

REPUBLIC OF AZERBAIJAN

On the rights of the manuscript

ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

**LOCAL AND GLOBAL BIFURCATION OF SOLUTIONS TO
SOME NONLINEAR EIGENVALUE PROBLEMS**

Specialty: 1201.01 – Analysis and functional analysis

Field of science: Mathematics

Applicant: **Gunay Mammad kizi Mammadova**

Baku – 2025

The work was performed at the department of Mathematical Analysis of Baku State University.

Scientific supervisor: doctor of math. sci., prof.
Ziyatkhan Seyfaddin oglu Aliyev

Official opponents: doctor of phys.-math.sci.,prof



Hamdulla İsrafil oglu Aslanov

doctor of math. sci., prof.

Elnur Hasan oglu Xalilov

doctor of math. sci.,

Elchin Camal oglu İbadov

One time Dissertation Council ED 2.17 of the Higher Attestation Commission under the President of the Republic of Azerbaijan, operating on the basis of Baku State University

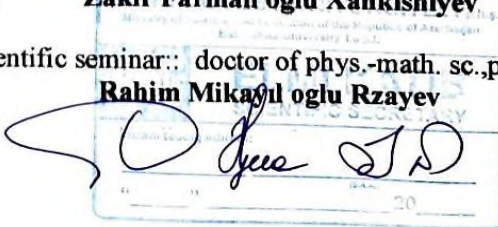
Chairman of the

Dissertation council: *[Signature]* academian, doctor of phys.-math. sc.,prof.
Magomed Farman oglu Mekhtiyev

Scientific secretary

of the Dissertation council: *[Signature]* candidate of phys.-math.sc.,ass.prof.
Zakir Farman oglu Xankishiyev

Chairman of the scientific seminar: *[Signature]* doctor of phys.-math. sc.,prof.
Rahim Mikayıl oglu Rzayev



GENERAL CHARACTERISTICS OF THE WORK

Relevance of the topic and degree of processing. Bifurcation theory is a mathematical science and the foundations of the mathematical theory of bifurcations were created by A. Poincaré and A. M. Lyapunov at the beginning of the 20th century, and then developed by many mathematicians. The behavior of nonlinear mappings or nonlinear differential equations, in particular the regimes established over time, may depend on some parameters. It turns out that with a slow change in the parameter, qualitative changes in the established modes can occur. The study of such rearrangements with variations in parameters in mappings and differential equations is the subject of bifurcation theory. Bifurcation theory finds applications in various sciences, from physics and chemistry to biology and sociology.

In the modern development of bifurcation theory, the bifurcation of nonlinear eigenvalue problems plays an important role. Many specific problems in mechanics, physics and biology lead to problems of this type. The main results of the theory of bifurcation of nonlinear eigenvalue problems are the local bifurcation theorem of M.A. Krasnoselski¹ and the global bifurcation theorem of P.H. Rabinowitz² in the presence of the Frechet derivative at zero of nonlinear perturbations. Based on these results, fairly general results have now been obtained for a wide class of nonlinear eigenvalue problems, which have been included in a systematic presentation of bifurcation theory.

Later, in the papers of A.P. Makhmudov and Z.S. Aliyev³, R. Chiappinelli⁴ and G. Przybycin⁵ investigated the bifurcation of

¹ Krasnoselski, M.A. Topological methods in the theory of nonlinear integral equations /, M.A. Krasnoselski. – New York: Macmillan, – 1965, – 395

² Rabinowitz, P.H. Some global results for nonlinear eigenvalue problems // Journal of Functional Analysis, – 1971, v.7, no. 3, – p. 487-513.

³ Makhmudov, A. P., Aliyev, Z. S. Global bifurcation of solutions of some nonlinearizable eigenvalue problems., – 1989. v. 25, no. 1, – p. 71–76

⁴Chiappinelli, R. Remarks on Krasnoselskii bifurcation theorem// Commentationes Mathematicae Universitatis Carolinae, – 1989. v. 30, no. 2, – p. 236-241.

solutions of nonlinear eigenvalue problems in Banach and Hilbert spaces, without the assumption of Fréchet differentiability of nonlinear perturbations. They studied the structure of bifurcation points and established that bifurcation occurs from intervals of the line of trivial solutions. However, only in the paper of A.P. Makhmudov and Z.S. Aliyev³ was shown the existence of global continua of solutions emanating from these bifurcation intervals.

For the study of local and global bifurcations of solutions of nonlinear eigenvalue problems for ordinary differential equations, the oscillatory properties of the corresponding linear problems play an important role. The oscillatory properties of eigenfunctions of linear ordinary differential operators have been studied by many authors using various methods. Local and global bifurcation of solutions of nonlinear Sturm-Liouville and Dirac problems, and nonlinear eigenvalue problems for ordinary differential equations of fourth and higher orders with the use of oscillatory properties of eigenfunctions has been investigated in many works, where the existence of two families of unbounded continua of solutions that have usual oscillatory properties and bifurcating from points and intervals of the line of trivial solutions has been proven.

In the work of T.I. Allahverdiyev⁶ a nonlinear Sturm-Liouville problem with a spectral parameter in the boundary condition is considered, where the results of P.G. Rabinowitz² and H. Berestycki⁷ are developed. It should be noted that there is a gap in, this work since the fractional-linear function in the boundary condition is strictly increasing in each of the intervals not containing zero of the denominator, but is not strictly increasing on \mathbb{R} . Therefore, this author, using the method of H. Berestycki⁷, did not investigate the behavior of all connected components of the set of solutions that

⁵ Przybycin, J. Some theorems of Rabinowitz type for nonlinearizable eigenvalue problems // *Opuscula Mathematica*, – 2004, v. 24, № 1, – p. 115–121.

⁶ Allahverdiyev T.I. Study of some linear and nonlinear Sturm-Liouville problems with a spectral parameter in boundary conditions / *Diss. Cand. Phys.–Math. Sciences.* / – Baku, 1991, – 106 p.

⁷ Berestycki, H. On some nonlinear Sturm-Liouville problems // *Journal of Differential Equations*, – 1977. v.26, no. 3, – p. 375–390.

have oscillatory properties of the eigenfunctions of the corresponding linear problem and bifurcate from points and intervals of the line of trivial solutions. This is also due to the fact that there is a natural number N_1 such that the eigenfunctions of the linear Sturm-Liouville problem with a spectral parameter in the boundary condition with ordinal numbers N_1 and $N_1 + 1$ have the same number of zeros.

The half-linear and half-linearizable Sturm-Liouville problems were completely investigated by H. Berestitsky⁷. The oscillatory properties of half-eigenfunctions of half-linear Sturm-Liouville problems with a spectral parameter in the boundary condition were studied by P.J. Browne⁸, using angular functions. But he failed to accurately determine the ordinal numbers of half-eigenfunctions that have the same number of zeros. Therefore, it was not possible to fully study the bifurcation of solutions of nonlinearizable perturbations of the semilinear Sturm-Liouville problem with a spectral parameter in the boundary condition.

Thus, based on the above reasoning, it can be said that the global bifurcation of solutions to nonlinear eigenvalue problems in Banach and Hilbert spaces with nonlinear nondifferentiable operators mapping bounded sets into bounded sets has not been studied in detail. Have not been studied completely also behavior of connected components of the set of solutions of non-linearizable perturbations of linear and half-linear Sturm-Liouville problems with a spectral parameter in the boundary condition that have oscillatory properties of eigenfunctions and half-eigenfunctions of linear and half-linear problems, respectively, and bifurcating from points and intervals of the line of trivial solutions.

Object and subject of the study. The object of the study is nonlinear eigenvalue problems in Banach and Hilbert spaces with nondifferentiable operators, as well as nonlinearizable perturbations of linear and p-fda-linear Sturm-Liouville problems with a spectral parameter in the boundary condition. The subject of the study is the

⁸ Browne, P.J., A Puffer approach to half-linear Sturm-Liouville problems // Proc. Edinburgh Math. Soc., – 1998. v.41, no. 3, – p. 573–583.

behavior and structure of global continua branching from points and intervals of the line of trivial solutions.

Goal and tasks of the study. The main goal and tasks of the dissertation is to study the global bifurcation of solutions to nonlinearizable eigenvalue problems in Banach and Hilbert spaces; the behavior of connected components of the set of solutions to nonlinearizable perturbations of linear Sturm-Liouville problems with a spectral parameter in the boundary condition; the behavior of continua of solutions to linearizable perturbations of half-linear Sturm-Liouville problems with a spectral parameter in the boundary condition.

Investigation methods. The work uses methods of function theory and functional analysis, operator theory, topology, differential equations, spectral analysis, spectral theory of differential operators, nonlinear analysis and bifurcation theory.

Basic statements to be defended. The following main points are put forward for defense:

- study the global bifurcation of solutions to nonlinear eigenvalue problems without the assumption of differentiability of nonlinear perturbations;
- investigate the structure of the global connected component of the solution set bifurcating from an interval containing a simple eigenvalue of a linear problem;
- find bifurcation intervals of a nonlinear Sturm-Liouville problem with a spectral parameter in the boundary condition;
- show the existence of a pair of unbounded connected components of the set of solutions to the nonlinear Sturm-Liouville problem with a spectral parameter in the boundary condition branching from the bifurcation intervals and having oscillatory properties of the eigenfunctions of the linear problem;
- study the oscillatory properties of half-eigenfunctions of the half-linear Sturm-Liouville problem with a spectral parameter in the boundary condition;
- prove the existence of four families of unbounded continua of the half-linearizable Sturm-Liouville problem with a spectral parameter in the boundary condition, bifurcating from half-

eigenvalues and having oscillatory properties of semi-eigenfunctions of the corresponding half-linear problem.

Scientific novelty of the study. The dissertation obtained the following main results:

- a theorem on global bifurcation (from the interval of the line of trivial solutions containing an eigenvalue of odd multiplicity of a linear problem) of solutions of nonlinear eigenvalue problems without the assumption of differentiability of nonlinear perturbations was proved;

- the structure of the global connected component of the set of solutions, branching from the interval of the line of trivial solution containing a simple eigenvalue of the linear problem, is studied in detail;

- found bifurcation intervals of the nonlinear Sturm-Liouville problem with a spectral parameter in the boundary condition;

- it is proved the existence of a pair of unbounded connected components of the set of solutions to the nonlinear Sturm-Liouville problem with a spectral parameter in the boundary condition, branching from bifurcation intervals and possessing oscillatory properties of eigenfunctions of the linear problem;

- the oscillatory properties of half-eigenfunctions of the half-linear Sturm-Liouville problem with a spectral parameter in the boundary condition have been studied;

- it is shown the existence of a pair of unbounded continua of solutions of the semilinearizable Sturm-Liouville problem with a spectral parameter in the boundary condition, bifurcating from semi-eigenvalues and possessing oscillatory properties of semi-eigenfunctions of the corresponding half-linear problem.

Theoretical and practical value of the study. The results obtained in the dissertation are of a theoretical nature. They can be used in various questions of functional analysis, bifurcation theory, spectral theory of differential operators, in studying various problems of mechanics and physics.

Approbation and application. The results of the dissertation were reported at the seminars of the departments of "Mathematical Analysis" (headed by prof. S.S. Mirzoyev) and "Theory of Functions

and Functional Analysis" (headed by prof. A.A. Akhmedov) of Baku State University, at the seminars of the departments of "Functional Analysis" (headed by prof. G.I. Aslanov) and "Differential Equations" (headed by prof. A.B. Aliyev) of the IMM NAS of Azerbaijan, at the International Conference "Recent Achievements in Pure and Applied Mathematics" (ICRAPAM-2014, Antalya, Turkey, 2014), at the International Conference "Mathematical Analysis, Differential Equations and Their Applications" (MADEA-7, Baku, 2015), at the I International Scientific Conference of Young Scientists (Ganja, 2016), at the International Scientific Conference "Theoretical and Applied Problems of Mathematics" dedicated to 55th anniversary of Sumgait State University (Sumgait, 2017), at the international Voronezh winter mathematical school "Modern methods of function theory and related problems" (Voronezh, Russia, 2021).

The author's personal contribution consists of formulating the research objective. In addition, all obtained research results belong to the author.

Publications of the author. The author's publications in scientific journals recommended by the Higher Attestation Commission under the President of the Republic of Azerbaijan are 5 (including 2 WOS, 2 SCOPUS), conference materials are 4 (2 international conferences, 2 republican, 1 of which were held abroad).

The institution where the dissertation work was performed. The work was performed at the Department of Mathematical Analysis of Baku State University.

Structure and volume of the dissertation work (in signs, indicating the volume of each structural unit separately). The total volume of the dissertation is 140769 characters (title page – 332 characters, table of contents – 1441 characters, introduction – 33606 characters, 1st chapter – 35100 characters, 2nd chapter – 36500 characters, 3rd chapter – 32400 characters, conclusion – 1790 characters). The list of references consists of 80 titles.

THE MAIN CONTENT OF THE DISSERTATION

The dissertation consists of an introduction, three chapters, including 11 paragraphs, a conclusion and a list of references.

The introduction substantiates the relevance of the research subject and shows the degree of its development, formulates the goal and objectives of the research, provides scientific novelty, notes the theoretical and practical value of the research, and provides information on the testing of the work.

The first chapter consists of four sections and is devoted to the study of global bifurcation of solutions to nonlinearizable eigenvalue problems in Banach and Hilbert spaces.

Let E be the real Banach space with the norm $\|\cdot\|$, and $L: E \rightarrow E$ be the linear operator with the domain—линейный оператор с областью определения $D(L)$, which is dense in E .

In 1.1 consider the following nonlinear eigenvalue problem

$$Lu = \lambda u + F(\lambda, u) + G(\lambda, u), \quad (1)$$

where $\lambda \in R$ is an eigenvalue parameter, $F, G: R \times E \rightarrow E$ are continuous operators which maps bounded sets to bounded sets, $F(\lambda, 0) = 0$ for any $\lambda \in R$, and G satisfies the condition

$$G(\lambda, u) = o(\|u\|) \text{ as } \|u\| \rightarrow 0, \quad (2)$$

uniformly for $\lambda \in R$, for every bounded interval $\Lambda \subset R$.

The norm in $R \times E$ is defined as follows:

$$\|(\lambda, u)\| = \{|\lambda|^2 + \|u\|^2\}^{1/2}.$$

We denote by $B_\varepsilon(\lambda)$ the ball in $R \times E$ with center at point $(\lambda, 0)$ and radius $\varepsilon > 0$, and by B_ε the ball in E with center at point and radius ε .

The point $(\mu, 0)$ is called a bifurcation point of problem (1) with respect to the line of trivial solutions, or more briefly a bifurcation point, if for any $\varepsilon > 0$ there exists a nontrivial solution $(\lambda_\varepsilon, u_\varepsilon)$ of problem (1) such that $(\lambda_\varepsilon, u_\varepsilon) \in B_\varepsilon(\mu)$, i.e.

$$| |(\lambda_\varepsilon, u_\varepsilon) - (\mu, 0) | | < \varepsilon.$$

Section 1.1 provides the definition of an L -compact operator (if $L^{-1}: E \rightarrow E$ exists, this means that L^{-1} is compact) and the topological concept of the degree of coincidence of two operators

Throughout what follows we will assume that the operator L is a Fredholm operator of index zero and the identity operator $I: E \rightarrow E$ is compact. Then, each eigenvalue of L is isolated and has finite multiplicity, and the entire spectrum $\sigma(L)$ consists of such points.

The following statements about the existence of bifurcation points of problem (1) with $F \equiv \mathbf{0}$ hold.

Lemma 1. Let $F \equiv \mathbf{0}$. If $(\mu, \mathbf{0})$ is a bifurcation point of problem (1), then $\mu \in \sigma(L)$.

Theorem 1. Let $F \equiv \mathbf{0}$. Is $\mu \in \sigma(L)$ has an odd multiplicity, then $(\mu, \mathbf{0})$ is a bifurcation point of problem (1) and this point corresponds to a continuous branch of non-trivial solutions.

Let Y denote the closure of the set of non-trivial solutions to problem (1).

The following theorem shows that bifurcation is a global phenomenon, not a local one.

Теорема 2. Let $F \equiv \mathbf{0}$ and $\mu \in \sigma(L)$ has an odd multiplicity. Then there exists connected component Y_μ of the set Y , which contain the point $(\mu, \mathbf{0})$ and either (i) Y_μ is unbounded $\mathbf{R} \times E$, or (ii) Y_μ contain the point $(\bar{\mu}, \mathbf{0})$, where $\mu \neq \bar{\mu} \in \sigma(L)$.

Let $F \equiv \mathbf{0}$ and $\mu \in \sigma(L)$ be simple. Then Y_μ can be decomposed into two subcontinua Y_μ^+ and Y_μ^- according to the construction proposed by E. Dacer.

The following theorem holds.

Theorem 3. Either the sets Y_μ^+ and Y_μ^- are unbounded in $\mathbf{R} \times E$, or $Y_\mu^+ \cap Y_\mu^- \neq \{(\mu, \mathbf{0})\}$.

In 1.2, the local bifurcation of solutions to problem (1) is studied. For $r > \mathbf{0}$ we put

$$k(F; r) = \sup_{\|u\| \leq r, \lambda \in \mathbb{R}} \frac{\|F(\lambda, u)\|}{\|u\|}, \quad (3)$$

under the condition that it is finite, at least for small r . Since $k = k(F; r)$ is a non-decreasing function of the parameter r we also set

$$k_0(F) = \lim_{r \rightarrow 0} k(F; r). \quad (4)$$

For $\lambda \in \sigma(L)$ let

$$c(\lambda) = \inf\{\|Lu - \lambda u\| : u \in D(L), \|u\| = 1\} = \|(L - \lambda I)^{-1}\|. \quad (5)$$

Throughout what follows we will assume that μ is an eigenvalue of the operator L of odd multiplicity.

Theorem 4. Let there exist $\underline{\lambda}$ and $\bar{\lambda}$ such that

$$\underline{\lambda} < \mu < \bar{\lambda}, \sigma(L) \cap [\underline{\lambda}, \bar{\lambda}] = \{\mu\} \text{ and } \min\{c(\underline{\lambda}), c(\bar{\lambda})\} > k_0(F), \quad (6)$$

where $k_0(F)$ and $c(\lambda)$ are defined by formulas (4) and (5), respectively. Then for any sufficiently small $r > 0$ problem (1) has a solution (λ_r, u_r) such that $\lambda \in [\underline{\lambda}, \bar{\lambda}]$ and $\|u_r\| = r$.

By B we denote the set of bifurcation points of the nonlinear problem (1) with respect to the line of trivial solutions.

Corollary 1. $([\underline{\lambda}, \bar{\lambda}] \times \{0\}) \cap B \neq \emptyset$.

Suppose that H is a real Hilbert space and $L: D(L) \subset H \rightarrow H$ is a self-adjoint operator. Then the following theorem holds.

Theorem 5. Let $\mu \in \sigma(L)$ has odd multiplicity and the

distance from it to the spectrum $\sigma(L)$ is more than $2k_0(F)$, i.e.

$$\text{dist}(\mu, \sigma(L) \setminus \{\mu\}) > 2k_0(F). \quad (7)$$

Then there exist small numbers $\tilde{r}_0 > 0$ and $\varepsilon_0 > 0$ such that for

$$0 < r \leq r_0$$

any problem (1) has a solution (λ_r, u_r) satisfying relations $\|u_r\| = r$ and $\lambda \in [\mu - k(r) - \varepsilon_0, \mu + k(r) + \varepsilon_0]$, $k(r) = k(F; r)$.

Corollary 2. The following relation holds:

$$(B \cap ((\mu - k_0(F), \mu + k_0(F)) \times \{0\})) \subset [\mu - k_0(F), \mu + k_0(F)] \times \{0\}.$$

In 1.3 we study the global bifurcation of solutions to problem (1). By \tilde{D}_μ we denote the union of all components of the closure of the set of nontrivial solutions of (1) emanating from the bifurcation points $(\lambda, 0) \in B \cap (I_\mu \times \{0\})$, where $I_\mu = [\mu - k_0(F), \mu + k_0(F)]$.

Let $D_\mu = \tilde{D}_\mu \cup (I_\mu \times \{0\})$. Note that the set D_μ is connected in $\mathbf{R} \times E$, but the set \tilde{D}_μ may not be connected in $\mathbf{R} \times E$.

The following result on the global bifurcation of solutions of the nonlinear problem (1) generalizes the well-known theorem 1.3 of the work of P.G. Rabinowitz².

Theorem 6. Let $\mu \in \sigma(L)$ has odd multiplicity and condition (8) holds. Then connected component D_μ of the set Y , containing $I_\mu \times \{0\}$ either i) is unbounded in $\mathbf{R} \times E$ or ii) contains an interval $I_\nu \times \{0\}$, where $\mu \neq \nu \in \sigma(L)$.

This section also provides an application of the obtained results to the study of bifurcation of nonlinear eigenvalue problems for ordinary differential equations of the second order.

In 1.4 the structure of the global connected component D_μ of the set of solutions to problem (1) is studied. Suppose that in (1) the nonlinear operator F satisfies the condition: there exists the positive number M such that

$$\|F(\lambda, u)\| \leq M\|u\|, (\lambda, u) \in \mathbf{R} \times E, \|u\| \leq 1. \quad (8)$$

In this section we obtain stronger results on the bifurcation of solutions from the interval $I_\mu \times \{0\}$ of the nonlinear problem (1) in

the case when μ is a simple eigenvalue of the operator L .

Additionally, suppose that

(A) for any sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty} \subset \mathbb{R} \times E$ that converges to $(\lambda, 0)$ there exists the subsequence $\{(\lambda_{n_k}, u_{n_k})\}_{n=1}^{\infty}$ and the number $m \in [-M, M]$ such that

$$\frac{F(\lambda_{n_k}, u_{n_k}) - mu_{n_k}}{\|u_{n_k}\|} \rightarrow 0 \quad \text{in } E \text{ as } k \rightarrow \infty. \quad (9)$$

Consequently, the set $A = \{m \in [-M, M]: \text{there exists a sequence } \{(\lambda_n, u_n)\}_{n=1}^{\infty} \text{ converging to } (\lambda, 0) (\lambda \in \Lambda) \text{ such that relation (9) holds}\}$ is nonempty. This condition is satisfied, in particular, when F is Frechet differentiable at zero and $F'(0) = \tilde{m}I$. Then it is obvious that $A = \{\tilde{m}\}$.

Lemma 2. Let $\mu \in \sigma(L)$ has odd multiplicity, the relation

$$\text{dist}(\mu, \sigma(L) \setminus \{\mu\}) > 2M, \quad (10)$$

holds and satisfies condition (A). Then

$$B \cap (I_\mu \times \{0\}) = \{(\mu + m, 0): m \in A\}.$$

Now suppose that $\mu \in \sigma(L)$ is simple. Let $\vartheta \in E \setminus \{0\}$ and $\varphi \in E^*$ such that

$$L\vartheta = \mu\vartheta, \quad L^*\varphi = \mu\varphi,$$

where L^* is the operator adjoint to the operator L , $\varphi(\vartheta) = 1$ and $\varphi(\vartheta)$ is the value of the functional φ at the point ϑ . We denote $E_1 = \{u \in E: \varphi(u) = 0\}$. Then $E = R \oplus E_1$ and each vector $u \in E$ can be represented as $u = \alpha\vartheta + w$, where $\alpha = \varphi(u)$ and $w \in E_1$.

For each $\xi, \eta \in (0, 1)$ we define a set

$$Q_{\xi, \eta} = \left\{ (\lambda, u) \in \mathbb{R} \times E : \text{dist}\{\lambda, I_\mu\} < \xi, |\varphi(u)| > \eta\|u\| \right\}.$$

It is obvious that $Q_{\xi, \eta}$ is an open subset of $\mathbb{R} \times E$ consists of two disjoint subsets

$$Q_{\xi, \eta}^+ = \left\{ (\lambda, u) \in \mathbb{R} \times E : \text{dist}\{\lambda, I_\mu\} < \xi, \varphi(u) > \eta\|u\| \right\}.$$

$$Q_{\xi, \eta}^- = \left\{ (\lambda, u) \in \mathbb{R} \times E : \text{dist} \{ \lambda, I_\mu \} < \xi, \varphi(u) < -\eta \|u\| \right\}.$$

Lemma 3. Let $\mu \in \sigma(L)$ be simple, condition (10) holds and F satisfies condition (A). Then there exists $\tau_0 > 0$ such that for all positive $\tau < \tau_0$ the relation

holds. Moreover, if then $u = \alpha \vartheta + w$, where $|\alpha| > \eta \|u\|$ and $w = o(|\alpha|)$ as $\alpha \rightarrow 0$.

Corollary 3. Let the conditions of Lemma 3 be satisfied. Then for each $\lambda \in \{\mu - m : m \in A\}$ the set $D_{\mu, \lambda} = \tilde{D}_{\mu, \lambda} \cup (I_\mu \times \{0\})$ is decomposed into two subcontinua $D_{\mu, \lambda}^+$ and $D_{\mu, \lambda}^-$ such that $(I_\mu \times \{0\}) \subset D_{\mu, \lambda}^\pm$ and

$$D_{\mu, \lambda}^+ \subset Q_{\xi, \eta}^+ \cup (I_\mu \times \{0\}), \quad D_{\mu, \lambda}^- \subset Q_{\xi, \eta}^- \cup (I_\mu \times \{0\}).$$

More over, if $(\lambda, u) \in D_{\mu, \lambda}^+ (D_{\mu, \lambda}^-) \cap (B_{(\mu, \tau)}(I_\mu \times \{0\}), \tau < \tau_0$, then $u = \alpha \vartheta + w$, where $w = o(|\alpha|)$ as $\alpha \rightarrow 0$.

By virtue of Corollary 3, the set D_μ can be divided into two subcontinua D_μ^+ and D_μ^- , where

$$D_\mu^+ = \bigcup_{B \cap (I_\mu \times \{0\})} D_{\mu, \lambda}^+ \quad \text{and} \quad D_\mu^- = \bigcup_{B \cap (I_\mu \times \{0\})} D_{\mu, \lambda}^-.$$

The main result of this section is the following theorem.

Theorem 7. Let $\mu \in \sigma(L)$ be simple, condition (11) holds and satisfies condition (A). Then either the sets D_μ^+ and D_μ^- are unbounded in $\mathbb{R} \times E$, or $D_\mu^+ \cap D_\mu^- \neq I_\mu \times \{0\}$.

Chapter II considers nonlinear Sturm-Liouville problems with a spectral parameter in the boundary condition. The structure of bifurcation points is investigated and the behavior of a pair of families of continua of the set of nontrivial solutions of problems under consideration, contained in classes of functions possessing oscillatory properties of eigenfunctions of the corresponding linear problem and branching from points and intervals of the line of trivial solutions, is studied.

Section 2.1 outlines the formulation of the problem. Consider the following nonlinear eigenvalue problem

$$\mathfrak{L}(y)(x) \equiv -(p(x)y')' + q(x)y = \lambda r(x)y + \mathbf{h}(x, y, y', \lambda), x \in (0, \pi), \quad (11)$$

$$b_0 y(0) = d_0 y'(0), \quad (12)$$

$$(a_1 \lambda + b_1) y(\pi) = (c_1 \lambda + d_1) y'(\pi), \quad (13)$$

where

$p(x) \in C^1([0, \pi]; R)$, $q(x), r(x) \in C([0, \pi]; R)$, $p(x), r(x) > 0$, $x \in [0, \pi]$, $a_1, b_0, b_1, c_1, d_0, d_1$ are real constants such that $b_0^2 + d_0^2 > 0$ and $\sigma_1 = a_1 d_1 - b_1 c_1 > 0$. The nonlinear term \mathbf{h} has the form $\mathbf{h} = f + g$, where f and g are continuous functions on $[0, \pi] \times R^3$ and satisfy the conditions:

$$\left| \frac{f(x, u, s, \lambda)}{u} \right| \leq M, x \in [0, \pi], u, s \in R, 0 < |u| \leq 1, |s| \leq 1, \lambda \in R; \quad (14)$$

$$g(x, u, s, \lambda) = o(|u| + |s|) \quad (15)$$

in the neighborhood of the point $(u, s) = (0, 0)$ uniformly for $x \in [0, \pi]$ and $\lambda \in \Lambda \subset R$, for each bounded interval Λ .

It should be noted that P.G. Rabinowitz² gave a nonlinear version of the classical results for linear Sturm-Liouville problems, where the existence of a pair of families of unbounded continua of solutions to problem (11)-(13) for $f \equiv 0$ and $a_1 = c_1 = 0$ bifurcating from the eigenvalues of the linear problem and contained in classes of functions possessing the usual Sturmian oscillatory properties was established.

Recall that due to the presence of a nonlinear term F , problem (11)-(13) is nonlinearizable in the neighborhood of $y = 0$ and therefore the bifurcation points of this problem are not located discretely on the real axis, but are contained in the intervals that surround the eigenvalues of the linear problem.

Later, the global bifurcations of solutions of nonlinearizable Sturm-Liouville problems (11)-(13) for $a_1 = c_1 = 0$ (i.e., when the spectral parameter is not contained in the boundary conditions), in various formulations, were studied by many authors. They proved the existence of two families of unbounded continua of the set of

solutions, possessing the usual Sturmian oscillation properties and bifurcating from intervals of the line of trivial solutions..

Recall that T.I. Allahverdiyev⁶ was unable to completely investigate the behavior of the continua of solutions to problem (11)-(13), which have oscillatory properties of the eigenfunctions of the linear problem and branching from the bifurcation points and intervals of the line of trivial solutions.

In this chapter, the structure of bifurcation points and global continua of solutions to problem (11)-(13) has been completely studied.

In 2.2, auxiliary facts and statements are given for the linear Sturm-Liouville problem with a spectral parameter in the boundary condition.

For $c_1 \neq 0$ we define a natural number N_1 from the inequality $\mu_{N_1-1} < -d_1/c_1 \leq \mu_{N_1}$, where $\mu_k, k \in N$, is the k -th eigenvalue of the linear Sturm-Liouville problem which obtained from (11)-(13) for $h \equiv 0$ and $a_1 = c_1 = d_1 = 0$.

The following oscillation theorem holds for problem (11)-(13) with $h \equiv 0$.

Theorem 8. The eigenvalues of the linear problem (11)-(13) with $h \equiv 0$ are real, simple and forms unboundedly increasing sequence $\lambda_1, \lambda_2, \dots, \lambda_k, \dots$. The corresponding eigenfunctions $y_1(x), y_2(x), \dots, y_k(x), \dots$ have the following oscillatory properties: (i) if $c_1 = 0$, then the eigenfunction $y_k(x), k \in N$, has exactly $k - 1$ simple zeros in the interval $(0, \pi)$; (ii) if $c_1 \neq 0$, then the eigenfunction $y_k(x), k \in N$, for $k \leq N_1$ has exactly $k - 1$ and for $k > N_1$ has exactly $k - 2$ simple zeros in the interval $(0, \pi)$.

In 2.3, the global bifurcation of solutions of the nonlinear problem (11)-(13) for $f \equiv 0$ is studied.

Let $E = C^1[0, \pi] \cap \{y: b_0 y(0) = d_0 y'(0)\}$ be the Banach space with the norm $\|y\|_1 = \max_{x \in [0, \pi]} |y(x)| + \max_{x \in [0, \pi]} |y'(x)|$.

By $S_k^+, k \in N$, denote the set of functions $y \in E$ such that the

function $y(x)$ has exactly $k - 1$ simple zeros in the interval $(0, \pi)$ and is positive in the right punctured neighborhood of the point $x = 0$. Let

$$S_k^- = -S_k^+ \text{ and } S_k = S_k^+ \cup S_k^-, k \in N.$$

It follows from Theorem 8 that if $c_1 = 0$, then $y_k \in S_k$ for any $k \in N$, and if $c_1 \neq 0$, then $y_k \in S_k$ for $k \leq N_1, y_k \in S_{k-1}$ for $k > N_1$. Consequently, the sets $S_k, S_k^v, k \in N, v \in \{+, -\}$, are nonempty. From the definition of these sets it follows that they are open subsets of E and $\tilde{S}_k^v \cap \tilde{S}_{k'}^{v'} = \emptyset$ for $(k, v) \neq (k', v')$. Moreover, if $y \in \partial S_k, k \in N$, then $y(x)$ has at least one double zero in $[0, \pi]$.

We denote by \mathfrak{S} the closure of the set of nontrivial solutions to problem (11)-(13) in $R \times E$. Let

$$T_k^v = \begin{cases} S_k^v & \text{if } c_1 = 0 \text{ or } c_1 \neq 0, k \leq N_1, \\ S_{k-1}^v & \text{if } c_1 \neq 0, k > N_1, \end{cases} \quad T_k = T_k^+ \cup T_k^-, \\ \mathfrak{S}_k^v = \mathfrak{S} \cap (R \times T_k^v), \mathfrak{S}_k = \mathfrak{S}_k^+ \cup \mathfrak{S}_k^-, k \in N, v \in \{+, -\}.$$

Lemma 4. *If $(\lambda, y) \in R \times E$ is a solution of problem (11)-(13) and $y \in \partial S_k^v, k \in N, v \in \{+, -\}$, then $y \equiv 0$.*

In a Hilbert space $H = L_2((0, \pi); r) \oplus C$ with scalar product $(\bar{y}, \bar{u}) = (\{y(x), m\}, \{u(x), n\}) = (y, u)_{L_{2,r}} + |\sigma_1| \Gamma^{-1} m \bar{n}$,

define the operator

$$A\bar{y} = A\{y(x), m\} = \left\{ \frac{1}{r(x)} \mathfrak{e}(y)(x), d_1 p(\pi) y'(\pi) - b_1 y(\pi) \right\},$$

on the domain

$$D(A) = \{ \{y(x), m\} \in H : y, py^{\dagger'} \in AC[0, \pi], r^{\dagger}(-1) \mathfrak{e}(y) \in L_1(0, \pi), \\ b_1 y(0) = d_1 p(0) y^{\dagger'}(0), m = a_1 y(\pi) - c_1 p(\pi) y^{\dagger'}(\pi) \}.$$

where $(y, u)_{L_{2,r}} = \int_0^\pi r(x) y(x) \overline{u(x)} dx$ and is a class of absolutely continuous functions on $[0, \pi]$. It is obvious that the operator A is correctly defined in H . Then problem (11)-(13) with $h \equiv 0$ takes the form

$$A\bar{y} = \lambda \bar{y}, \bar{y} \in D(A),$$

i.e. the eigenvalues λ_k of problem (12)-(14) with $h \equiv 0$ and the

operator A coincide together with their multiplicities, and between the eigenfunctions there is a one-to-one correspondence

$$y_k(x) \leftrightarrow \{y_k(x), m_k\}, \quad m_k = a_1 y_k(\pi) - c_1 y'_k(\pi).$$

We define the operators $F, G: \mathbf{R} \times D(A) \rightarrow H$ as follows:

$$G(\lambda, y^*) = G(\lambda, \{y(x), m\}) = \{(r(x))^\dagger (-1) g(x, y(x), y'(x)), \lambda), 0\},$$

$$F(\lambda, \hat{y}) = F(\lambda, \{y(x), m\}) = \{(r(x))^{-1} f(x, y(x), y'(x)), \lambda), 0\}.$$

Then problem (11)-(13) reduces to the following nonlinear problem

$$A\hat{y} = \lambda\hat{y} + F(\lambda, \hat{y}) + G(\lambda, \hat{y}), \quad (16)$$

i.e. there is a correspondence between the solutions to these problems $(\lambda, \hat{y}) \leftrightarrow (\lambda, y)$.

Let $\tilde{E} = E \oplus C$ be the Banach space with the norm $\|\hat{y}\| = \|y\|_1 + |m|$, where $\hat{y} = \{y, m\}$.

Note that, if $\hat{y} = \{y, m\} \in D(A)$, then $y' \in AC[0, \pi]$, because $p \in C^1([0, \pi]; R)$. Then $y \in C^1([0, \pi]; R)$, and consequently, $D(A) \subseteq E$.

By $\tilde{S}_k^\nu, k \in N, \nu \in \{+, -\}$, denote the set of vectors $\hat{y} = \{y, m\} \in \tilde{E}$ such that $y(x) \in S_k^\nu; S_k = S_k^+ \cup S_k^-$. From the definition of sets $\tilde{S}_k, \tilde{S}_k^\nu, k \in N, \nu \in \{+, -\}$, it follows that these sets are open subsets of \tilde{E} and $\tilde{S}_k^\nu \cap \tilde{S}_{k'}^{\nu'} = \emptyset$ for $(k, \nu) \neq (k', \nu')$. Moreover, if $\hat{y} = \{y, m\} \in \partial\tilde{S}_k$, then $y(x)$ has at least 1 double 0 in the interval $[0, \pi]$.

Remark 1. It follows from Lemma 4 that if $(\lambda, \hat{y}) \in \mathbf{R} \times \tilde{E}$ is a solution of problem (16) and $y \in \partial S_k^\nu, k \in N, \nu \in \{+, -\}$, then

$\hat{y} = \tilde{0}$, where

By $\tilde{\mathfrak{S}}$ denote the closure in $\mathbf{R} \times \tilde{E}$ of the set of nontrivial solutions of problem (17).

$$\begin{aligned}\widehat{T}_k^v &= \begin{cases} \widehat{S}_k^v & \text{if } c_1 = 0 \text{ or } c_1 \neq 0, k \leq N_1, \\ \widehat{S}_{k-1}^v & \text{if } c_1 \neq 0, k > N_1, \end{cases} & \widehat{T}_k &= \widehat{T}_k^+ \cup \widehat{T}_k^-, \\ \widehat{\mathfrak{S}}_k^v &= \mathfrak{S} \cap (R \times \widehat{T}_k^v), \widehat{\mathfrak{S}}_k &= \widehat{\mathfrak{S}}_k^+ \cup \widehat{\mathfrak{S}}_k^-, k \in N, v \in \{+, -\}.\end{aligned}$$

Now suppose that $f \equiv \mathbf{0}$ in equation (11) (in fact, we assume that the nonlinear term itself satisfies condition (15)). Then problem (11)-(13) will be equivalent to the nonlinear problem

$$A\mathcal{Y} = \lambda\mathcal{Y} + G(\lambda, \mathcal{Y}). \quad (17)$$

Theorem 9. Let the condition $f \equiv \mathbf{0}$ be satisfied. Then for each $k \in N$ and each $v \in \{+, -\}$ there exists the continuum \widehat{C}_k^v of solutions of (17) that contain $(\lambda_k, \widehat{\mathbf{0}})$, is contained in $(\mathbf{R} \times \widehat{T}_k^v) \cup \left\{ (\lambda_k, \widehat{\mathbf{0}}) \right\}$ and is unbounded in $\mathbf{R} \times \widehat{E}$

Since there is a one-to-one correspondence between the solutions of problems (17) and (11)-(13) with $f \equiv \mathbf{0}$, then from Theorem 9 we obtain that the following result holds.

Theorem 10. Let $f \equiv \mathbf{0}$ in (12). Then for each $k \in N$ and each $v \in \{+, -\}$ there exists the continuum C_k^v of solutions of (17) that contain $(\lambda_k, \mathbf{0})$, is contained in $(\mathbf{R} \times T_k^v) \cup \{(\lambda_k, \mathbf{0})\}$ and is unbounded in $\mathbf{R} \times E$.

In 2.4, the global bifurcation of solutions of problem (11)-(13) is studied in the general case. Despite the fact that the nonlinear problem (11)-(13) is not linearizable at zero, it is still related to some linear problems.

In studying the bifurcation of problem (11)-(13), we first approximate this problem by linearizable problems to which Theorem 9 is applicable. We then pass to the limit using a priori estimates that are obtained using the max-min property of the eigenvalues of some linear problems.

A point $(\lambda, \mathbf{0})$ is said to be a bifurcation point of problem (11)-(13) with respect to the set $\mathbf{R} \times S_k^v, k \in N, v \in \{+, -\}$, if in each small neighborhood of this point there exists a nontrivial solution of problem (11)-(13) that is contained in $\mathbf{R} \times S_k^v$.

Lemma 5. For each $k \in N$, each $v \in \{+, -\}$ and any

sufficiently small $\tau > 0$ there exists a solution $(\zeta_{\tau,k}, \widehat{w}_{\tau,k})$ of problem (16) such that $\widehat{w}_{\tau,k} \in \mathbf{R} \times \widehat{S}_k^v$, $\|\widehat{w}_{\tau,k}\|_1 = \tau$.

Corollary 4. The set of bifurcation points of problem (16) with respect to the set $\mathbf{R} \times \widehat{S}_k^v$ is not empty.

Let $\varepsilon > 0$. Along with problem (11)-(13), we consider the following so-called approximation problem

$$\begin{cases} \mathbf{e}(y) = \lambda r(x)y + f(x, |y|^\varepsilon y, y', \lambda) + g(x, y, y', \lambda), x \in (0, \pi), \\ b_0 y(0) = d_0 y'(0), (a\lambda + b)y(\pi) = (c\lambda + d)y'(\pi). \end{cases} \quad (18)$$

It is clear from (17) that problem (19) can be rewritten in the equivalent form

$$A\mathcal{Y} = \lambda\mathcal{Y} + F_\varepsilon(\lambda, \widehat{y}) + G(\lambda, \mathcal{Y}), \quad (19)$$

where

$$F_{\downarrow\varepsilon}(\lambda, \mathcal{Y}) = F_{\downarrow\varepsilon}(\lambda, \{y(x), m\}) = \{f(x, |y(x)|^\varepsilon y(x), y'(x), \lambda), 0\}.$$

By condition (14) the function $f(x, |u|^\varepsilon u, s, \lambda)$ satisfies condition

(15). Consequently, $\|F_\varepsilon(\lambda, \widehat{y})\|_0 = o(\|\widehat{y}\|_1)$ near uniformly

for $\lambda \in \Lambda \subset \mathbf{R}$, where $\|\widehat{y}\|_0 = \|\{y, m\}\|_0 = \max_{x \in [0,1]} |y(x)| + m$ is the

norm in the space $C[0,1] \oplus C$. Then, based on Theorem 9, for each $k \in N$ and each $v \in \{+, -\}$ there exists an unbounded continuum $\widehat{C}_{k,\varepsilon}^v$ of solutions of problem (19) such that

$$(\lambda_k, \widehat{0}) \in \widehat{C}_{k,\varepsilon}^v \subset (\mathbf{R} \times T_k^v) \cup \{(\lambda_k, \widehat{0})\}.$$

Hence for each $\varepsilon \in (0,1]$ there exists a solution $(\zeta_{\tau,k,\varepsilon}, \widehat{w}_{\tau,k,\varepsilon}) \in \mathbf{R} \times \widehat{E}$ of problem (19) such that $\widehat{w}_{\tau,k,\varepsilon} \in \partial B_\tau \cap \widehat{T}_k^v$, where ∂B_τ is the boundary of the sphere $\widehat{B}_\tau \subset \widehat{E}$.

We introduce the following notation:

$$J_k = [\lambda_k - M/r_0, \lambda_k + M/r_0], k \in N,$$

where $r_0 = \min_{x \in [0,\pi]} r(x)$. Let

$$\widehat{I}_k = \begin{cases} J_k & \text{if } k < N_1, \\ J_{N_1} \cup J_{N_1+1} & \text{if } k = N_1, \\ J_{k+1} & \text{if } k > N_1. \end{cases}$$

Lemma 6. Let $\{\varepsilon_n\}_{n=1}^\infty \subset (0,1]$ and $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. If

$(\zeta_n, \widehat{w}_n) \in \mathbb{R} \times S_k^\nu$ is a solution of problem (19) for $\varepsilon = \varepsilon_n$ and $\left\{ (\zeta_n, \widehat{w}_n) \right\}_{n=1}^\infty$ converges $(\zeta, \widehat{0})$ in $\mathbb{R} \times \widehat{E}$, then $\zeta \in \widehat{I}_k$.

Corollary 5. If $(\lambda, \widehat{0})$ a bifurcation point of problem (16) with respect to $\mathbb{R} \times \widehat{S}_k^\nu$, then $\lambda \in \widehat{I}_k$.

Now let

$$I_k = \begin{cases} \widehat{I}_k & \text{for } k \neq N_1, \\ \left[\lambda_{N_1} - M/r_0, \lambda_{N_1+1} + M/r_0 \right] & \text{for } k = N_1. \end{cases}$$

For each $k \in N$ and $\nu \in \{+, -\}$ we denote by \widehat{D}_k^ν the union of all connected components of the set $\widehat{\mathfrak{S}}$ emanating from the bifurcation points $(\lambda, \widehat{0}) \in \widehat{I}_k \times \{0\}$ with respect to the set $\mathbb{R} \times S_k^\nu$. It is obvious that \widehat{D}_k^ν is a connected subset of $\mathbb{R} \times E$.

Theorem 11. For each $k \in N$ and $\nu \in \{+, -\}$, the connected component \widehat{D}_k^ν of the set $\widehat{\mathfrak{S}}$, containing $I_k \times \{\widehat{0}\}$ is contained in $(\mathbb{R} \times \widehat{S}_k^\nu) \times (I_k \times \{\widehat{0}\})$ and is unbounded in $\mathbb{R} \times \widehat{E}$.

Since there is a one-to-one correspondence $(\lambda, \widehat{y}) \leftrightarrow (\lambda, y)$ between the solutions of problems (16) and (11)-(13), the validity of the following theorem follows from Theorem 11.

Theorem 12. For each $k \in N$ and $\nu \in \{+, -\}$, the connected component D_k^ν of the set \mathfrak{S} , containing $I_k \times \{0\}$ is contained in $(\mathbb{R} \times S_k^\nu) \times (I_k \times \{0\})$ and is unbounded in $\mathbb{R} \times E$.

The following theorem also holds.

Theorem 13. Suppose that $g \equiv 0$ and f satisfies the condition (14) for all $(x, u, s, \lambda) \in [0, 1] \times \mathbb{R}^3$. Then for each $k \in N$ and $\nu \in \{+, -\}$ the following relations hold:

$$\widehat{D}_k^\nu \subset J_k \times \left(\widehat{S}_k^\nu \cup \{\widehat{0}\} \right) \quad \text{and} \quad D_k^\nu \subset J_k \times \left(S_k^\nu \cup \{0\} \right).$$

Chapter III considers half-linear Sturm-Liouville problems

with a spectral parameter in the boundary condition. The structure of the set of bifurcation points is studied and the behavior of continua of solutions of the problems under consideration, bifurcating from the points of the line of trivial solutions, is investigated.

In 3.1 the problem statement is given and information about previously known results is provided.

Consider the following nonlinear eigenvalue problem

$$\ell(y)(x) \equiv -(p(x)y')' + q(x)y = \lambda r(x)y + H(x, y, y', \lambda), 0 < x < \pi, \quad (20)$$

$$b_0 y(0) = d_0 y(0), \quad (21)$$

$$(a_1 \lambda + b_1) y(\pi) = (c_1 \lambda + d_1) y(\pi), \quad (22)$$

where $\lambda \in R$ is a spectral parameter, the coefficients in the equation and in the boundary conditions satisfy the conditions of Chapter II.

Nonlinear term H has the form $H = \alpha y^+ + \beta y^- + \mathcal{H}$ where $\alpha, \beta \in C([0, \pi]; R)$, $y^+ + (x) = \max\{y(x), 0\}$, $y^+ - (x) = (-y)^+ + (x)$.

The function $\mathcal{H} \in C([0, \pi] \times R^3)$ satisfies the condition:

$$\mathcal{H}(x, u, s, \lambda) = o(|u| + |s|) \text{ when } |u| + |s| \rightarrow 0, \quad (23)$$

uniformly for $x \in [0, \pi]$ and $\lambda \in \Lambda$, for any bounded interval $\Lambda \subset R$.

The purpose of this chapter is to study the structure of the set of bifurcation points on the real axis and to more accurately describe the structure and behavior of connected branches of solutions to problem (20)–(22) bifurcating from the line of trivial solutions.

Section 3.2 studies the oscillatory properties of the half-linear eigenvalue problem

$$\begin{cases} \ell(y) = \lambda r(x)y + \alpha(x)y^+ + \beta(x)y^-, x \in (0, \pi), \\ b_0 y(0) = d_0 y(0), (a_1 \lambda + b_1) y(\pi) = (c_1 \lambda + d_1) y(\pi). \end{cases} \quad (24)$$

Problem (24) is nonlinear, but positively homogeneous (in the sense that if y is a solution to this problem, then $c y$ is also a solution for all $c > 0$) and linear in the cones $y > 0$ and $y < 0$.

It is said that λ is a half-eigenvalue of problem (24) if there exists a non-trivial solution (λ, y_λ) of this problem; the function y_λ is called a half-eigenfunction. In this situation the set $\{(\lambda, t y_\lambda) : t > 0\}$ is a half-line of nontrivial solutions to problem (24). The number λ is called prime if all solutions (λ, ϑ) of problem (24), where ϑ and

y_λ have the same sign in the punctured neighborhood of the point $x = 0$, are located on the half-line $\{(\lambda, t y_\lambda) : t > 0\}$. There may be another half-line of solutions $\{(\lambda, t \vartheta_\lambda) : t > 0\}$, but then λ is said to be simple if ϑ_λ and y_λ have different signs in the neighborhood of $x = 0$, and all solutions (λ, ϑ) of problem (24) lie on these two half-lines.

The half-linear problem (24) in the case of $|a_1| + |c_1| > 0$ was studied in the paper of P.J. Browne⁸, where the author showed that for each $\nu \in \{+, -\}$ there exists an infinitely increasing sequence of real, simple half-eigenvalues $\{\lambda_k^\nu\}_{k=1}^\infty$ of this problem. The corresponding half-eigenfunctions $y_k^\nu(x), k = 1, 2, \dots$, have the following oscillatory properties: (i) $\nu y_k^\nu(x) > 0$ in the pierced neighborhood of the point $x = 0$; (ii) if $c_1 = 0$, then the function $y_k^\nu(x), k \in N$, has exactly $k - 1$ simple zeros in $(0, \pi)$; (iii) if $c_1 \neq 0$, then the function $y_k^\nu(x)$ has exactly $k - 1$ simple zeros for $k \leq N_1^\nu$, has exactly $k - 2$ simple zeros for $k > N_1^\nu$ in $(0, \pi)$, where $N_1^\nu \in N$ is determined from the inequality $\mu_{N_1^\nu - 1}^\nu < -d_1/c_1 \leq \mu_{N_1^\nu}^\nu$, and μ_k^ν is the k -th half-eigenvalues of the spectral (20), (21), $y(\mathbf{1}) = 0$ for $h \equiv 0$.

It should be noted that In the above-mentioned work of P.J. Browne⁸ in the case of $c_1 \neq 0$ the connection between the natural numbers $N_1^\nu, \nu \in \{+, -\}$, and N_1 is not indicated. Therefore, in this case, using the results of H. Berestycki⁷, it is impossible to study the structure of all continua of solutions to problem (20)-(22), bifurcating from trivial solutions corresponding to the half-eigenvalues of problem (24).

As in Chapter II, we introduce the following notation:

$$I_k = \left[\lambda_k - M/r_0, \lambda_k + M/r_0 \right], k \in N,$$

where $M = \max_{x \in [0, \pi]} |\alpha(x)| + \max_{x \in [0, \pi]} |\beta(x)|$. Moreover, let the intervals $I_k, k \in N$, are determined as in Chapter II.

The numbers $\gamma_k, k \in N \cup \{0\}$, are defined as follows:

$$\gamma_k = \lambda_{k+1} - \lambda_k, k \in N, \gamma_0 = \{\gamma_k : k \in N\}.$$

Throughout the following we will assume that for $c_1 \neq 0$ the following condition is satisfied: $M < r_0 \gamma_0 / 2$. Consequently, for any $k, l \in N, k \neq l$, we have $J_k \cap J_l = \emptyset$, and by Theorem we obtain that for each $k \in N$ and $v \in \{+, -\}$ the connected component D_k^v of the set of solutions of problem (24) contain $J_k \times \{0\}$, is contained in $(R \times S_k^v) \cup (J_k \times \{0\})$ and is unbounded in $\mathbb{R} \times E$.

The following oscillation theorem holds for problem (24).

Theorem 14. Let for $c_1 \neq 0$ the condition $M^* < r_0 \gamma_0 / 2$ holds, where $M^* = \max_{x \in [0, \pi]} \{|\alpha(x)| + |\beta(x)|\}$. Then there exist there exist two sequences $\{\lambda_k^+\}_{k=1}^\infty$ and of real, simple half-eigenvalues of problem (24) such that

$$\lambda_1^+ < \lambda_2^+ < \dots < \lambda_k^+ \rightarrow +\infty, \lambda_1^- < \lambda_2^- < \dots < \lambda_k^- \rightarrow +\infty;$$

if $k' > k \geq 1$, then $\lambda_{k'}^{v'} > \lambda_k^v$ for each $v, v' \in \{+, -\}$; the corresponding half-lines of solutions are located in $\{\lambda_k^+\} \times T_k^+$ $a\{\lambda_k^-\} \times T_k^-$, $k \in N$. In addition, besides these solutions and trivial ones, there are no other solutions to problem (24).

In 3.3, the global bifurcation of solutions to problem (20)-(22) is studied.

Lemma 7. The set of bifurcation points of problem (20)-(22) is nonempty.

Lemma 8. If $(\lambda, 0)$ is a bifurcation point of problem (20)-(22), then λ is a half-eigenvalue of problem (24).

By Lemmas 7, 8 and Corollary 4 the following results hold.

Lemma 9. The set of bifurcation points of problem (20)-(22) with respect to the set $\mathbb{R} \times S_k^v$ is nonempty.

Lemma 10. If $(\lambda, 0)$ is a bifurcation point of problem (20)-(22) with respect to the set $\mathbb{R} \times S_k^v$, then $\lambda \in I_k$. Moreover, $\lambda = \lambda_k^v$, if $k < N_1$, $\lambda = \lambda_{k+1}^v$, if $k > N_1$, and either $\lambda = \lambda_{N_1}^v$, or $\lambda = \lambda_{N_1+1}^v$ in the case of $k = N_1$.

Lemma 12. For each $\nu \in \{+, -\}$ the points $(\lambda_{N_1}^\nu, 0)$ and $(\lambda_{N_1+1}^\nu, 0)$ are bifurcation points of problem (20)-(22) $\mathbf{R} \times S_{N_1}^\nu$.

We denote by \mathfrak{S} the closure in $\mathbf{R} \times E$ of the set of nontrivial solutions to problem (20)-(22) and by \mathfrak{R}_k^ν the closure in $R \times E$ of the set of all solutions (λ, y) of problem (20)-(22) such that $y \in T_k^\nu$.

The following theorem describes the structure and behavior of global continua of the set of solutions to problem (20)-(22), bifurcating from the line of trivial solutions.

Theorem 14. For each $k \in N$ each $\nu \in \{+, -\}$ there exists unbounded continuum D_k^ν of solutions of problem (20)-(22) such that $(\lambda_k^\nu, 0) \in D_k^\nu \subset (\mathbf{R} \times T_k^\nu) \cup \{(\lambda_k^\nu, 0)\}$.

Conclusion

The following main results were obtained in the dissertation:

- a theorem on the global bifurcation of solutions from a line segment of trivial solutions, which contains an odd-multiple eigenvalue of a linear problem, nonlinear eigenvalue problems without the assumption of differentiability of nonlinear perturbations, was proved;
- the structure and behavior of the connected component of the set of solutions, bifurcating from the interval of the line of trivial solutions containing a simple eigenvalue of the linear problem, are studied in detail;
- found bifurcation intervals of the nonlinear Sturm-Liouville problem with a spectral parameter in the boundary condition;
- the existence of a pair of unbounded connected components of the set of solutions to the nonlinear Sturm-Liouville problem with a spectral parameter in the boundary condition, branching from bifurcation intervals and possessing oscillatory properties of eigenfunctions of the linear problem, is proved;
- the oscillatory properties of half-eigenfunctions of the half-linear Sturm-Liouville problem with a spectral parameter in the

boundary condition were studied;

– the existence of a pair of unbounded continua of solutions of the half-linearizable Sturm-Liouville problem with a spectral parameter in the boundary condition, bifurcating from half-eigenvalues and possessing oscillatory properties of half-eigenfunctions of the corresponding half-linear problem, is proved.

Main results of the dissertation were published in the following papers:

1. Aliyev Z.S., Mamedova G.M. Global bifurcation of solutions to nonlinearizable Sturm-Liouville problems with spectral parameter in boundary conditions // Materials of the Republican scientific conference of master's, doctoral and young researchers "Actual problems of mathematics and mechanics" dedicated to the 91st anniversary of the birth of the national leader of the Azerbaijani people Heydar Aliyev, - Baku: - 2014, - 12–13 May, - pp. 21-24. Mamedova, G.M. Local and global bifurcation for some nonlinearizable eigenvalue problems // – Baku: Proceedings of IMM of NAS of Azerbaijan, – 2014. v. 40, no. 2, – p. 45-51.
2. Mamedova, G.M. On the bifurcation of solutions from a simple eigenvalue for some nonlinearizable spectral problems // Materials of the Republican scientific conference “Actual problems of mathematics and mechanics” dedicated to the 92nd anniversary of the birth of the national leader of the Azerbaijani people Heydar Aliyev, – Baku: – 20–21 May, – 2015, – pp. 80–81.
3. Aliyev, Z.S., Mamedova G.M. Some global results for nonlinear Sturm-Liouville problems with spectral parameter in the boundary condition // Annales Polonici Mathematici, – 2015. т.115, № 1, – с. 75–87.
4. Mamedova, G.M. Global bifurcation from zero for some nondifferentiable mappings // – Baku: Transactions of NAS of

The defense will be held on 31 october at 12:00 at the meeting of the Dissertation council ED 2.17 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at Baku State University.

Address: AZ 1148, Baku city, Z.Khalilov str., 23.

Dissertation is accessible at the library of the Baku State University.

Electronic versions of the dissertation and its abstract are available on the official website of the Baku State University.

Abstract was sent to the required addresses 30 09 2025

LS

Signed for print: 22.09.2025
Paper format: 60x841/16
Volume: 37030
Number of hard copies: 20