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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

APPROXIMATIONS OF HYPERSINGULAR INTEGRAL OPERATORS AND THEIR APPLICATIONS

Speciality: 1202.01 -Analysis and functional analysis
Field of science: Mathematics
Applicant: **Chinara Arif gizi Hajieva**

Baku – 2021

The work was performed at Baku State University the department of "Mathematical analysis".

Scientific supervisor: doctor of sciences of math., associate professor
Rashid Avazaga oglu Aliyev

Official opponents: doctor of sciences of math., professor
Bahram Ali oglu Aliev

doctor of sciences of math., associate professor

Vugar Elman oglu Ismailov

cand. of phys.-math. sc., assoc. prof.

Elnur Hasan oglu Khalilov

Dissertation council ED 1.04 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at the Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan

Chairman of the Dissertation council:

corr.-member of NASA, doc. of ph. and math. sc., prof.

Misir Jumail oglu Mardanov

Scientific secretary of the Dissertation council:

cand. of ph. and math. sc.

Abdurrahim Farman oglu Guliyev

Chairman of the scientific seminar:

corr.-member of NASA, doc. of ph. and math. sc., prof.

Bilal Telman oglu Bilalov

GENERAL CHARACTERISTIC

Actuality of the work and the degree of elaboration. An active development of numerical methods for solving hypersingular integral equations is of considerable interest in modern numerical analysis. This is due to the fact that hypersingular integral equations have numerous applications in acoustics, aerodynamics, fluid mechanics, electrodynamics, elasticity, fracture mechanics, geophysics and etc. Therefore the construction and justification of numerical schemes for approximate solutions of hypersingular integral equations is a topical issue and numerous works are devoted to their development.The development of constructive methods for solving hypersingular integral equations is impossible without studying the properties of hypersingular integral operators contained in these equations, and is associated with the approximation of such operators, which indicates the actuality of the subject of dissertation research.Hypersingular integrals were introduced by J. Hadamard for the solution of the Cauchy problem for a linear partial differential equations of a hyperbolic type.They also arise in solving Neumann problem for the Laplace equation, in solving integral equations of the linear theory of a bearing surface, in inverting generalized Riesz potentials, when presenting some classes of pseudo-differential operators and in other areas of mathematics and mechanics.

Approximations of hypersingular integrals and the construction of constructive methods for solution of hypersingular integral equations with Cauchy kernel and Hilbert kernel, the theory of which is well described in monographs are devoted to the works of A.Yu. Anfinogenov, I. K. Lifanov, P.I. Lifanov , R. B. Babaev, B. V. Boykov , G. M. Vainikko, I. K. Lifanov, L.N. Poltavsky, K. Buhring, Z. Chen, Y. Zhou , D. Chien, K. Atkinson , M. De Bonis, D. Occorsio , V. Ervin, E. Stephan, S. Fata, L. Gray, H. Feng, X. Zhang , J. Huang, Z. Wang, R.Zhu , AVKostenko, R. Kress, A. Sidi and other authors. In 2006 a new constructive method for solution of the singular integral equations with the Cauchy kernel was worked out by R.A.Aliev, in which singular integral operator is approximated by operators preserving main properties of the singular integral operator and that enables to obtain more exact results than traditional methods in terms of the convergence rate and requires less computational cost because it allows to find approximate solutions explicitly (and not at individual points), herewith the coefficients of the corresponding systems of linear algebraic equations are easily calculated. In present dissertation work this constructive method is worked out and is justified for solution of hypersingular integral equations with Hilbert kernel and Cauchy kernel in the space of the square-integrable functions and and in the Hölder spaces.

The aim and objectives of the research. The aim of the work is construction of approximation hypersingular integral operator with Hilbert kernel and Cauchy kernel, construction and justification of constructive method for solution of hypersingular integral equations with Cauchy kernel and Hilbert kernel.

Research methods. In order to justificate the results obtained in the dissertation the methods of the theory of functions of a real and complex variable, the theory of singular integral equations, functional analysis, linear algebra and the general theory of approximate methods are used.

Key points of the dissertation which will be defended.

- the error estimates of the approximation of hypersingular integral operators with Cauchy kernel and with Hilbert kernel in the space of the square-integrable functions and in the Hölder spaces;
- presentation and justification of the constructive method for solution of hypersingular integral equations with Hilbert kernel and with Cauchy kernel in the space of the square-integrable functions;
- the application of given constructive method to the 2D inner Neumann problem for Laplace equation and using numerical experiments to show the efficiency of this method;

Scientific novelty of the research.

- the error estimates of the approximation of hypersingular integral operators with Cauchy kernel and with Hilbert kernel in the space of the square-integrable functions and in the Hölder spaces;
- the constructive method for solution of hypersingular integral

equations with Hilbert kernel and with Cauchy kernel is presented and justificated in the space of the square-integrable functions;

- the application of given constructive method to the 2D inner Neumann problem for Laplace equation is described and results of numerical examples confirming the efficiency of the proposed method are given.

Scientific and practical value of the research. The dissertation is mainly theoretical in nature. However, the obtained results can find application in the further development of numerical methods for solution of the singular integral equations and other problems of analysis. They can also be applied in solving various theoretical and applied problems, which are reduced to hypersingular integral equations considered in the dissertation.

Presentation and application of the work. The main results of the dissertation have been presented at the seminars of the chair of "Mathematical analysis" in BSU (Chief of the chair D.p.-m.s., prof. Mirzoev S.S), at the seminars of the chair of "Theory of functions and functional analysis" in BSU (Chief of the chair D.p.-m.s., prof. Akhmedov A.M), at the seminars of the department "Theory of functions" in IMM NAS of Azerbaijan (Chief of the dep. prof. Ismailov V.E), at international conference devoted to the 85^{th} anniversary of prof. Y.Mammadov, Baku 2015, at international conference on "Actual problems of theoretical and practical mathematics" devoted to the 100^{th} anniversary of M.L.Rasulov, Baku 2016, at the republican conference on "Functional analysis and its applications"
devoted to the 100^{th} anniversary of A.Habibzade, Baku 2016, at international conference on "Modern problems of Mathematics and Mechanics" dedicated to the 80-th anniversary of academician A.Gadjiev, Baku 2017, at 1st İnternational Science and Engineering Coference, Baku 2018, at international conference on "Operators, Functions and Systems of Mathematical Physics" , Baku 2019, at international conference on "Operators in general Morrey-Type Spaces and Applications" (OMTSA 2019), Turkey,2019.

Personal contribution of the author. All the results obtained in the dissertation belong to the author.

The name of the institution where the thesis is performed. The work was performed at Baku State University the department of " Mathematical Analysis"

Publications of the author. The main results of the dissertation work have been published in 13 author's publications given at the end of the report.

Structure and volume of the dissertation (in signs, indicating the volume of each structural subsection separately). The total volume of the dissertation –211669 signs (the title page-1706 and content 570 signs, introduction – 36000 signs, chapter I – 64000 signs, chapter II – 42000 signs, chapter III – 66000 signs Conclusion – 1393 signs). The list of references consists of 71 names

CONTENT OF DISSERTATION

The disertation is divided into introduction, three chapters, conclusion and references. The first chapter describes the approximation of hypersingular integral operator with Cauchy kernel in the space of the square-integrable functions and in the Hölder spaces. The main results of this chapter were published in the following publications of the author [1, 2, 3, 4,5, 10,11].
Consider the following integral

$$\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau - t)^2} d\tau, \quad t \in \gamma_0 , \qquad (1)$$

where the function $\varphi(t)$ is Lebesgue integrable on the unit circle $\gamma_0 = \{t \in C : |t| = 1\}$. If we define the integral (1) similar to the Cauchy integral, even if $\varphi \equiv 1$, we get the divergent integral. Therefore, using the idea of Hadamard finite part integral, we will define the integral (1) as follows.

Definition 1. If a finite limit

$$\lim_{\varepsilon\to 0+}\left(\int_{\gamma_\varepsilon}\frac{\varphi(\tau)}{(\tau-t)^2}d\tau+\frac{2i\varphi(t)}{\varepsilon\cdot t}\right),$$

exists, then the value of this limit is referred to as the hypersingular integral of the function $\frac{\varphi(\tau)}{(\tau-t)^2}$ on the circle γ_0 and is denoted by $\int_{\gamma_0}\frac{\varphi(\tau)}{(\tau-t)^2}d\tau$, where $\gamma_\varepsilon=\{\tau\in\gamma_0:|\tau-t|>\varepsilon\}$.

In 1.1 the hypersingular integral (1) is investigated, as well as the integrals of the form

$$\int_{\gamma_0}\frac{\varphi(\tau)}{(\tau-t)^m}d\tau\ ,\ m\geq 3,\ m\in N\ ,\ t\in\gamma_0\ , \tag{3}$$

$$\int_{\gamma_0}\frac{\varphi(\tau)}{|\tau-t|^{m+\lambda}}d\tau\ ,\ m\in N,\ \lambda\in[0,1),\ t\in\gamma_0\ , \tag{4}$$

where the function $\varphi(t)$ is Lebesgue integrable on the circle γ_0.

Theorem 1. If the function φ is absolutely continuous on γ_0 , then the hypersingular integral (1) exists for almost all $t\in\gamma_0$ and the following integration by parts formula holds:

$$\int_{\gamma_0}\frac{\varphi(\tau)}{(\tau-t)^2}d\tau=\int_{\gamma_0}\frac{\varphi'(\tau)}{\tau-t}d\tau\ .$$

Theorem 2. If the function φ has absolutely continuous $(m-2)$ order derivatives on γ_0 , then the hypersingular integral (0.3) exists for almost all $t\in\gamma_0$ and the following equality holds:

$$\int_{\gamma_0}\frac{\varphi(\tau)}{(\tau-t)^m}d\tau=\frac{1}{m-1}\int_{\gamma_0}\frac{\varphi'(\tau)}{(\tau-t)^{m-1}}d\tau\ .$$

Corollary 1. If the function φ has absolutely continuous $(m-2)$ order derivatives on γ_0 , then the following equality holds:

$$\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau - t)^m} d\tau = \frac{1}{(m-1)!} \int_{\gamma_0} \frac{\varphi^{(m-1)}(\tau)}{\tau - t} d\tau , \tag{4}$$

where the integral standing in the right side is understood in the sense of the Cauchy principal value.

Corollary 2. If the function φ is differentiable $(m-1)$ times at the point t, then the following equality holds:

$$\int_{\gamma_0} \frac{\varphi(\tau)}{(\tau - t)^m} d\tau = \int_{\gamma_0} \frac{\varphi(\tau) - \sum_{k=0}^{m-2} \frac{\varphi^{(k)}(t)}{k!}(\tau - t)^k}{(\tau - t)^m} d\tau ,$$

where the integral standing in the right side is understood in the sense of the Cauchy principal value.

Theorem 3. If the function $\varphi(t)$ is differentiable $(m-1)$ times on the circle γ_0 and $(m-1)$th derivative of the function $\varphi(t)$ is Hölder continuous with exponent $\lambda < \alpha \leq 1$ in γ_0, then the hypersingular integral (3) exists for all $t \in \gamma_0$.

Let $L_2(\gamma_0)$ be the space of the functions square-integrable on γ_0 with the norm

$$\|\varphi\|_{L_2(\gamma_0)} = \left(\frac{1}{2\pi} \int_{\gamma_0} |\varphi(\tau)|^2 |d\tau| \right)^{\frac{1}{2}},$$

and let $W_2^m(\gamma_0)$ be the space of functions having $(m-1)$ th order absolutely continuous derivative on γ_0 and whose m th order derivative belongs to $L_2(\gamma_0)$, with the norm

$$\|\varphi\|_{W_2^m(\gamma_0)} = \|\varphi\|_{L_2(\gamma_0)} + \sum_{k=1}^{m} \left\|\varphi^{(k)}\right\|_{L_2(\gamma_0)} .$$

Since the singular integral operator with Cauchy kernel is bounded in $L_2(\gamma_0)$, then from the Theorem 1 and Corollary 1 it follows that the following hypersingular integral operator with Cauchy kernel

$$\left(H^{(m)}\varphi\right)(t) \equiv \frac{1}{\pi i}\int_{\gamma_0}\frac{\varphi(\tau)}{(\tau-t)^m}d\tau\ ,\ t\in\gamma_0\ ,\ m=2,3,...$$

(in the case $m=2$ we will denote $H\varphi \equiv H^{(2)}\varphi$) is bounded from the space $W_2^{m-1}(\gamma_0)$ into the space $L_2(\gamma_0)$, and

$$\left\|H^{(m)}\right\|_{W_2^{m-1}(\gamma_0)\to L_2(\gamma_0)} \le \frac{1}{(m-1)!}.$$

In 1.2 the approximations of hypersingular integral operator $H^{(m)}$ with Cauchy kernel are given in the space $L_2(\gamma_0)$ and is described that these approximations preserve the main properties of the hypersingular integral operator and is obtained an appropriate estimate of the convergence.

Consider the sequence of operators

$$(H_n\varphi)(t) = \frac{1}{\pi i}\sum_{k=0}^{n-1}\frac{\varphi\left(\tau_{2k+1}^{(t)}\right)-\varphi(t)}{\left(\tau_{2k+1}^{(t)}-t\right)^2}\Delta\tau_{2k+1}^{(t)}\ ,\ t\in\gamma_0,\ n=1,2,..,$$

where $\tau_k^{(t)} = e^{k\theta i}\cdot t$, $\Delta\tau_k^{(t)} = \left(\tau_{k+1}^{(t)}-\tau_{k-1}^{(t)}\right)\frac{\theta}{\sin\theta} = 2ie^{k\theta i}\cdot t\cdot\theta$, $k=\overline{0,2n}$, $\theta = \frac{\pi}{n}$.

Theorem 4. The operators H_n, $n=1,2,...$ is bounded from the space $W_2^1(\gamma_0)$ on the space $L_2(\gamma_0)$, then the following relation holds:

$$\left\|H_n\right\|_{W_2^1(\gamma_0)\to L_2(\gamma_0)} \le 1,$$

and for any algebraic polynomial $q(t) = \sum_{k=-n+1}^{n-1} q_k t^k$ with degree less than $n-1$ the following relation holds

$$(H_n q)(t) = (Hq)(t).$$

Suppose that $E_n\left(\varphi;W_2^m\right)=\inf_{q\in T_n}\left\|\varphi(\cdot)-q_n(\cdot)\right\|_{W_2^m(\gamma_0)}$ – is the best approximation of the function $\varphi\in W_2^m\left(\gamma_0\right)$ by polynomials T_n, where T_n – is the set of polynomials of the form $\sum_{k=-n}^{n}\alpha_k t^k$, $\alpha_k\in C$.

Theorem 5. The sequence of operators $\{H_n\}$ strongly converges to the operator H, and, for any $\varphi\in W_2^1\left(\gamma_0\right)$ the following estimate holds:

$$\left\|H\varphi-H_n\varphi\right\|_{L_2(\gamma_0)}\le 2E_n\left(\varphi;W_2^1\right).$$

Now consider the sequence of operators

$$\left(H_n^{(3)}\varphi\right)(t)=\left(H_nS_nH_n\right)\varphi(t),$$

$$\left(H_n^{(m+1)}\varphi\right)(t)=\left(H_nS_nH_n^{(m)}\right)\varphi(t),\quad m=3,4,5,\dots,$$

where

$$\left(S_n\varphi\right)(t)=\frac{1}{\pi i}\sum_{k=0}^{n-1}\frac{\varphi\left(\tau_{2k+1}^{(t)}\right)}{\tau_{2k+1}^{(t)}-t}\cdot\Delta\tau_{2k+1}^{(t)},\qquad t\in\gamma_0,\ n=1,2,..$$

Theorem 6. The operators $H_n^{(m)}$, $n=1,2,..$ is bounded from the space $W_2^{m-1}\left(\gamma_0\right)$ into the space $L_2\left(\gamma_0\right)$, then

$$\left\|H_n^{(m)}\right\|_{W_2^{(m-1)}(\gamma_0)\to L_2(\gamma_0)}\le 1,$$

and for $n\ge m-1$ for any algebraic polynomial $q(t)=\sum_{k=-n+m-1}^{n-m+1}q_k t^k$ with degree less than $n-m+1$ the following relation holds:

$$\left(H_n^{(m)}q\right)(t)=\left(H^{(m)}q\right)(t).$$

Theorem 7. The sequence of operators $\left\{H_n^{(m)}\right\}_{n=1}^{\infty}$ strongly converges to the operator $H^{(m)}$, and, for any $\varphi\in W_2^{m-1}\left(\gamma_0\right)$ the following estimate holds: $\left\|H^{(m)}\varphi-H_n^{(m)}\varphi\right\|_{L_2(\gamma_0)}\le 2E_{n-m+1}\left(\varphi;W_2^{m-1}\right)$.

Let $\Lambda_\alpha(\gamma_0)$, $0<\alpha\le 1$ be the space of Hölder continuous functions with exponent α in γ_0 i.e. the space of the functions which satisfies the following condition

$$\exists M>0 \quad \forall t_1,\, t_2\in\gamma_0: \ \ |\varphi(t_1)-\varphi(t_2)|\le M\cdot|t_1-t_2|^\alpha,$$

with the norm

$$\|\varphi\|_\alpha=\|\varphi\|_\infty+h(\varphi;\alpha),$$

where

$$\|\varphi\|_\infty=\max_{t\in\gamma_0}|\varphi(t)|, \quad h(\varphi;\alpha)=\sup\left\{\frac{|\varphi(t_1)-\varphi(t_2)|}{|t_1-t_2|}: t_1,t_2\in\gamma_0,\, t_1\ne t_2\right\}.$$

From the Theorem 3 it follows that, if $\varphi\in\Lambda_\alpha(\gamma_0)$ then hypersingular integral (4) exists for all $t\in\gamma_0$, where $\lambda<\alpha\le 1$.

In 1.3 the approximations of hypersingular integral operator with Cauchy kernel are given :

$$\left(H^{(\lambda)}\varphi\right)(t)=\frac{1}{\pi i}\int_{\gamma_0}\frac{\varphi(\tau)}{|\tau-t|^{1+\lambda}}d\tau\,,\ 0\le\lambda<1,$$

in the space $\Lambda_\alpha(\gamma_0)$, where $\varphi\in\Lambda_\alpha(\gamma_0)$, $\lambda<\alpha\le 1$ and is obtained an appropriate estimate of the convergence.

The second chapter describes the approximations of hypersingular integral operator with Hilbert kernel in the space of the square-integrable functions and in the Hölder spaces. The main results of this chapter were published in the following publications of the author [7,12, 13].

Consider the integral

$$\int_0^{2\pi}\csc^2\frac{\tau-t}{2}\varphi(\tau)d\tau\,, \quad t\in T_0=[0,2\pi], \qquad (5)$$

where $\varphi(t)$ is Lebesgue integrable on T_0 and 2π -periodic function.

Therefore, using the idea of Hadamard , we will define the integral (5) as follows:

Definition 2. If a finite limit

$$\lim_{\varepsilon\to 0+}\left(\int_{t-\pi}^{t-\varepsilon}\csc^2\frac{\tau-t}{2}\varphi(\tau)d\tau+\int_{t+\varepsilon}^{t+\pi}\csc^2\frac{\tau-t}{2}\varphi(\tau)d\tau-\frac{8\varphi(t)}{\varepsilon}\right),$$

exist, then the value of this limit is referred to as the hypersingular Hilbert integral of the function $\csc^2\frac{\tau-t}{2}\varphi(\tau)$ on T_0, and is denoted by $\int_0^{2\pi}\csc^2\frac{\tau-t}{2}\varphi(\tau)d\tau$.

If according to the definition we will calculate the hypersingular integral $\int_0^{2\pi}\csc^2\frac{\tau-t}{2}d\tau$, where $t\in T_0$, then we get

$$\int_0^{2\pi}\csc^2\frac{\tau-t}{2}d\tau=0. \qquad (6)$$

From the equality (6) follows that existence of hypersingular integral $\int_0^{2\pi}\csc^2\frac{\tau-t}{2}\varphi(\tau)d\tau$ is equivalent to existence of the integral $\int_0^{2\pi}\csc^2\frac{\tau-t}{2}[\varphi(\tau)-\varphi(t)]d\tau$ in the sense of the Cauchy principal value, and the following equation holds:

$$\int_0^{2\pi}\csc^2\frac{\tau-t}{2}\varphi(\tau)d\tau=\int_0^{2\pi}\csc^2\frac{\tau-t}{2}[\varphi(\tau)-\varphi(t)]d\tau,$$

where the integral standing in the right hand side is understood in the sense of the Cauchy principal value.

In 2.1 the hypersingular integral (5) is investigated as well as the integrals of the form

$$\int_0^{2\pi}\csc^m\frac{\tau-t}{2}\varphi(\tau)d\tau,\quad m\geq 3,\quad m\in N,\quad t\in T_0, \qquad (7)$$

$$\int_0^{2\pi}\left|\csc\frac{\tau-t}{2}\right|^{m+\lambda}\varphi(\tau)d\tau,\quad m\in N,\ \lambda\in[0,1),\ t\in T_0, \qquad (8)$$

where the function $\varphi(t)$ is Lebesgue integrable on T_0.

Theorem 8. If the 2π-periodic function $\varphi(t)$ is absolutely continuous on T_0, then the hypersingular Hilbert integral (5) exists for almost all $t \in T_0$, and the following integration by parts formula holds:

$$\int_0^{2\pi} \csc^2 \frac{\tau - t}{2} \varphi(\tau) d\tau = 2 \int_0^{2\pi} ctg \frac{\tau - t}{2} \varphi'(\tau) d\tau .$$

Let $L_2 = L_2(T_0)$ be the space of the functions square-integrable on T_0 with the norm

$$\|\varphi\|_{L_2} = \left(\frac{1}{2\pi} \int_0^{2\pi} |\varphi(\tau)|^2 \, d\tau \right)^{\frac{1}{2}},$$

and let $W_2^m(T_0)$ – be the space of 2π-periodic functions having $(m-1)$th order absolutely continuous derivative and whose mth order derivative belongs to $L_2(T_0)$, with the norm

$$\|\varphi\|_{W_2^m(T_0)} = \|\varphi\|_{L_2(T_0)} + \sum_{k=1}^{m} \left\|\varphi^{(k)}\right\|_{L_2(T_0)} .$$

Since the hypersingular integral operator with Hilbert kernel is bounded in the space $L_2(T_0)$, from the Theorem 8 it follows that, the hypersingular integral operator with Hilbert kernel

$$(\widetilde{H}\varphi)(t) = \frac{1}{4\pi} \int_0^{2\pi} \csc^2 \frac{\tau - t}{2} \varphi(\tau) d\tau , \; t \in T_0$$

is bounded from the space $W_2^1(T_0)$ into the space $L_2(T_0)$, then

$$\left\|\widetilde{H}\right\|_{W_2^1(T_0) \to L_2(T_0)} \le 1 .$$

In 2.2 the approximations of hypersingular integral operator $\widetilde{H}$ with Hilbert kernel are given in the space $L_2(T_0)$ and is described that these approximations preserve the main properties of the hypersingular integral operator and is obtained an appropriate estimate of the convergence.

Consider the sequence of operators

$$\left(\widetilde{H}_n\varphi\right)(t)=\frac{1}{2n}\sum_{k=0}^{n-1}\csc^2\frac{\pi(2k+1)}{2n}\left(\varphi\left(t+\frac{\pi(2k+1)}{n}\right)-\varphi(t)\right),\quad t\in T_0,$$
$$n=1,2,...$$

Theorem 9. The operators $\widetilde{H}_n, n=1,2,...$ is bounded from the space $W_2^1(T_0)$ on the space $L_2(T_0)$, then

$$\left\|\widetilde{H}_n\right\|_{W_2^1(T_0)\to L_2(T_0)}\le 1,$$

and for any trigonometric polynomial $q(t)=\sum_{k=-n}^{n}q_k e^{ikt}$ with degree less than n the following relation holds

$$\left(\widetilde{H}_n q\right)(t)=\left(\widetilde{H}q\right)(t).$$

Suppose that $E_n\left(\varphi;W_2^1\right)=\inf_{q\in T_n}\left\|\varphi(\cdot)-q_n(\cdot)\right\|_{W_2^1(T_0)}$ – is the best approximation of the function $\varphi\in W_2^1(T_0)$ by polynomials from T_n, where T_n– is the set of trigonometric polynomials of the form $\sum_{k=-n}^{n}\alpha_k e^{ikt}$, $\alpha_k\in C$.

Theorem 10. The sequence of operators $\left\{\widetilde{H}_n\right\}$ strongly converges to the operator $\widetilde{H}$, and, for any $\varphi\in W_2^1(T_0)$ the following estimate holds:

$$\left\|\widetilde{H}\varphi-\widetilde{H}_n\varphi\right\|_{L_2(T_0)}\le 2E_n\left(\varphi;W_2^1\right).$$

Let $\Lambda_\alpha(T_0)$, $0<\alpha\le 1$ be the space of 2π-periodic, Hölder continuous functions with exponent α on the real axis. Note that, if $\varphi\in\Lambda_\alpha(T_0)$, $0\le\lambda<\alpha\le 1$, then hypersingular integral $\int_0^{2\pi}\left|\csc\frac{\tau-t}{2}\right|^{\lambda+1}\varphi(\tau)d\tau$ exists for all $t\in T_0$.

In 2.3 the approximations of hypersingular integral operator with Hilbert kernel are given:

$$\left(\widetilde{H}^{(\lambda)}\varphi\right)(t)=\frac{1}{4\pi}\int_0^{2\pi}\left|\csc\frac{\tau-t}{2}\right|^{1+\lambda}\varphi(\tau)d\tau\,,\ 0\le\lambda<1,$$

in the space $\Lambda_\alpha(T_0)$, where $\varphi\in\Lambda_\alpha(T_0)$, $\lambda<\alpha\le 1$, and is obtained an appropriate estimate of the convergence.

The third chapter is devoted to the constructive solution of hypersingular integral equations with Hilbert kernel and Cauchy kernel in the space of the square-integrable functions. The main results of this chapter were published in the following publications of the author [6, 8, 9, 12].

In 3.1 the constructive solution of hypersingular integral equations with Cauchy kernel is given. At First the simple hypersingular integral equation of the first kind with Cauchy kernel is considered:

$$(H\varphi)(t)=f(t),\ \ t\in\gamma_0, \tag{9}$$

where $f\in L_2(\gamma_0)$. From the theorem 1 follows that, the equation (9) is solvable only if

$$\int_{\gamma_0} f(\tau)d\tau=0\,. \tag{10}$$

If the condition (10) is satisfies, then the equation (9) has infinitely many solutions in the general form

$$\varphi^*(t)=d_0-\sum_{k=-\infty}^{-2}\frac{c_k(f)}{k+1}t^{k+1}+\sum_{k=0}^{+\infty}\frac{c_k(f)}{k+1}t^{k+1}\in W_2^1(\gamma_0), \tag{11}$$

where $c_k(f)=\dfrac{1}{2\pi i}\displaystyle\int_{\gamma_0}\tau^{-k-1}f(\tau)d\tau$ - are Fourier's coefficients of the function $f\in L_2(\gamma_0)$ and d_0 - is a constant.

Therefore if we consider the following equation

$$(H\varphi)(t)=f(t)-\frac{1}{2\pi it}\int_{\gamma_0} f(\tau)d\tau\,,\ t\in\gamma_0, \tag{12}$$

$$\frac{1}{2\pi i}\int_{\gamma_0}\frac{\varphi(\tau)}{\tau}d\tau = d_0 \tag{13}$$

then the equation (12) - (13) is unique solvable for any $(f;d_0)\in L_2(\gamma_0)\times C$ and the solution of (12) - (13) is the function $\varphi^*(t)$ defined by (11).

We will look for approximate solution of the equation (12) - (13) in the form $\sum_{k=0}^{2n-1}\alpha_k^{(n)}(t)f\left(\tau_k^{(t)}\right)$, where $\alpha_k^{(n)}(t)$, $k=\overline{0,2n-1}$ -are continuous functions, $n\in N$.

Therefore consider the following equation

$$(H_n\varphi)(t) = f(t)\ ,\ t\in\gamma_0, \tag{14}$$

where $f\in L_2(\gamma_0)$. For any $t\in\gamma_0$ if the following equality holds

$$\sum_{p=0}^{2n-1}(H_n\varphi)(\tau_p^{(t)})\cdot\tau_p^{(t)} = 0$$

then we obtain that, the equation (14) is solvable only if

$$\sum_{p=0}^{2n-1}f(\tau_p^{(t)})\cdot\tau_p^{(t)} = 0. \tag{15}$$

If the condition (15) is satisfies, then the equation (14) has infinitely many solutions in the form

$$\sum_{k=-\infty}^{+\infty}d_k t^{2kn} + \sum_{k=-\infty}^{+\infty}\frac{c_k(f)}{\mu_{k+1}^{(n)}}t^{k+1}\in L_2(\gamma_0)\ . \tag{16}$$

From the above mentioned it follows that the equation

$$(H_n\varphi)(t) = f(t) - \frac{1}{2n}\sum_{p=0}^{2n-1}e^{p\theta i}f\left(\tau_p^{(t)}\right)\quad ,\ t\in\gamma_0\ , \tag{17}$$

$$\frac{1}{2n}\sum_{p=0}^{2n-1}\varphi\left(\tau_p^{(t)}\right) = d_0 \tag{18}$$

is unique solvable for any $(f;d_0)\in L_2(\gamma_0)\times C$ and the solution of (17) - (18) is the function $\varphi_n^*(t)$ defined by the following equality

$$\varphi_n^*(t) = d_0 + \sum_{\substack{k=-\infty \\ k \neq -1(\mathrm{mod}\, 2n)}}^{+\infty} \frac{c_k(f^*)}{\mu_{k+1}^{(n)}} t^{k+1} \in L_2(\gamma_0) .$$

Theorem 11. For any $(f; d_0) \in L_2(\gamma_0) \times C$ the equation (17)-(18) is unique solvable and the solution of this system $\varphi_n^*(t)$ converges to the solution of the equation (12)-(13) $\varphi^*(t)$ in the norm of $L_2(\gamma_0)$, and the following estimate is holds

$$\left\| \varphi_n^* - \varphi^* \right\|_{L_2(\gamma_0)} \leq \left(1 + \frac{1}{n} \right) E_n(f; L_2).$$

Now consider the hypersingular integral equation of the first kind with Cauchy kernel

$$(R\varphi)(t) = (H\varphi)(t) + (\mathcal{K}\varphi)(t) = f(t) \text{ , } t \in \gamma_0 \qquad (19)$$

where $(\mathcal{K}\varphi)(t) = \int_{\gamma_0} K(t,\tau)\varphi(\tau)d\tau$ and $K(t,\tau) = \frac{\partial}{\partial t} F(t,\tau)$-is continuous function. Since for any $\varphi \in W_2^1(\gamma_0)$ the following equalities hold:

$$\int_{\gamma_0} (H\varphi)(\tau)d\tau = 0, \quad \int_{\gamma_0} (\mathrm{K}\varphi)(\tau)d\tau = 0,$$

then we obtain that the equation (19) is solvable only if the condition (10) holds. Therefore we will consider the following equation:

$$(R\varphi)(t) = f(t) - \frac{1}{2\pi i t} \int_{\gamma_0} f(\tau)d\tau \text{ , } t \in \gamma_0, \qquad (20)$$

$$\frac{1}{2\pi i} \int_{\gamma_0} \frac{\varphi(\tau)}{\tau} d\tau = d_0. \qquad (21)$$

Theorem 12. If for any $(f; d_0) \in L_2(\gamma_0) \times C$ the equation (20)-(21) is unique solvable, then for large values of n the following equation

$$(H_n\varphi)(t)+(\mathsf{K}_n\varphi)(t)-\frac{1}{2n}\sum_{p=0}^{2n-1}e^{p\theta i}\sum_{k=0}^{2n-1}K(\tau_p^{(t)},\tau_k^{(t)})\times$$
$$\times\varphi(\tau_k^{(t)})\left(\frac{1}{2}\Delta\tau_k^{(t)}\right)=f(t)-\frac{1}{2n}\sum_{p=0}^{2n-1}e^{p\theta i}f(\tau_p^{(t)}) \tag{22}$$

$$\frac{1}{2n}\sum_{p=0}^{2n-1}\varphi(\tau_k^{(t)})=d_0, \tag{23}$$

where $(\mathsf{K}_n\varphi)(t)=\sum_{k=0}^{2n-1}K(t,\tau_k^{(t)})\varphi(\tau_k^{(t)})\left(\frac{1}{2}\Delta\tau_k^{(t)}\right)$, are also unique solvable for any $(f;d_0)\in L_2(\gamma_0)\times C$; $\varphi_n^*(t)$ of the equation (22)-(23) converge in the norm of the space $L_2(\gamma_0)$ to the solution $\varphi^*(t)$ of the equation (20)-(21) , and the following estimat holds

$$\left\|\varphi_n^*-\varphi^*\right\|_{L_2(\gamma_0)}\le const\cdot\Big\{E_n(f;L_2)+E_n(K\varphi^*;L_2)+4\pi\|K\|_\infty E_{n-1}(\varphi^*;L_2)+$$
$$+4\pi E_{n-1}(K)\Big[E_{n-1}(\varphi^*;L_2)+\left\|\varphi^*\right\|_{L_2(\gamma_0)}\Big]\Big\}.$$

In 3.2 the constructive solution of hypersingular integral equations with Hilbert kernel is given. First the simple hypersingular integral equation of the first kind with Hilbert kernel is considered

$$(\widetilde{H}\varphi)(t)=f(t),\quad t\in T_0=[0,2\pi], \tag{24}$$

where $f\in L_2(T_0)$. From the theorem 8 follows that, the equation (24) is solvable only if

$$\int_0^{2\pi}f(\tau)d\tau=0\,. \tag{25}$$

If the condition (25) is satisfies then the equation (24) has infinitely many solutions in the general form

$$\varphi^*(t)=d_0-\sum_{k\in z\backslash\{0\}}\frac{c_k(f)}{|k|}e^{ikt}\in W_2^1(T_0), \tag{26}$$

where $c_k(f)=\frac{1}{2\pi}\int_0^{2\pi} e^{-ik\tau} f(\tau)d\tau$ - are Fourier's coefficients of the function $f \in L_2(T_0)$ and d_0 - is a constant.

From the above mentioned it follows that the equation

$$\left(\widetilde{H}\varphi\right)(t)=f(t)-\frac{1}{2\pi}\int_0^{2\pi} f(\tau)d\tau\,,\ t\in T_0\,, \tag{27}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi(\tau)d\tau = d_0 \tag{28}$$

is unique solvable for any $(f;d_0)\in L_2(T_0)\times C$ and the solution of (27)-(28) is the function $\varphi^*(t)$, defined by (26).

We will look for approximate solution of the equation (27)-(28) in the form $\sum_{k=0}^{2n-1}\alpha_k^{(n)}(t) f\left(t+\frac{\pi k}{n}\right)$, where $\alpha_k^{(n)}(t)$, $k=\overline{0,2n-1}$ - are continuous functions, $n\in N$.Therefore consider the following equation

$$\left(\widetilde{H}_n\varphi\right)(t)=f(t)\ ,\ t\in T_0\,, \tag{29}$$

where $f\in L_2(T_0)$. For any $t\in T_0$ if the following equality holds

$$\sum_{p=0}^{2n-1}\left(\widetilde{H}_n\varphi\right)(t+\frac{\pi k}{n})=0$$

then we obtain that, the equation (29) is solvable only if

$$\sum_{p=0}^{2n-1} f\left(t+\frac{\pi p}{n}\right)=0\,. \tag{30}$$

If the condition (30) is satisfies, then the equation (29) has infinitely many solutions in the form

$$\sum_{k=-\infty}^{+\infty} d_k e^{2knit} + \sum_{\substack{k=-\infty\\ k\neq 0(\mathrm{mod}\,2n)}}^{+\infty}\frac{c_k(f)}{\widetilde{\mu}_k^{(n)}}e^{ikt}\in L_2(T_0)\,. \tag{31}$$

From the above mentioned it follows that the equation

$$\left(\widetilde{H}_n\varphi\right)(t)=f(t)-\frac{1}{2n}\sum_{p=0}^{2n-1}f\left(t+\frac{\pi p}{n}\right), \qquad (32)$$

$$\frac{1}{2n}\sum_{k=0}^{2n-1}\varphi\left(t+\frac{\pi k}{n}\right)=d_0 \qquad (33)$$

is unique solvable for any $(f;d_0)\in L_2(T_0)\times C$ and the solution of (32)-(33) is the function $\varphi_n^*(t)$, defined by the following equality

$$\varphi_n^*(t)=d_0+\sum_{\substack{k=-\infty \\ k\neq 0(\mathrm{mod}\,2n)}}^{+\infty}\frac{c_k(f)}{\widetilde{\mu}_k^{(n)}}e^{ikt}\in L_2(T_0)\ .$$

Theorem 13. For any $(f;d_0)\in L_2(T_0)\times C$ the equation (32)-(33) is unique solvable and the solution of this system $\varphi_n^*(t)$ converges to the solution $\varphi^*(t)$ of the equation (27)-(28) in the norm of $L_2(T_0)$, and the following estimate is holds

$$\left\|\varphi_n^*-\varphi^*\right\|_{L_2(T_0)}\leq E_n(f;L_2).$$

Now consider the hypersingular integral equation of the first kind with Hilbert kernel

$$\left(\widetilde{R}\varphi\right)(t)=\left(\widetilde{H}\varphi\right)(t)+\left(\widetilde{\mathcal{K}}\varphi\right)(t)=f(t)\ ,\ \ t\in T_0, \qquad (34)$$

where $\left(\widetilde{\mathcal{K}}\varphi\right)(t)=\frac{1}{2\pi}\int_0^{2\pi}\widetilde{K}(t,\tau)\varphi(t)d\tau$, and $\widetilde{K}(t,\tau)=\frac{\partial}{\partial t}\widetilde{F}(t,\tau)$- is continuous function. Since for any $\varphi\in W_2^1(\gamma_0)$ the following equalities hold:

$$\int_0^{2\pi}\left(\widetilde{H}\varphi\right)(\tau)d\tau=0\ ,\quad \int_0^{2\pi}\left(\widetilde{\mathrm{K}}\varphi\right)(\tau)d\tau=0\ ,$$

then we obtain that the equation (34) is solvable only if the condition (25) holds. Therefore we will consider the following equation:

$$\left(\widetilde{R}\varphi\right)(t) = f(t) - \frac{1}{2n}\sum_{p=0}^{2n-1} f\left(t + \frac{\pi p}{n}\right), \tag{35}$$

$$\frac{1}{2\pi}\sum_{k=0}^{2n-1}\varphi\left(t + \frac{\pi k}{n}\right) = d_0 . \tag{36}$$

Theorem 14. If for any $(f;d_0) \in L_2(T_0) \times C$ the equation (35)-(36) is unique solvable, then for large values of n the following equation

$$\left(\widetilde{H}_n\varphi\right)(t) + \left(\widetilde{\mathsf{K}}_n\varphi\right)(t) - \frac{1}{4n^2}\sum_{k=0}^{2n-1}\sum_{p=0}^{2n-1}\widetilde{K}\left(t + \frac{\pi p}{n}, t + \frac{\pi k}{n}\right)\varphi\left(t + \frac{\pi k}{n}\right) =$$

$$= f(t) - \frac{1}{2n}\sum_{k=0}^{2n-1} f\left(t + \frac{\pi k}{n}\right), \tag{37}$$

$$\frac{1}{2\pi}\sum_{p=0}^{2n-1}\varphi\left(t + \frac{\pi p}{n}\right) = d_0 , \tag{38}$$

where $\left(\widetilde{\mathsf{K}}_n\varphi\right)(t) = \frac{1}{2n}\sum_{k=0}^{2n-1}\widetilde{K}\left(t, t + \frac{\pi k}{n}\right)\varphi\left(t + \frac{\pi k}{n}\right)$, are also unique solvable for any $(f;d_0) \in L_2(T_0) \times C$; the solution $\varphi_n^*(t)$ of the equation (37)-(38) converge in the norm of the space $L_2(T_0)$ to the solution $\varphi^*(t)$ of the equation (35)-(36), and the following estimate holds

$$\left\|\varphi_n^* - \varphi^*\right\|_{L_2(T_0)} \le const \cdot \left\{E_n(f;L_2) + E_n\left(\mathcal{K}\varphi^*;L_2\right) + \right.$$

$$\left. + 2\left\|\widetilde{K}\right\|_\infty E_{n-1}\left(\varphi^*;L_2\right) + 2E_{n-1}\left(\widetilde{K}\right)\left[E_{n-1}\left(\varphi^*;L_2\right) + \left\|\varphi^*\right\|_{L_2(T_0)}\right]\right\}.$$

In 3.3 the results of numerical examples confirming the effectiveness of the proposed method are given.

CONCLUSION

In the present dissertation work the following main results were obtained:

- the error estimates of the approximation of hypersingular integral operators with Cauchy kernel and with Hilbert kernel in the space of the square-integrable functions and in the Hölder spaces are obtained;
- the constructive method for solution of hypersingular integral equations with Hilbert kernel and with Cauchy kernel in the space of the square-integrable functions is presented and justificated;
- the application of given constructive method to the 2D inner Neumann problem for Laplace equation is described and the results of numerical examples confirming the efficiency of the proposed method are given;

The main results of the dissertation have been published in the following papers:

1.Амрахова, А.Ф., Гаджиева, Ч.А. Об аппроксимации гиперсингулярного интегрального оператора на окружности // -Баку: Вестник БГУ, серия физ.-мат.наук, -2015, 4, -с.43-50.

2.Амрахова, А.Ф., Гаджиева, Ч.А. Аппроксимация гиперсингулярного интегрального оператора на окружности // Ə.Həbibzadənin 100 illik yubileyinə həsr olunmuş "Funksional analiz və onun tətbiqləri" adlı respublika elmi konfransının materialları, -Bakı, -2016, -səh.120-121.

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9. Gadjieva, Ch.A. The approximate solution of hypersingular integral equations of the first kind // M.L.Rəsulovun 100 illik yubileyinə həsr olunmuş "Nəzəri və tətbiqi riyaziyyatın aktual

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12. Gadjieva, Ch.A. The Approximate solution of hypersingular integral equations with Hilbert kernel of the first kind // 1st İnternational Science and Engineering Coference Proceedings, -Baku, -2018, -p.102-104.
13. Gadjieva, Ch., Approximation of Hypersingular integral operators with Hilbert kernel on Hölder spaces \\ Abstarct book Operators in general Morrey-Type Spaces and Applications (OMTSA 2019), - Turkey, Kutahya: Kutahya Dumlupinar University, -16-20 July, -2019, -p.98-100.

The defense will be held on 31 May 2021 at 11^{00} at the meeting of the Dissertation council ED 1.04 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at the Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan.

Address: AZ 1141, Baku, B.Vahabzadeh, 9.

Dissertation is accessible at the Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan Library

Electronic versions of dissertation and its abstract are available on the official website of the Institute of Mathematics and Mechanics of National Academy of Sciences of Azerbaijan.

Abstract was sent to the required addresses on 26 April 2021.

Signed for print: 26.04.2021
Paper format: 60x84 1/16
Volume: 80000
Number of hard copies: 20

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