective optimality.

Let $\left(u^0(t),\upsilon^0(t)\right)$be a singular case in the problem (21)-(24), (26). Introduce the following vector functions:

$$K_1(t,u(t))=f(t_1,t,u(t))+\int_{t}^{t_1} R(t_1,\tau)f(\tau,t,u(t))d\tau,$$

$$K_2(t,u(t)) = F(t_2,t)G\left[f(t_1,t,u(t)) + \int_t^{t_1} R(t_1,\tau) f(\tau,t,u(t))d\tau \right],$$

$$K_3(\tau,s) = F'(t_2,\tau)\frac{\partial^2 \varphi_2'(y^0(t_2))}{\partial y^2} F(t_2,s)$$

Second order growth formula for the functional is constructed and the following theorem is proved:

Theorem 8. For the singular case in the sense of Pontryagin's maximum principle $(u^0(t),\upsilon^0(t))$ to be an optimal control in the problem (21)-(24), (26), it is necessary that the inequalities

$$\left[K_1(\theta,u) - K_1(\theta,u^0(\theta))\right]' \frac{\partial^2 \varphi_1'(x^0(t_1))}{\partial x^2}\left[K_1(\theta,u) - K_1(\theta,u^0(\theta))\right] +$$

$$+\left[K_2(\theta,u) - K_2(\theta,u^0(\theta))\right]' \frac{\partial^2 \varphi_2'(y^0(t_2))}{\partial y^2}\left[K_2(\theta,u) - K_2(\theta,u^0(\theta))\right] \geq 0,$$

$$\left[g(\xi,\upsilon) - g(\xi,\upsilon^0(\xi))\right]' K_3(\xi,\xi)\left[g(\xi,\upsilon) - g(\xi,\upsilon^0(\xi))\right] \geq 0$$

hold for every $\theta \in [t_0,t_1)$, $u \in U$ and $\xi \in [t_1,t_2)$, $\upsilon \in V$ respectively.

Inthe last paragraph of Chapter 3, we consider the problem of finding the minimum of the terminal type functional

$$J(u,\upsilon) = \varphi_1(x(t_1)) + \varphi_2(y(t_2)) \tag{27}$$

under the conditions

$$\begin{aligned} u(t) &\in U \subset R^r,\ t \in T_1, \\ \upsilon(t) &\in V \subset R^q,\ t \in T_2 \end{aligned} \tag{28}$$

$$x(t) = \int_{t_0}^{t} f(t,s,x(s),u(s))ds, \quad t \in T_1, \tag{29}$$

$$\dot{y} = g(t,y,\upsilon), \quad t \in T_2, \tag{30}$$

$$y(t_1) = G(x(t_1)) \tag{31}$$

where $f(t,s,x,u)(g(t,y,\upsilon))$ is a given $n(m)$ -dimensional vector function continuous with respect to all of its variables together with its second order derivatives with respect to $(x,u)((y,\upsilon))$, $G(x)$is a given twice continuously differentiable m -dimensional vector function, $t_0 < t_1 < t_2$, $\varphi_1(x)$ and $\varphi_2(y)$ are the given twice continuously differentiable scalar functions, $u(t)(\upsilon(t))$ is an $r(q)$ -dimensional piecewise continuous vector function which takes values on the bounded, non-empty open set $U(V)$.

Let $(u^0(t), \upsilon^0(t))$ be some possible control.Denote

$$N(p^0, x) = p^{0'}(t_1)G(x),$$

$$H(t, x(t), u(t), \psi^0(t)) = -\frac{\partial\varphi_1'(x^0(t_1))}{\partial x} f(t_1, t, x(t), u(t)) +$$

$$+\frac{\partial N'(p^0, x^0(t_1))}{\partial x} f(t_1, t, x(t), u(t)) + \int_t^{t_1} \psi^{0'}(s) f(s, t, x(t), u(t)) ds,$$

$$M(t, y, \upsilon, p^0) = p^{0'} g(t, y, \upsilon)$$

where the n and m -dimensional vector functions $\psi^0(t)$ and $p^0(t)$ are the solutions of the equations

$$\psi^0(t) = H_x(t, x^0(t), u^0(t), \psi^0(t)),$$

$$\dot{p}^0(t) = -M_y(t, y^0(t), \upsilon^0(t), p^0(t)),$$

$$p^0(t_2) = -\frac{\partial\varphi_2(y^0(t_2))}{\partial y}$$

respectively.

As the control domains are open, the first and the second variations of quality criterion have been found.

As the control domains are open sets,according to the classical theory of calculus of variations, the first variations of the functional in the considered problem must be zero during the optimal control $(u^0(t), \upsilon^0(t))$, while the second variations must be non-negative.

There fore, a necessary condition in the form of the analogue of Euler's equation has been obtained first.

Theorem 9.(The analogue of Euler's equation) For the possible control $\left(u^0(t), \upsilon^0(t)\right)$ to be an optimal control in the problem (27)-(31), it is necessary that the equalities

$$H_u\left(\theta, x^0(\theta), u^0(\theta), \psi^0(\theta)\right) = 0,$$
$$M_\upsilon\left(\theta, y^0(\theta), \upsilon^0(\theta), p^0(\theta)\right) = 0$$

hold for every $\theta \in [t_0, t_1)$ and $\theta \in [t_1, t_2)$.

Every possible control $\left(u^0(t), \upsilon^0(t)\right)$ which satisfies the analogue of Euler's equation is called a classical extremal for the considered problem.

In the considered problem, the necessary conditions for optimality have been proved using the condition that the second variations of the functional throughout the optimal process are not negative.

By specifying the possible variations of the control functions,the following theorem is proved.

Theorem 10. For the classical extremal $\left(u^0(t), \upsilon^0(t)\right)$ to be an optimal control in the problem (27)-(31), it is necessary that the equalities

$$u' H_{uu}\left(\theta, x^0(\theta), u^0(\theta), \psi^0(\theta)\right) u \le 0,$$
$$\upsilon' M_{\upsilon\upsilon}\left(\theta, y^0(\theta), \upsilon^0(\theta), p^0(\theta)\right) \upsilon \le 0$$

hold for every $\theta \in [t_0, t_1)$, $u \in R^r$ and $\xi \in [t_1, t_2)$, $\upsilon \in R^q$ respectively.

Definition 4. If for $\theta \in [t_0, t_1)$, $u \in R^r$ and $\xi \in [t_1, t_2)$, $\upsilon \in R^q$ the equalities

$$u' H_{uu}\left(\theta, x^0(\theta), u^0(\theta), \psi^0(\theta)\right) u = 0,$$
$$\upsilon' M_{\upsilon\upsilon}\left(\theta, y^0(\theta), \upsilon^0(\theta), p^0(\theta)\right) \upsilon = 0$$

hold, then the classical extremal $\left(u^0(t), \upsilon^0(t)\right)$is called a singular casein the classical sense in the optimal control problem (27)-(31).

The optimality condition of the specific control in the classical sense has been proved.

CONCLUSION

This dissertation consists of Introduction, three Chapters, Conclusion and Reference list.

Introduction emphasizes the relevance and informs about the background of the work.

Chapter 1 considers the optimal control problems described by differential and Volterra type integral equations in two different time periods, respectively.

Necessary and sufficient condition for optimality in a linear optimal control problem is obtained in the form of Pontryagin's maximum principle.

Sufficient condition for optimalityin case where the quality criterion is nonlinear and convex is obtained.

Optimality conditions are found in case where the quality criterion is multipoint.

Chapter 2 considers the nonlinear analogue of the problem considered in Chapter 1.In case where the control domain is open, first order necessary condition for optimality is obtained in the form of the analogue of Euler's equation, followed by the second order necessary condition for optimality.

In Paragraph 2 of Chapter 2, the analogue of the linearized maximum condition is obtained in case where the control domain is convex.New necessary conditions are obtained also in case where it is degenerate.

In Paragraph 3 of Chapter 2,the analogue of Pontryagin's maximum principleis obtained in case where the control domains are arbitrary bounded sets, and the case where the maximum principle is degenerate (special case) is considered.

Chapter 3 consists of three paragraphs. It considers linear and nonlinear variable structure optimal control problems described by the system of Volterra type integral equations and the system of ordinary differential equations in a given period of time under various conditions.

Necessary and sufficient condition for optimality is obtained in linear case. The case where the system of equations and the connectedness condition are linear, while the quality criterion is nonlinear, is also considered.

In this case, necessary condition in the form of maximum principle is obtained first, and then the case where it is degenerate is considered using the constructed second order growth formula.

In Paragraph 3 of Chapter 3, nonlinear optimal control problem is considered in case where the control domains are bounded open sets,and constructively verifiable first and second order necessary conditions for optimality are obtained using the first and the second variations (in the classical sense) of the functional.

In the particular case considered, the analogue of the Legendre-Clebsch condition was proved and the optimality condition of the classical extremal was obtained in the classical sense.

The main results obtained in this dissertation have been reported in the following publications:

1. Mansimov K.B., Alekberov A.A. One optimal control problem described bythe system of differential and integral equations // Vestnik BGU, ser. fiz.-mat. nauk, 2017, №4, p. 5-12. (*Russian*)
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Dissertation defence will be held on **24 June 2025** year at **14^{00}** pm at the meeting of Dissertation Council ED 2.17 of the Baku State University.

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The dissertation work is available in the library of the Baku State University.

Electronic versions of the dissertation and its abstract are available on the official website of the Baku State University.

The copies of Dissertation Abstract have been sent to whom it may concern on 20 May 2025.

Authorized for printing: 08.05.2025
Format: 60x84, 1/16
Volume: 38848
Circulation: 20 copies