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# ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

# SOME PROPERTİES OF INTEGRAL TRANSFORMS OF HARMONIC ANALYSIS

Specialty: 1202.01 – Analysis and functional analysis

Field of science: Mathematics

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**Baku – 2023**

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## GENERAL CHARACTERISTICS OF THE WORK

**Rationale and the degree of development of the topic.**

The dissertation work is devoted to the study of the properties of Ahlfors-Berling, Riesz and complex Riesz transforms, which are the main integral transforms of harmonic analysis. Many problems of mechanics and physics are solved by introducing partial differential equations. The solutions of these partial differential equations are often found in the form of potentials. The potentials used here are mainly singular integral operators based on the above mentioned integral transforms. This shows how this topic is rational and practical importance.

The Alfors-Beurling transform was created when reducing the two-variable quadratic form to the canonical form and constructing quasiconform mappings on a complex plane. So, these problems are solved by reducing them to Beltrami system of differential equations. L. Ahlfors showed that the system of Beltrami differential equations has a solution in the form of a Cauchy integral on the complex plane, and the desired function is a solution of the integral equation involving the Ahlfors-Beurling transform. The existence of this integral equation depends on the boundedness of the Alfors-Beurling transform in some space of functions and other properties.

The Riesz transform is a multidimensional analogue of the Hilbert transform and is widely used in solving the Dirichlet problem and the Neumann problem for elliptic partial differential equations usyng the Fourier transform.

The Hilbert transform, a one-dimensional singular integral operator, appears in problems of summation of Fourier series and in solving Riemann and Hilbert boundary value problems for analytic functions. The boundedness of the Hilbert transform in the space of square summable functions was shown by D. Hilbert in 1905.. Later, in 1922, the existence of the Hilbert transform for Lebesgue integrable functions was proved by A. Plessner, and in 1928 the boundedness of the Hilbert transform in the space $L_p$, $1 < p < \infty$ was proved by M. Riess. İn case $p = 1$, the Hilbert transform does not act from the space $L_1$ to the space $L_1$; more precisely, the Hilbert transform of a Lebesgue integrable function may not be Lebesgue integrable. A.Kolmogorov has shown that for $p=1$ the Hilbert transform acts from the space $L_1$ to the weak space $L_1$.

In 1952 A.Calderon and A.Zigmund showed the almost everywhere existence of singular integrals of Lebesgue integrable functions in multidimensional case, the boundedness of multidimensional singular integral operators including the Alfors-Beurling, Riesz and complex Riesz transforms in the spaces $L_p$, $1<p<\infty$ and in the case $p=1$ they showed that they act from the space $L_1$ to the weak space $L_1$. Later R.Hunt, B.Muckenhoupt, R.Wheeden, F.Chiarenza, M.Frasca, J.Peetre, D.R.Adams, R.Coifman, E.Stein, G.Weiss, C.Fefferman, A.Cianchi, E.Nakai, S.Samko, V.Kokilashvilli, R.Banuelos, P.Janakiraman, V.Cruz, X.Tolsa, T.Iwaniec, V.Cruz, J.Mateu, J.Orobitg, E.Doubtsov, A.V.Vasin, O.Dragicevic, H.Kwok-Pun, M.Prats, X.Tolsa, Z.Guo, P.Li, L.Peng, A.V.Vasin and other researchers have proved the boundedness and other properties of singular integral operators including of Alfors-Beurling, Riesz and complex Riesz transforms in the weighted Lebesgue, Orlicz, Morrey, Sobolev, Besov, Companato, Hölder and other functions spaces.

In this field we can mention the works of Azerbaijan mathematicians A.A.Babayev, I.A.Aliyev, A.J.Gadjiev, S.R.Abdullayev, V.S.Guliyev, R.M.Rzayev, J.H.Hasanov, R.Mustafayev and others.

As it was noted, since in general the Alfors-Beurling and Riesz transforms of Lebesgue integrable functions are not Lebesgue integrable, using the notion of Lebesgue integral it is impossible to study completely the Alfors-Beurling transforms of functions from the space $L_1$. In 1929 E.Titchmarsh has introduced the generalization of the Lebesgue integral, the $Q$ and $Q'$ integrals. In his paper E.Titchmarsh has shown that when applying adjoint trigonometric series to the Fourier series of Lebesgue integrable functions, the $Q$

integral being the generalization of the Lebesgue integral enables to obtain more natural results.

But the main fact that complicates the application of $Q$ and $Q'$ integral notions to the theory of functions with real and complex variables is that these integrals do not satisfy the additivity property with respect to functions, that is $Q$ integrability of two functions does not mean that their sum is $Q$ integrable. Furthermore, even if the sum is $Q$ integrable, the sum of the integral might not be egual to the sum of integrals.

Adding the condition

$$m\{x \in [a,b] : |f(x)| > \lambda\} = o(1/\lambda), \qquad \lambda \to +\infty$$

to the definition of the $Q$ integral ($Q'$-integral) of functions measurable on the interval $[a;b]$, the notions of $Q$ and $Q'$ integrals coincides and the obtained functions satisfy the additivity condition with respect to the functions in the class of the obtained functions, here $m$ is Lebesgue measure of the set (in this case the function $f$ is called $A$-integrable in the interval $[a;b]$, the value of the $Q$ integral is called $A$-integral of the function $f$ ).

The properties and applications of $A$-integral were studied in detail in the works of P.L.Ulyanov, Yu.S.Ocha, I.I.Bondin, T.P.Lukashenko, I.A.Vinogradova, F.S.Vakher, G.A.Khuskivadze, K.Yoneda, V.I.Rybakov, O.D.Chereteli, V.A.Skvorsov, A.V.Rybkin, A.B.Alexandrov, T.S.Salimov, A.A.Saiyad and other authors, the properties of the $Q$- and $Q'$- integrals in the works of E.Titchmarch, T.S.Salimov, M.P.Yefimova and R.A.Aliyev.

In the dissertation work, the asymptotics of distribution functions of Lebesgue integrable functions, of finite complex measures with atomic discrete measure singular part on a complex plane, Riesz and complex Riesz transformations of Alfors-Beurling and also Lebesgue integrable functions and also modified Alfors-Beurling, Riesz and complex Riesz transformations were given, and using the $A$-, $Q$- and $Q'$- integral notions being the generalization of the Lebesgue integral, the analogues of the Riesz equalities were obtained for these transformations.

**Object and subject of the research.**

The object of the dissertation work is the integral transforms of harmonic analysis. The subject of the work is the study of these integral transforms using generalized integrals.

**The goal and objectives of the study.**

The main goal of the dissertation work is to give asymptotics of distribution functions of Alfors-Beurling, Riesz and complex Riesz transforms of Lebesgue integrable functions, to study the properties of modified Alfors-Beurling, Riesz and complex Riesz transforms using the integral notions being the generalized Lebesgue integral, and to obtain analogues of Riesz equation for them.

**Research methods.**

In the dissertation work, the methods of theory of functions with real and complex variables, of harmonic analysis, of theory of singular integral operators and functional analysis were used.

**The main thesis to be defended.**

- Asymptotic behavior of the distribution function of the Ahlfors-Beurling and modified Ahlfors-Beurling transforms of Lebesgue-integrable functions on the complex plane, as well as finite complex measures whose singular part is an atomic discrete measure, an analogue of the Riesz equation for the modified Ahlfors-Beurling transform of Lebesgue integrable functions in a bounded domain, as well as for finite complex measures whose singular part is an atomic discrete dimension.
- Asymptotic behavior of the distribution function of the Riesz and modified Riesz transforms of Lebesgue-integrable functions in space $R^d$, an analogue of the Riesz equation for the modified Riesz transform of Lebesgue integrable functions in a bounded domain.
- Asymptotic behavior of the distribution function of the complex Riesz and modified complex Riesz transforms of Lebesgue-integrable functions on a complex plane, an analogue of the Riesz equation for the modified complex Riesz transform of Lebesgue integrable functions in a bounded domain.

**Scientific novelty of the research.** The following results were obtained in the thesis:

- ❖ Asymptotics of the distribution function of the Ahlfors-Beurling transform and the modified Ahlfors-Beurling transform of Lebesgue integrable functions in the complex plane, as well as finite complex measures whose singular part is an atomic discrete measure is given.
- ❖ Using the notion of generalized integrals, the analogue of the Riesz equation was proved for modified Ahlfors-Beurling transform of functions integrable in the sense of Lebesgue in a bounded domain, as well as finite complex measures whose singular part is an atomic discrete measure.
- ❖ Asymptotics of the distribution function of the Riesz and modified Riesz transform of Lebesgue integrable functions is given.
- ❖ Using the notion of generalized integral, the analogue of the Riesz equation is proved for modified Riesz transform of Lebesgue integrable functions in a bounded domain.
- ❖ Asymptotics of the distribution function of complex Riesz and modified complex Riesz transform of Lebesgue integrable function on a complex domain, was given.
- ❖ Using the notion of generalized integral, the analogue of the Riesz equation for modified complex Riesz transform of Lebesgue integrable functions on a bounded domain, was proved.

**Theoretical and practical significance of the research.**

The results of the dissertation work mainly are of theoretical character. The results obtained in the dissertation work can be used in solving various problems of elliptic type partial differential equations, of mathematical physics and mechanics.

**Approbation and application of the work**. The results of the dissertation work were discussed at the scientific seminars of “Mathematics” department (head: PhD of math. sc. A.Huseynli) of Khazar University, of the chair of “Mathematical analysis” (head: prof. S.S.Mirzoyev) of BSU, the department of “Function theory” (head: doctor of math. sc. V.E.Ismayilov) of IMM. Furthermore, the results obtained in the work were reported in the following scientific conferences: The international scientific conferences “Operators, functions and systems of mathematical physics” (Baku, 2018, 2019), in the international mathematical conference “Complex analysis, mathematical physics and nonlinear equations” (Ufa, Russia 2021).

The obtained results can be used in constructing quasi-conformal mappings on the complex plane, in problems of reducing a two-dimensional quadratic form to canonical form, as well as in solving many problems of mathematical physics and mechanics related to solving elliptic-type differential equations.

The obtained results can be used in constructing quasi conform mappings on a complex plane in reducing two-variable quadratic form to the canonical form and when solving many mathematical physics and mechanics problems reduced to the solution of elliptic type partial differential equations.

**Author's personal contribution**. All the results obtained in the work belong to the author.

**Author's publications.**

In scientific editions recommended by Higher Attestation Commission under the President of the Republic of Azerbaijan – **6** (see: [1, 3, 5-7, 9]) (including scientific editions implicated in international database – **3** (see: [1, 5, 6]); no coauthors – **3** (see: [3, 7, 9])). Abstracts – **3** (see: [2, 4, 8]) (including whose results were published abroad– 1 (see: [8])).

**The name of the organization where the dissertation work was executed.** The work was executed at the department of “Mathematics” of the faculty of Natural Sciences, Art and Technology, Higher education of Khazar-University.

**Structure and volume of the dissertation (in signs indicating the volume of each structural subdivision separately).**

Total volume of the dissertation work– 177715 signs (title page – 318 signs, contents –1390 signs, introduction – 30761 signs, chapter I –64000 signs, chapter II – 42000 signs, chapter III -38000 signs, conclusion – 1246 signs).

## THE CONTENT OF THE DISSERTATION

The dissertation work consists of introduction, three chapters, conclusion and list of references.

**Chapter I** of the work was devoted to the properties of Ahlfors-Beurling transformation. In this chapter the asymptotics of the distribution function of the Ahlfors-Beurling transform and the modified Ahlfors-Beurling transform of Lebesgue integrable functions in the complex plane, as well as finite complex measures whose singular part is an atomic discrete measure is given; using the notion of generalized integrals, the analogue of the Riesz equation was proved for modified Ahlfors-Beurling transform of functions integrable in the sense of Lebesgue in a bounded domain, as well as finite complex measures whose singular part is an atomic discrete measure.

Let the function $f$ is integrable with degree $p \geq 1$ in the complex plane. The singular integral

$$(Bf)(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\{w \in C : |z-w| > \varepsilon\}} \frac{f(w)}{(z-w)^2} \, dm(w), \quad z \in C$$

is called Ahlfors-Beurling transform of the function $f$.

From the theory of singular integrals it is known that the Ahlfors–Beurling transform is a bounded operator in the space $L_p(C)$, $1 < p < \infty$, that is, if $f \in L_p(C)$, then $Bf \in L_p(C)$ and

$$\|Bf\|_{L_p(C)} \leq C_p \|f\|_{L_p(C)}.$$

In the case $f \in L_1(C)$ only the weak inequality holds:

$$m\{z \in C : |(Bf)(z)| > \lambda\} \leq \frac{C_1}{\lambda} \|f\|_{L_1(C)}, \quad \lambda > 0$$

where $m$ stands for the Lebesgue measure $C_p$, $C_1$ are constants independent of $f$.

In section 1.1 of chapter I the asymptotics of the distribution function of the Ahlfors-Beurling transform are given as $\lambda \to +\infty$ and as $\lambda \to 0+$.

**Theorem 1.** Assume that $f \in L_1(C)$. Then the asymptotic equation

$$\lim_{\lambda \to +\infty} \lambda m\{z \in C : |(Bf)(z)| > \lambda\} = 0$$

is satisfied.

**Theorem 2.** Assume that $f \in L_1(C)$. Then the asymptotic equation

$$\lim_{\lambda \to 0+} \lambda m\{z \in C : |(Bf)(z)| > \lambda\} = \left| \int_C f(z) dm(z) \right|$$

is satisfied.

Let $\Omega$ be a domain given on a complex plane and the function $f$ be a Lebesgue integrable function in the domain $\Omega$. The function

$$(B_\Omega f)(z) = B(\chi_\Omega f)(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\{w \in \Omega : |z-w| > \varepsilon\}} \frac{f(w)}{(z-w)^2} dm(w), \; z \in \Omega$$

is called modified Ahlfors-Beurling transform of the function $f$, where $\chi_\Omega$ is a characteristical function of the set $\Omega$.

In section 1.2 of chapter I the asymptotics of the distribution function of modified Ahlfors-Beurling transform are given as $\lambda \to +\infty$ and as $\lambda \to 0+$.

**Theorem 3.** Assume that $f \in L_1(\Omega)$. Then the asymptotic equation

$$\lim_{\lambda \to +\infty} \lambda m\{z \in \Omega : |(B_\Omega f)(z)| > \lambda\} = 0$$

is satisfied.

**Theorem 4.** Assume that $\Omega$ is a domain given in a complex plane, and $f \in L_1(\Omega)$. Then the asymptotic equations

$$\limsup_{\lambda\to 0+} \lambda m\{z\in\Omega : |(B_\Omega f)(z)| > \lambda\} = d^*(\Omega)\cdot\left|\int_\Omega f(z)dm(z)\right|,$$

$$\liminf_{\lambda\to 0+} \lambda m\{z\in\Omega : |(B_\Omega f)(z)| > \lambda\} = d_*(\Omega)\cdot\left|\int_\Omega f(z)dm(z)\right|$$

is satisfied, where

$$d^*(\Omega) = \limsup_{r\to+\infty}\frac{m(\Omega\cap U(0;r))}{m(U(0;r))}, \quad d_*(\Omega) = \liminf_{r\to+\infty}\frac{m(\Omega\cap U(0;r))}{m(U(0;r))}.$$

For a complex value function $f$, measurable in the bounded domain $\Omega\subset C$ we denote

$$[f(z)]_n = [f(z)]^n = f(z), \quad \text{for } |f(z)|\le n,$$

$$[f(z)]_n = n\,\mathrm{sgn}\, f(z), \ [f(z)]^n = 0, \ \text{for } \ |f(z)| > n,$$

where $n\in N$, $\mathrm{sgn}\, w = w/|w|$ for $w\ne 0$ and $\mathrm{sgn}\, 0 = 0$.

In 1929 E.Titchmarch has introduced the notions of $Q$ and $Q'$ integral for the functions measurable in the domain $\Omega$.

**Definition 1.** If there exists the finite limit

$$\lim_{n\to\infty}\int_\Omega [f(z)]_n\, dm(z) \ \left(\lim_{n\to\infty}\int_\Omega [f(z)]^n\, dm(z) \text{ respectively}\right),$$

then the function $f$ is said to be $Q$-integrable function ($Q'$-integrable, respectively) in the domain $\Omega$ and is denoted as $f\in Q(\Omega)$ $(f\in Q'(\Omega))$. The value of this limit is called $Q$-integral ($Q'$- integral) of the function $f$ in the domain $\Omega$ and is denoted as

$$(Q)\int_{\Omega} f(z)dm(z) \quad \left((Q')\int_{\Omega} f(z)dm(z)\right).$$

E.Titchmarsh has shown that when studying the series adjoint to the Fourier series of Lebesgue integrable functions, the $Q$ integral gives more natural results. But as it was noted above, the fact that complicates application of $Q$ integral and $Q'$ integral notions to the thory of real and complex variable functions is that these integrals do not satisfy the additivity property. But if we add to the definition of $Q$ integral (of $Q'$ integral) the condition

$$m\{z \in \Omega : |f(z)| > \lambda\} = o(1/\lambda), \qquad \lambda \to +\infty, \tag{1}$$

then the notions of $Q$ and $Q'$ integrals coincide and satisfy the additivity property in the obtained class of functions with respect to the functions.

**Definition 2.** If the function $f$ is $Q$ integrable (or $Q'$ integrable) in the domain $\Omega$ and condition (1) is satisfied, then the function $f$ is said to be $A$-integrable function in the domain $\Omega$ and is denoted as $f \in A(\Omega)$. In this case, the value of the limit $\lim\limits_{n\to\infty} \int_{\Omega} [f(z)]_n dm(z)$ (or $\lim\limits_{n\to\infty} \int_{\Omega} [f(z)]^n dm(z)$) is called $A$-integral of the function $f$ in the domain $\Omega$ and is denoted as

$$(A)\int_{\Omega} f(z)dm(z).$$

In section 1.3 of chapter I, the analogue of the Riesz equality for modified Ahlfors-Beurling transform of Lebesgue integrable functions in the bounded domain $\Omega \subset C$, is proved .

**Theorem 5.** Let $\Omega$ be a bounded domain in the complex plane, $f$ be a Lebesgue integrable function in the domain $\Omega$, the function $g(z)$ be a bounded function given in the function $\Omega$ such that the function $(B_\Omega g)(z)$ also is bounded in the domain $\Omega$. Then the function $g(z)\cdot(B_\Omega f)(z)$ is $A$-integrable in the domain $\Omega$ and ther equation

$$(A)\int_\Omega g(z)(B_\Omega f)(z)dm(z) = \int_\Omega f(z)(B_\Omega g)(z)dm(z)$$

is satisfied.

**Corollary 1.** Assume that $\Omega$ is a bounded domain in a complex plane whose boundary is Lyapunov curve and the function $f$ is Lebesgue integrable function in the domain $\Omega$. Then the function $(B_\Omega f)(z)$ is $A$-integrable in the domain $\Omega$ and the equation

$$(A)\int_\Omega (B_\Omega f)(z)dm(z) = \int_\Omega f(z)(B_\Omega 1)(z)dm(z)$$

is satisfied.

In section 1.4 of chapter I the properties of Ahlfors-Beurling transform of a finite complex measure with an atomic discrete measure singular part, are studied.

**Definition 3.** Assume that the set $X$ is given on a complex plane. If there exist such number $\delta > 0$ that the inequality $|x-y| \geq \delta$ is satisfied for arbitrary elements $x, y \in X$, then the set $X$ is called an atomic set.

It is clear that atomic set can consist of no more than a countable number of elements.

**Definition 4.** If the measure $\nu$ given in the complex plane is concentrated in the atomic set, then the measure $\nu$ is called an atomic discrete measure.

By $M_a$ we denote a set of finite complex measures whose singular part is an atomic discrete measure on a complex plane.

**Definition 5.** Let $\mu \in M_a$. The function

$$(B\mu)(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\{w \in C : |z-w| > \varepsilon\}} \frac{d\mu(w)}{(z-w)^2}, \quad z \in C$$

is called Ahlfors-Beurling transform of the measure $\mu$.

It is clear that Ahlfors-Beurling transform of the measure $\mu \in M_a$ is almost everywhere defined on a complex plane and if

$$d\mu(z) = f(z)dm(z) + d\mu_s(z),$$
$$\mathrm{supp}\mu_s = X = \{z_j\}_{j \in J}, \ \ \mu_s(z_j) = \alpha_j, \ j \in J,$$

then the equation

$$(B\mu)(z) = (Bf)(z) - \frac{1}{\pi} \sum_{j \in J} \frac{\alpha_j}{(z_j - z)^2}$$

is satisfied almost everywhere, where $\mu_s$ denotes a singular part of the measure $\mu$.

**Theorem 6.** Assume that $\mu \in M_a$. Then the asymptotic equation

$$\lim_{\lambda \to +\infty} \lambda m\{z \in C : |(B\mu)(z)| > \lambda\} = \|\mu_s\|$$

is satisfied, where $\mu_s$ denotes a singular part of the measure $\mu$ and $\|\mu_s\|$ is a full variation of the measure $\mu_s$.

Assume that $\Omega$ is a domain in the complex plane. Denote by $M_a(\Omega)$ the set of finite complex measures whose singular part is an atomic discrete measure on $\Omega$. For $\mu \in M_a(\Omega)$ the function

$$(B_\Omega \mu)(z) = -\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\{w \in \Omega : |z-w| > \varepsilon\}} \frac{d\mu(w)}{(z-w)^2}, \quad z \in C$$

is called modified Ahlfors-Beurling transform of the measure $\mu$.

**Theorem 7.** Assume that $\Omega$ is a bounded domainn in the complex plane, whose boundary is a Liapunov curve and $\mu \in M_a(\Omega)$. İf the function $g$ is Hölder continuous function in the closure of $\Omega$, then the function $(B_\Omega \mu)(z)g(z)$ is $Q'$-integrable ($Q$-integrable) in the domain $\Omega$, and the equation

$$(Q')\int_\Omega g(z)(B_\Omega \mu)(z)dm(z) = \int_\Omega (B_\Omega g)(z)d\mu(z)$$

$$\left( (Q)\int_\Omega g(z)(B_\Omega \mu)(z)dm(z) = \int_\Omega (B_\Omega g)(z)d\mu(z) \right)$$

is satisfied.

Chapter II was devoted to the study of properties of the Riesz transform. In this chapter the asymptotics of distribution function of modified Riesz and Riesz transforms of Lebesgue integrable functions are obtained and using the notion of generalized integral, the analogy of the Riesz equation for modified Riesz transform of Lebesgue integrable functions on a bounded domain was proved.

Let the function $f$ is integrable with degree $p \geq 1$ in the space $R^d$. The singular integral

$$R_j(f)(x) = \gamma(d) \lim_{\varepsilon \to 0} \int_{\{y \in R^d : |x-y| > \varepsilon\}} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)dy, \quad j \in \{1, 2, \ldots .d\}$$

is called Riesz transform of the function $f$ with respect to the $j$-th variable, where $\gamma(d) = \dfrac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}$, $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$ is Euler’s Gamma function.

It is well known that the Riesz transform is a bounded operator in the space $L_p\left(R^d\right)$, $1<p<\infty$, that is, if $f\in L_p(C)$, then $R_j(f)\in L_p(C)$ and

$$\left\|R_j f\right\|_{L_p\left(R^d\right)}\le \tilde{C}_p\|f\|_{L_p\left(R^d\right)}.$$

In the case $f\in L_1\left(R^d\right)$ only the weak inequality holds:

$$m\left\{x\in R^d:\left|\left(R_j f\right)(x)\right|>\lambda\right\}\le\frac{\tilde{C}_1}{\lambda}\|f\|_{L_1\left(R^d\right)},\quad \lambda>0$$

where $\tilde{C}_p$, $\tilde{C}_1$ are constants independent of $f$.

In section 2.1 of chapter II the asymptotics of distribution function of the Riesz transformation is given as $\lambda\to+\infty$ and as $\lambda\to 0+$.

**Theorem 8.** Assume that $f\in L_1\left(R^d\right)$. Then the asymptotic equation

$$\lim_{\lambda\to+\infty}\lambda m\left\{x\in R^d:\left|\left(R_j f\right)(x)\right|>\lambda\right\}=0$$

is satisfied.

**Theorem 9.** Assume that $f\in L_1\left(R^d\right)$. Then the asymptotic equation

$$\lim_{\lambda\to 0+}\lambda m\left\{x\in R^d:\left|\left(R_j f\right)(x)\right|>\lambda\right\}=\gamma(d)\theta(d)\left|\int_{R^d} f(x)dx\right|$$

is satisfied, where $\theta(d)=\frac{2^d}{d\cdot(d-1)!!}\left(\frac{\pi}{2}\right)^{\left[\frac{d-1}{2}\right]}$ and $\left[\frac{d-1}{2}\right]$ note the entire part of the number $\frac{d-1}{2}$.

Assume that $\Omega$ is a domain in the space $R^d$ and the function $f$ is a Lebesgue integrable in the domain $\Omega$. The function

$$\left(R_{j,\Omega} f\right)(x) = R_j\left(\chi_\Omega f\right)(x) =$$

$$\gamma(d) \lim_{\varepsilon\to 0} \int_{\{y\in\Omega:|x-y|>\varepsilon\}} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)dy, \; j = 1,2,\ldots,d, \quad z\in\Omega$$

is said to be modified Riesz transform of the function $f$ .

In section 2.3 of chapter II the analogue of the Riesz equation for Riesz transform of Lebesgue integrable function in the bounded domain $\Omega\subset C$, is proved.

**Theorem 10.** Assume that $\Omega$ is a bounded domain in the space $R^d$, $f\in L_1(\Omega)$, and $g(z)$ function is bounded function in the domain $\Omega$ such that the function $\left(R_{j,\Omega} g\right)(x)$ is also bounded in the domain $\Omega$. Then the function $g(x)\cdot\left(R_{j,\Omega} f\right)(x)$ is $A$-integrable in the domain $\Omega$ and the equation

$$(A)\int_\Omega g(x)\left(R_{j,\Omega} f\right)(x)dx = -\int_\Omega f(x)\left(R_{j,\Omega} g\right)(x)dx$$

is satisfied.

**Corollary 2.** Assume that $\Omega$ is a bounded domain in the space $R^d$, whose boundary is a Lyapunov plane and the function $f$ is a Lebesgue integrable function in the domain $\Omega$. Then the function $\left(R_{j,\Omega} f\right)(x)$ is $A$-integrable in the domain $\Omega$ and the equation

$$(A)\int_\Omega \left(R_{j,\Omega} f\right)(x)dx = -\int_\Omega f(x)\left(R_{j,\Omega} 1\right)(x)dx$$

is satisfied.

Chapter III studies the properties of the complex Riesz transforms. In this chapter, the asymptotics of distribution function of Lebesgue integrable functions on a complex plane is obtained, and using the notion of generalized integral, the analogue of the Riesz equation for the modified complex Riesz transform of Lebesgue integrable functions on a bounded domain is proved.

Let the function $f$ is integrable with degree $p \geq 1$ on the complex plane. For arbitrary $k \in Z$, $k \neq 0$ the singular integral

$$\left(R^{(k)} f\right)(z) = \frac{|k|}{2\pi i^{|k|}} \lim_{\varepsilon \to 0} \int_{\{w \in C : |z-w| > \varepsilon\}} \frac{(\bar{z} - \bar{w})^k}{|z - w|^{k+2}} f(w) dm(w), \quad z \in C$$

is called $k$-th order complex Riesz transform of the function $f$. For $k = 0$, $R^{(0)}$ is taken as an identity operator, that is $R^{(0)} = I$. Since for $k = 2$ the complex Riesz transform coincides with Ahlfors-Beurling transform, we can consider the complex Riesz transform as a generalization of the Ahlfors-Beurling transform.

In section 3.1 of chapter III the asymptotics of distribution function of the complex Riesz transform are given as $\lambda \to +\infty$ and as $\lambda \to 0+$.

**Theorem 11.** Assume $f \in L_1(C)$. Then the asymptotic equation

$$\lim_{\lambda \to +\infty} \lambda m\left\{z \in C : \left|\left(R^{(k)} f\right)(z)\right| > \lambda\right\} = 0$$

is satisfied.

**Theorem 12.** Assume, $f \in L_1(C)$. Then the asymptotic equation

$$\lim_{\lambda \to 0+} \lambda m\left\{z \in C : \left|\left(R^{(k)} f\right)(z)\right| > \lambda\right\} = \frac{|k|}{2} \cdot \left|\int_C f(z) dm(z)\right|$$

is satisfied.

Assume that $\Omega$ is a domain in the complex plane and the function $f$ is a Lebesgue integrable function on the domain $\Omega$. The function

$$\left(R_\Omega^{(k)} f\right)(z) = R^{(k)}(\chi_\Omega f)(z) =$$

$$= \frac{|k|}{2\pi i^{|k|}} \lim_{\varepsilon \to 0} \int_{\{w \in \Omega : |z-w| > \varepsilon\}} \frac{(\bar{z} - \bar{w})^k}{|z - w|^{k+2}} f(w) dm(w), \quad z \in \Omega$$

is called a modified complex Riesz transform of the function $f$.

İn section 3.3 of chapter III we prove the analogue of the Riesz equation for a complex Riesz transform of Lebesgue integrable function in the bounded domain $\Omega \subset C$.

**Theorem 13.** Assume that $\Omega$ is a bounded domain in the complex plane, the fiunction $f$ is a Lebesgue integrable function in the domain $\Omega$ and the function $g$ is bounded function in the domain $\Omega$ such that the function $R_{\Omega}^{(k)} g$ is also bounded in the domain $\Omega$. Then the function $g(z) \cdot \left(R_{\Omega}^{(k)} f\right)(z)$ is $A$-integrable in the domain $\Omega$ and the equation

$$(A)\int_{\Omega} g(z)\left(R_{\Omega}^{(k)} f\right)(z)dm(z) = (-1)^k \int_{\Omega} f(z)\left(R_{\Omega}^{(k)} g\right)(z)dm(z)$$

is satisfied.

**Corollary 3.** Assume that $\Omega$ is a bounded domain whose boundary is a Lyapunov curve and the function $f$ is a Lebesgue integrable function in the domain $\Omega$. Then the function $\left(R_{\Omega}^{(k)} f\right)(z)$ is $A$-integrable in the domain $\Omega$, and the equality

$$(A)\int_{\Omega} \left(R_{\Omega}^{(k)} f\right)(z)dm(z) = (-1)^k \int_{\Omega} f(z)\left(R_{\Omega}^{(k)} 1\right)(z)dm(z)$$

is satisfied.

The author expresses her deep respect and sincere gratitude to her supervisor prof. R.A.Aliyev for putting problems, discussing the results and paying constant attention to the work.

# CONCLUSION

The dissertation work is devoted to the study of the properties of Ahlfors-Berling, Riesz and complex Riesz transforms, which are the main integral transforms of harmonic analysis. The following results were obtained in the thesis:

- ❖ Asymptotics of the distribution function of the Ahlfors-Beurling transform and the modified Ahlfors-Beurling transform of Lebesgue integrable functions in the complex plane, as well as finite complex measures whose singular part is an atomic discrete measure is given.
- ❖ Using the notion of generalized integrals, the analogue of the Riesz equation was proved for modified Ahlfors-Beurling transform of functions integrable in the sense of Lebesgue in a bounded domain, as well as finite complex measures whose singular part is an atomic discrete measure.
- ❖ Asymptotics of the distribution function of the Riesz and modified Riesz transform of Lebesgue integrable functions is given.
- ❖ Using the notion of generalized integral, the analogue of the Riesz equation is proved for modified Riesz transform of Lebesgue integrable functions in a bounded domain.
- ❖ Asymptotics of the distribution function of complex Riesz and modified complex Riesz transform of Lebesgue integrable function on a complex domain, was given.
- ❖ Using the notion of generalized integral, the analogue of the Riesz equation for modified complex Riesz transform of Lebesgue integrable functions on a bounded domain, was proved.

**The main results of the dissertation work are published in the following works:**

1. Aliev, R.A., Nebiyeva, Kh.I. The $A$-integral and restricted Ahlfors–Beurling transform // Integral Transforms and Special Functions, – 2018. vol. 29:10, – p. 820-830.
2. Aliev, R.A., Nebiyeva, Kh.I. The $A$-integral and Ahlfors-Beurling transform // Operators, Functions, and Systems of Mathematical Physics Conference, Khazar University, -Baku: Azerbaijan, -21-24 May, – 2018, – p. 60.
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5. Aliev, R.A., Nebiyeva, Kh.I. The $A$-integral and restricted complex Riesz transform // -Baku: Azerbaijan Journal of Mathematics, – 2020. vol. 10:2, – p. 209-221.
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9. Nebiyeva, Kh.I. On properties of Ahlfors-Beurling transform of finite complex measures // -Baku: Caspian Journal of Applied Mathematics, Ecology and Economics, – 2021. vol. 9:1, – p. 40-48.

The defense will be held on **22 September 2023** year at **$14^{00}$** at the meeting of the Dissertation council ED 1.04 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan.

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Dissertation is accessible at the library of the Institute of Mathematics and Mechanics of the library.

Electronic versions of the dissertation and its abstract are available on the official website of the Institute of Mathematics and Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan.

Abstract was sent to the required addresses on **06 June 2023**.

Signed for print: 03.05.2023
Paper format: 60x84 1/16
Volume: 35790
Number of hard copies: 30