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ABSTRACT

of the dissertation for the degree of Doctor of Science

GEOMETRY OF COFRAME BUNDLE

Speciality: 1204.01 – Geometry

Field of science: Mathematics

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Baku-2024

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ELMİ KATİB
SCIENTIFIC SECRETARY
İmzanı təsdiq edirəm:
"05" 04 2025

GENERAL CHARACTERISTICS OF THE WORK

Rationale of the theme and development degree. One of the most important classes of fiber spaces is the class of vector bundles. The tangent, cotangent, and tensor bundles of smooth manifolds belong to this class. The study of differential-geometric structures on tangent bundles began in the 1960s. S.Sasaki, K.Yano, S.Kobayashi, S.Ishihara defined various metrics, vector and tensor fields, affine connections and they are called extensions (lifts) of the corresponding structures given in the bundle base. A. Morimoto and A.P. Shirokov interpreted the tangent bundle as a manifold over the algebra of dual numbers and based on this they constructed a different way of defining the lifts of some structures. VV Vishnevskii developed this idea while studying semitangent bundles. The study of structures on cotangent bundles revealed differences between these bundles and tangent bundles. For example, it was not possible to determine the complete and horizontal lifts of covariant tensor fields, as well as the complete lift of the affine connection using the traditional relation to the cotangent bundle. K.Yano and E.M.Patterson used the symplectic structure, which naturally exists in this bundle, when constructing lifts of differential geometric structures in cotangent bundles. S.L.Druta-Romaniuc, V.Oproiu, D.A.Poroşniuc, K.P.Mok, A.A.Salimov investigated new classes of natural Riemannian metrics and natural anti-Hermitian metrics.

Tensor bundles are generalizations of tangent and cotangent bundles. Differential geometric structures on tensor bundles have been studied by I. Sato, A. A. Salimov, N. Çengiz and others. A.A. Salimov proposed a new method for constructing lifts in these bundles by applying operator theory to the study of tensor fields on sections of tensor bundles.

When studying the geometry of a bundle of frames, it is assumed that this bundle is a principal bundle and the tangent bundle is associated with this bundle. Based on this relationship, F.Okubo, K.P.Mok, L.A.Cordero, M. de Leon, J.Kürek, A.Ya.Sultanov, O.Kowalski, M.Sekizawa, K.Niedzialomski constructed lifts of differential geometric structures in the bundle of frames and identified various structures in this bundle. Reviews of papers devoted to the study of differential-geometric structures on frame bundles are given in the monographs [1] byL.A. Co rdero, M. de Leon and C.T. Dodson. Due to the fact that the frame bundle is a principal bundle, in contrast to the tangent bundle, it became possible to study the lifts of differential-geometric structures along the sections of the frame bundle only in the case of a parallelizable manifold.

The geometry of the coframe bundle, which plays an important role among principal bundles, is associated with the geometry of the cotangent bundle and differs sharply from the geometry of the frame bundle. The application of coreframe bundles in the general theory of relativity is described in detail in the monograph [2] by E.Prugovecki. Some differential-geometric structures on coreframe bundles were studied by M. de Leon, I. Mendez, M. Salgado [3] and M. Khodjiev [4] and after that there was a stagnation in this direction. It should also be noted that the construction of lifts of basic differential-geometric structures, including vector and tensor fields, affine connections, the study of these lifts along sections, the study of various types of natural (natural) Riemannian metrics, their curvature properties, integrability conditions consistent with these metrics of affinor structures have not been done yet. In our opinion, the reason for the occurrence of stagnation is that the methods used for tensor bundles (for example, the theory of $\Phi-$operators, the construction of Sasaki-type lifts, etc.) were not applied to coframe bundles.

In the dissertation work, the reasons that impede the solution of the above issues in the bundle of corepers are eliminated by the methods we have indicated, so the topic of the dissertation is relevant.

S**ubject and objects of research.** Study of lifts of vector and tensor fields, affine connections, almost complex and paracomplex structures and $f-$structures, Sasaki and Cheeger-Gromoll

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[1] Cordero, L.A. Differential geometry of frame bundles / L.A.Cordero, C.T.Dodson, de M. Leon. – Dordrech: Kluwer Academic Publishers, - 1989, - 234 p.

[2] Prugovecki, E. Principles of Quantum General Relativity / E. Prugovecki. - World Scientific Publishing Company. – 1995, - 376 p.

metrics, curvature properties of these metrics in the coframe bundle, and some structures in the tensor frame bundle of type (1,1).

The goal and tasks of the study. The main goal and task of the dissertation work is the construction of lifts of differential-geometric structures given on the basis of bundles, as well as various Riemannian metrics, some affinor structures, determining the conditions for their integration, as well as applying the experience gained to solving some similar problems in the bundle of tensor frames of type (1,1).

Research methods. When studying the issues considered in the dissertation, the tensor method with indices and the invariant tensor method on smooth manifolds were used. In some cases, the rules of covariant differentiation were applied.

The basic aspects to be defended. The following main provisions are made for citation:

1. Construction of vertical lifts of functions and 1-forms, complete lifts vector and tensor fields, derivations, almost complex structures and f – structures in a bundle of coframes.

2. Construction of horizontal lifts of vector and tensor fields, derivations, almost complex structures and f – structures in coreper bundles.

3. Implementation of an adapted frame in the bundle of coframes, construction of horizontal and complete lifts of symmetric affine connection, study of their geodesics.

4. Investigation of lifts of vector and tensor fields, affine connection and almost complex structure on sections of coreper bundles.

5. Determination of the Sasaki metric in the bundle of coframes and study of the Levi-Civita connection, curvature properties and geodesics of this metric.

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[3] Leon, de M., Mendez, I., Salgado, M. On the differential geometry of the coframe bundle // Proceedings of the conference "Differential geometry and its applications", -Brno, Czechoslovakia, -August 24-30, - 1986, - p. 213-224.

[4] Ходжиев, М.Б. Расслоение кореперов и кокасательное расслоение // Тезисы докл. Межд. Научн. Конф. «Лобачевский и современная геометрия», I часть, Казань: - 1992, - с. 106-107.

6. Determination of structures of various ranks in the bundle of corepers and study of their connection with the Sasaki metrics.

7. Construction of a homogeneous lift of the Riemannian metric in the coreper bundle, study of its Levi-Civita connection and curvature properties.

8. Determination of the Cheeger-Gromoll metric in the coreper bundle, investigation of the Levi-Civita connection and curvature properties of this metric.

9. Study of almost complex and paracomplex structures in coreper bundles with the Cheeger-Gromoll metric.

10. Construction of the Sasaki metric and complete and horizontal lifts with an affine connection, study of their properties in a bundle of tensor frames of type.

Scientific novelty of the research. The following main results were obtained in the dissertation:

1. Theorems on complete and horizontal lifts of almost complex structure and $f-$structure, affine connections and derivations are proved.

2. Theorems are proved on the behavior of lifts of differential-geometric structures along sections of the bundle of coframes.

3. In a bundle of corepers over a Riemannian manifold, the Sasaki metric is defined and theorems on the Levi-Civita connection, connection coefficients, properties of curvature, and geodesics of this metric are proved.

4. In a bundle of corepers over a Riemannian manifold, a homogeneous lift of the Riemannian metric is defined, theorems on the Levi-Civita connection, connection coefficients, curvature properties, and geodesic lines of this metric are proved.

5. $f-$structures of various ranks are defined in the coreframe bundle, and theorems on the compatibility of the Sasaki metric with these structures are proved.

6. Theorems on lifts of tensor fields in the bundle of coframes of a Riemannian manifold are proved.

7. In the coreper bundle, the Cheeger-Gromoll metric is defined, Levi-Civita connection theorems, connection coefficients, curvature

properties of this metric, and integrability conditions for almost complex and paracomplex structures of a special form, consistent with the Cheeger-Gromoll metric, are proved.

8. Theorems are proved on the Sasaki metric, on the horizontal and complete lifts of the affine connection in the bundle of tensor frames of type $(1,1)$.

The theoretical and practical value of the research. The results of the dissertation are theoretical. These results can be used in the theory of smooth manifolds and in the theory of bundles, as well as in teaching special courses in geometry.

Approbation and application. The main results of the dissertation were reported at the Republican scientific conferences dedicated to the 87th and 94th anniversary of the National Leader of the Azerbaijani People Heydar Aliyev (Baku, 2010, 2014), at the International Scientific Conference dedicated to the 80th anniversary of Corresponding Member. ANAS, prof. Ya.J.Mamedova (Baku, 2010), at the International Scientific Conference dedicated to the 100th anniversary of Academician Z.I.Khalilov (Baku, 2011), at the III and IV Congresses of the Mathematical Society of the Turkic World (Almaty, 2009 Baku, 2011), at the International scientific conferences dedicated to the 55th and 60th anniversary of IMM NAS Azerbaijan (Baku, 2014, 2019), at the 8th Eurasian International Conference on Mathematical Sciences and Applications dedicated to the 100th anniversary of Baku State University (Baku, 2019), at the International Conference "Innovative technologies for oil and gas production and modern problems of applied mathematics", dedicated to the 90th anniversary of academician A.Kh. Mirzajanzade (Baku, 2018) , at the 18th International Geometric Symposium (Malatya, 2021), at the I International Congress on Natural Sciences (Erzurum, 2021), at the I International Scientific and Technical Conference "Actual Problems of Science and Technology" (Sarapul, 2021 .), at the faculty-wide seminar of the Faculty of Mechanics and Mathematics of the Belarusian State University (headed by prof. Z.S. Aliyev), at the seminar of the department "Algebra and Geometry" of BSU (headed by Prof. A.A. Salimov).

Institution where the dissertation work was executed. The work was done at the Algebra and Geometry Department of the Faculty of Mechanics and Mathematics of the Baku State University.

Personal contribution of the author lies in the formulation of the goal and the choice of the research direction. In addition, all the conclusions and results obtained in the dissertation belong to the author personally.

Pblications of the author. Publications in editions recommended by HAC under President of the Republic of Azerbaijan – 20. Of these, 13 articles were published in journals of the Web of Science database and 1 article in a journal of the Scopus database. Materials of conferences - 14. 12 of them are conferences of the international level, 2 of them are of the republican level. In addition, 5 abstracts were published in conference books published abroad.

Structure and volume of the dissertation (in signs, indicating the volume of each structural subsection separately). The total volume of the dissertation work is 473331 characters (title page - 314 characters, table of contents - 4319 characters, introduction - 38678 characters, first chapter - 66000 characters, second chapter - 76000 characters, third chapter - 30000 characters, fourth chapter - 136000 characters, fifth chapter - 88000 characters, the sixth chapter - 34000 characters). The list of used literature consists of 141 titles.

THE MAIN CONTENT OF THE DISSERTATION

The dissertation consists of an introduction, six chapters and a list of references.

The introduction substantiates the relevance of the topic of the dissertation work, provides a brief overview of the relevant issues and outlines the main results of the dissertation.

Chapter I is devoted to the definition of a coframe bundle and the construction of vertical and complete lifts in this bundle. The main results of this chapter are in the author's papers [2, 10, 11, 13-15, 32].

In Section 1.1 of Chapter I, a bundle of coframes of a smooth manifold is defined, information is given about local coordinates, their transformation formulas, global vector fields, and presymplectic structures in this bundle.

A coframe bundle $F^*(M)$ over a smooth manifold, M being a principal bundle, is the set of coframes $u^* = (X^1,..., X^n)$ at all points $x \in M$ of this manifold, whthe coframe u^* is a basis of the cotangent space $T^*_x(M)$. The projection $\pi : F^*(M) \to M$ of the coframe bundle $F^*(M)$ for $\forall(x,u^*) \in F^*(M)$ acts according to the rule $\pi(x,u^*) = x \in M$. $F^*(M)$ is a $n+n^2$-dimensional smooth manifold. If a local coordinate system $\{U, x^i\}$ is given on a manifold M, then a local coordinate system $\{\pi^{-1}(U),(x^i, X^\alpha_i)\}$ is defined in the coframe bundle $F^*(M)$. Local coordinates are transformed according to the rule

$$\begin{cases} x^{i'} = x^{i'}(x^i), \\ X^\alpha_{i'} = A^i_{i'} X^\alpha_i, \end{cases}$$

where $i,\alpha = 1,2,..., n, i_\alpha = n+1,..., n+n^2.$

In Section 1.2 vertical lifts of a function and a 1-form to a coreper bundle are con structed.

Let f be a diffrentiable function on a smooth manifold M, i.e., $f \in \Im^0_0(M)$. The vertical lift of a function f to a coframe bundle $F^*(M)$ is a function defined by the rule

$$^V f = f \circ \pi .$$

Let $\omega \in \Im^0_1(M)$ be a 1-form and $\forall Y \in \Im^1_0(M)$ be a vector field.

Definition 1. *Vertical lifts of a 1-form* ω *to a coframe bundle* $F^*(M)$ *are vector fields* $^{V_\lambda}\omega$ *such that for* $\forall Y \in \Im^1_0(M)$,

$$^{V_\lambda}\omega = (i^\mu Y) = \omega(Y)\delta^\lambda_\mu$$

where $\lambda, \mu = 1,2,..., n$ *and* δ^λ_μ *is the Kronecker delta.*

Note that $i^\mu Y = X^\mu_m Y^m$ is a function on the coframe bundle $F^*(M)$, acting according to the rule $(I^\mu y)(x,u^*) = X^\mu(Y)$.

It is established that vertical lifts $^{V_\lambda}\omega$ have components

$$^{V_\lambda}\omega = \begin{pmatrix} 0 \\ \omega_i \delta_\beta^\lambda \end{pmatrix}$$

in the local coordinate system (x^i, X_i^β).

Lemma 1. *For arbitrary 1-forms (covector fields), $\omega, \theta \in \Im_1^0(M)$ the equality* $\left[{}^{V_\beta}\omega, {}^{V_\gamma}\theta \right] = 0$

Is true, where, $[,]$ *is the commentator (bracket Li) of vector fields.*

In Section 1.3 we consider the question of the complete lift of vector fields. A complete lift ${}^C V \in \Im_0^1(F^*(M))$ to the coframe bundle $F^*(M)$ of a vector field V is a vector field defined as

$${}^C V(i^\mu Y) = i^\mu (L_V Y) = X_m^\mu (L_V Y)^m$$

for all vector fields $\forall Y \in \Im_0^1(M)$,

It is established that the complete lift ${}^C V$ of the vector field $\forall V \in \Im_0^1(M)$ to the coframe bundle $F^*(M)$ has the components

$${}^C V = \begin{pmatrix} {}^C V^k \\ V^{k_\mu} \end{pmatrix} = \begin{pmatrix} V^k \\ -X_m^\mu \partial_k V^m \end{pmatrix}$$

in the local coordinate system (x^k, X_k^μ). where $L_V Y$ is the Lie derivative along the vector field V.

Theorem 2. *For arbitrary two vector fields $V, W \in \Im_0^1(M)$ the equality*

$${}^C[V,W] = [{}^C V, {}^C W]$$

is true. .

Lemma 3. *(a) Assume that $\tilde{X}, \tilde{Y}$ $F^*(M)$ are vector fields on a coframe bundle $F^*(M)$ such that for any function f and a vector field Z on M,*

$$\tilde{X}\, {}^V f = \tilde{Y}\, {}^V f, \quad \tilde{X}({}^V\gamma^\alpha Z) = \tilde{Y}({}^V\gamma^\alpha Z), \quad 1 \le \alpha \le n.$$

Then $\tilde{X} = \tilde{Y}$..

(b) Let $\tilde{S}, \tilde{T}$ – ensor fields of type $(r,s), s > 0$ on the

coframe bundle $F^*(M)$ *be such that for arbitrary vector fields* $X_1,\dots,X_s$ *on* M *,*

$$\tilde{S}({}^C X_1,\dots,{}^C X_s)=\tilde{T}({}^C X_1,\dots,{}^C X_s).$$

Then $\tilde{S}=\tilde{T}.$

In Section 1.4, complete lifts of tensor fields of type $(1,p)$ are defined and their properties are studied. It is established that the complete lift ${}^C\varphi$ of the affinor field $\varphi=\varphi^i_j\partial_i\otimes dx^j$ to the coframe bundle $F^*(M)$ has the components

$$ {}^C\varphi^i_j=\varphi^i_j, \qquad {}^C\varphi^i_{j_\beta}=0,$$
$$ {}^C\varphi^{i_\alpha}_j=X^\alpha_k(\partial_j\varphi^k_i-\partial_i\varphi^k_j), \qquad {}^C\varphi^{i_\alpha}_{j_\beta}=\delta^\alpha_\beta\varphi^j_i.$$

It is established that the complete lift ${}^CS$ of the skew-symmetric tensor field S of type $(1,2)$ given on a smooth manifold M to the coframe bundle $F^*(M)$ has the components

$$ {}^CS^k_{ij}=S^k_{ij},\quad {}^CS^{k_\gamma}_{ij}=-X^\gamma_m(\partial_i S^m_{jk}+\partial_j S^m_{ki}+\partial_k S^m_{ij}),\quad {}^CS^{k_\gamma}_{i_\alpha j}=\delta^\gamma_\alpha S^i_{kj},$$
$$ {}^CS^{k_\gamma}_{ij_\beta}=\delta^\gamma_\beta S^j_{ik},\quad {}^CS^k_{i_\alpha j}={}^CS^k_{ij_\beta}={}^CS^k_{i_\alpha j_\beta}={}^CS^{k_\gamma}_{i_\alpha j_\beta}=0$$

in the local coordinate system (x^k, X^μ_k).

In a similar way it turns out that the complete lift ${}^CS$ of a vector-valued $p-$forn S given on a smooth manifold M to the coframe bundle $F^*(M)$ has the components

$$ {}^SS^k_{ij\cdot l}=S^k_{ij\cdot l},$$
$$ {}^CS^{k_\gamma}_{ij\cdot l}=X^\gamma_m(\partial_i S^m_{kj\cdot l}+\partial_j S^m_{ik\cdot l}+\cdot+\partial_l S^m_{ij\cdot k}-\partial_k S^m_{ij\cdot l}),$$
$$ {}^CS^{k_\gamma}_{i_\alpha j\cdots l}=\delta^\gamma_\alpha S^i_{kj\cdots l},\ {}^CS^{k_\gamma}_{ij_\beta\cdots l}=\delta^\gamma_\beta S^j_{ik\cdots l},\dots,{}^CS^{k_\gamma}_{ij\cdots l_\lambda}=\delta^\gamma_\lambda S^l_{ij\cdots k}.$$

Section 1.5 considers questions related to complete lifts of derivations.

Definition 2. *A derivation of an algebra* $\Im(M)$ *is a mapping* $D:\Im(M)\to\Im(M)$ *that satisfies the following conditions:*

(a) D *linear mapping with respect to a constant coefficient;*

(b) For all numbers r, s , $D(\Im^r_s(M)) \subset \Im^r_s(M)$;

(c) For all tensor fields $S, T \in \Im(M)$ *(d),*

$$D(S \otimes T) = DS \otimes T + S \otimes DT;$$

(d) The mapping D *commutes with each convolution of the tensor field.*

The complete lift of differentiation $D = L_X + i_\varphi$ to the coframe bundle $F^*(M)$ is defined in the form ${}^C D = L_{{}^C X} + i_{{}^C\varphi}$.

Theorem 4. *The mapping* $D \to {}^C D$ *is a homomorphism of the Lie algebra* $D(M)$ *onto the Lie algebra* $D(F^*(M))$.

In Section 1.6 we study complete lifts of almost complex structures and $f-$structures.

Theorem 5. *Suppose that* φ *is an almost complex structure given on a smooth manifold* M *. Then the complete lift* ${}^C\varphi$ *is an almost complex structure on the coframe bundle* $F^*(M)$ *if and only if* φ *is integrable.*

According to the definition introduced by K.Yano, a $f-$ structure on an $n = 2m$dimensional manifold M is a tensor field f of type (1,1) such that the relation

$$f^3 + f = 0$$

is satisfied and the rank of the affinor field f on the manifold M is everywhere equal to the number r ю

A necessary and sufficient condition is established for determining the $f-$structure of the complete lift ${}^C\varphi$ of the f – structure φ given on the manifold M to the coframe bundle $F^*(M)$.

Chapter II is devoted to the construction of horizontal lifts of differential-geometric structures in a coframe bundle. The main results of this chapter were published in the author's papers [2, 10, 13, 14, 18, 24, 30].

In Section 2.1 we consider the question of the horizontal lift of a vector field.

A horizontal lift of a vector field V to the coframe bundle $F^*(M)$ is a vector field ${}^H V \in \Im_0^1(F^*(M))$ defined in the form

$$ {}^H V(i^\mu Y) = i^\mu (\nabla_V U) = X^\mu_m (\nabla_V Y)^m $$

for all vector fields $Y \in \Im_0^1(M)$, where ∇_V is the covariant derivative along the vector field V.

It is proved that a horizontal lift ${}^H V$ of an arbitrary vector field $V \in \Im_0^1(M)$ to the coframe bundle $F^*(M)$ has components

$$ {}_H V = \begin{pmatrix} V^k \\ X^\mu_m \Gamma^m_{lk} V^l \end{pmatrix} $$

with respect to the local coordinates (x^k, X^μ_k).

Theorem 6. *For arbitrary two vector fields* $V, W \in \Im_0^1(M)$, *an arbitrary 1-form (covector field)* $\omega \in \Im_1^0(M)$ *and* , $\forall A \in GL(n;R)$,

a) $[{}^H V, {}^{V\beta}\omega] = {}^{V\beta}(\nabla_V \omega)$,

b) $[{}^H V, {}^H W] = {}^H[V,W] + \gamma(R(V,W))$,

c) $[{}^H V, \lambda A] = 0$,

where $\lambda A = Y = A^\beta_\alpha X^\alpha_j \partial_{j_\beta}$.

Section 2.2 is devoted to the construction of horizontal lifts of tensor fields of type . $(1, p)$. It is proved that the horizontal lift ${}^H\varphi$ of an affinor field φ, given on a smooth manifold M, to the coframe bundle $F^*(M)$ has components

$$ ({}^H\varphi^I_J) = \begin{pmatrix} \varphi^i_j & 0 \\ -X^\alpha_m \Gamma^m_{jl} \varphi^l_i + X^\alpha_m \Gamma^m_{il} \varphi^l_j & \delta^\alpha_\beta \gamma^j_i \end{pmatrix} $$

with respect to local coordinates (x^k, X^μ_k), where φ^i_j are components of the affinor field φ on the manifold M, Γ^k_{ij} are the coefficients of the affine connection ∇.

The horizontal lift ${}^H S$ of vector-valued 2-form S given on a smooth manifold M to the coframe bundle $F^*(M)$ is defined as

$$^{H}S = {}^{C}S + \gamma(\nabla S)$$

where $\gamma(\nabla S)$ is the tensor field of type $(1,2)$ on the coframe bundle $F^*(M)$ with nonzero components

$$\gamma(\nabla S)_{ij}^{k_\gamma} = X_m^\gamma (\nabla_i S_{kj}^m + \nabla_j S_{ik}^m - \nabla_k S_{ij}^m).$$

In a similar way, using relation

$$^{H}S = {}^{C}S + \gamma(\nabla S),$$

the horizontal lift ^{H}S of the vector-valued $p-$form S to the coframe bundle $F^*(M)$ is determined, where ^{C}S is the complete lift of the vector-valued $p-$form S to the coframe bundle $F^*(M)$ and $\gamma(\nabla S)$ is the tensor field of type $(1,p)$ on the coframe bundle $F^*(M)$ with nonzero components

$$\gamma(\nabla S)_{i_1 \cdots i_p}^{k_\gamma} = X_m^\gamma (\nabla_{i_1} S_{k i_2 \cdots i_p}^m + \cdots + \nabla_{i_p} S_{i_1 \cdots i_{p-1} k}^m - \nabla_k S_{i_1 \cdots i_p}^m).$$

In Section 2.3 we define an adapted to connection ∇ frame $\{D_J\} = \{D_j, D_{j_\beta}\}$ of the coframe bundle . $F^*(M)$. It is established that the horizontal lift ^{H}V and vertical lifts $^{V_\alpha}\omega$, respectively, of the vector field V and 1-form ω have components

$$^{H}V = \begin{pmatrix} V^\alpha \\ 0 \end{pmatrix}$$

and

$$^{V_\alpha}\omega = \begin{pmatrix} 0 \\ \omega_i \delta_\alpha^\beta \end{pmatrix}$$

цith respect to the adapted frame $\{D_J\}$,

It is shown that the object of nonholonomy $\Omega_{IJ}{}^K$ of the adapted frame $\{D_J\}$ has nonzero components in the form

$$\begin{cases} \Omega_{ij_\beta}{}^{k_\gamma} = -\Omega_{j_\beta i}{}^{k_\gamma} = -\delta_\beta^\gamma \Gamma_{ik}^j, \\ \Omega_{ij}{}^{k_\gamma} = X_m^\gamma R_{ijk}^m. \end{cases}$$

In Section 2.4 we study horizontal lifts of almost complex structures and $f-$structures.

Theorem 7. *Assume that on an* $-n=2m-$*dimensional differentiable manifold* M *with a symmetric affine connection* ∇, *an* $f-$*structure* F *of rank* r *is given. Then the horizontal lift* ${}^{H}F$ *is a* $f-$*structure which has rank* $r(1+n)$ *in the coframe bundle* $F^{*}(M)$.

Section 2.5 considers the problem of constructing a horizontal lift of an affine connection.

Definition 5. *A horizontal lift of a symmetric affine connection* ∇ *to a coframe bundle* $F^{*}(M)$ *is an affine connection* ${}^{H}\nabla$ *defined in the form*

$$ {}^{H}\nabla_{{}^{H}X}{}^{H}Y={}^{H}(\nabla_{X}Y), \qquad {}^{H}\nabla_{{}^{H}X}{}^{V\beta}\theta={}^{V\beta}(\nabla_{X}\theta), $$

$$ {}^{H}\nabla_{{}^{V\alpha}\omega}{}^{H}Y=0, \qquad {}^{H}\nabla_{{}^{V\alpha}\omega}{}^{V\beta}\theta=0 $$

for arbitrary vector fields $X,Y\in\Im_{0}^{1}(M)$ *and 1-forms (covector fields)* $\omega,\theta\in\Im_{1}^{0}(M)$.

Theorem 8. *The horizontal lift* ${}^{H}\nabla$ *of a symmetric affine connection* ∇, *defined on a smooth manifold* M *to the coreframe bundle* $F^{*}(M)$ *has components (coefficients)*

$$ {}^{H}\Gamma_{ij}^{l}=\Gamma_{ij}^{l}, \quad {}^{H}\Gamma_{ij}^{l_\sigma}={}^{H}\Gamma_{ij_\beta}^{l}=0, \quad {}^{H}\Gamma_{ij_\beta}^{l_\sigma}=-\delta_{\beta}^{\sigma}\Gamma_{il}^{j}, $$

$$ {}^{H}\Gamma_{i_\alpha j}^{l}={}^{H}\Gamma_{i_\alpha j}^{l_\sigma}={}^{H}\Gamma_{i_\alpha j_\beta}^{l}={}^{H}\Gamma_{i_\alpha j_\beta}^{l_\sigma}=0 $$

with respect to the adapted frame $\{D_I\}=\{D_i,D_{i_\alpha}\}$.

It is proved that the horizontal lift ${}^{H}\nabla$ of a symmetric affine connection ∇ given on a smooth manifold M, to the coframe bundle $F^{*}(M)$ has components (coeficients)

$$ {}^{H}\Gamma_{ij}^{k}=\Gamma_{ij}^{k}, \quad {}^{H}\Gamma_{ij}^{k_\gamma}=X_{m}^{\gamma}(\Gamma_{kl}^{m}\Gamma_{ij}^{l}+\Gamma_{ik}^{l}\Gamma_{lj}^{m}-\partial_{i}\Gamma_{kj}^{m}), $$

$$ {}^{H}\Gamma_{i_\alpha j}^{k}=-\delta_{\alpha}^{\gamma}\Gamma_{jk}^{i}, \quad {}^{H}\Gamma_{ij_\beta}^{k_\gamma}=-\delta_{\beta}^{\gamma}\Gamma_{ik}^{j}, \quad {}^{H}\Gamma_{i_\alpha j}^{k}=0, $$

$$ {}^{H}\Gamma_{ij_\beta}^{k}=0, \quad {}^{H}\Gamma_{i_\alpha j_\beta}^{k}=0, \quad {}^{H}\Gamma_{i_\alpha j_\beta}^{k_\gamma}=0 $$

with respect to the natural frame $\{\partial_J\} = \left\{ \frac{\partial}{\partial x^j}, \frac{\partial}{\partial X_j^{\beta}} \right\}$

In Section 2.6 we study the question of the complete lift of a symmetric affine connection to a coframe bundle..

It is proved that the complete lift ${}^C\nabla$ of the symmetric affine connection ∇ to the coframe bundle $F^*(M)$, for $\forall X,Y \in \Im_0^1(M)$ satisfies the relation

$${}^C\nabla_{{}^C X}{}^C Y = {}^C(\nabla_X Y) + \gamma(\tilde{L}),$$

where $\gamma(\tilde{L})$ is a vertical vector field on the coframe bundle $F^*(M)$ such that

$$\gamma(\tilde{L}) = \begin{pmatrix} 0 \\ X_l^{\gamma}[(\nabla_k X^i \nabla_i Y^l + \nabla_k Y^i \nabla_i X^l) + X^i Y^m (R^l_{kim} + R^l_{kmi})] \end{pmatrix}$$

and R is the curvature tensor of the affine connection ∇.

The following expressions of non-zero components (coefficients) of the complete lift ${}^C\nabla$ with respect to the natural frame $\{\partial_J\} = \left\{ \frac{\partial}{\partial x^j}, \frac{\partial}{\partial X_j^{\beta}} \right\}$ were calculated:

$${}^C\Gamma_{ij}^k = \Gamma_{ij}^k, \qquad {}^C\Gamma_{i_\alpha j}^{k_\gamma} = -\delta_\alpha^\gamma \Gamma_{kj}^j, \qquad {}^C\Gamma_{ij_\beta}^{k_\gamma} = -\delta_\beta^\gamma \Gamma_{ik}^j,$$

$${}^C\Gamma_{ij}^{k_\gamma} = X_m^\gamma(\Gamma_{kl}^m \Gamma_{ij}^l + \Gamma_{ik}^l \Gamma_{lj}^m - \partial_i \Gamma_{kj}^m) - X_m^\gamma(\partial_j \Gamma_{ki}^m - \partial_k \Gamma_{ji}^m + \Gamma_{ki}^l \Gamma_{jl}^m -$$

$$-\Gamma_{ji}^l \Gamma_{kl}^m) = X_m^\gamma(\partial_k \Gamma_{ji}^m - \partial_j \Gamma_{ki}^m - \partial_i \Gamma_{kj}^m + 2\Gamma_{ij}^l \Gamma_{kl}^m).$$

In paragraph 2.7. In **Chapter II** we study geodesic curves of horizontal and complete lifts of an affine connection to a coframe bundle $F^*(M)$.

Theorem 9. *The geodesic curve* $\tilde{C}$ *of a horizontal lift* ${}^H\nabla$ *of a symmetric affine connection* ∇ *defined on a smooth manifold* M *to the coframe bundle* $F^*(M)$ *has the equations*

$$\begin{cases} \frac{d^2x^k}{dt^2} + \Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0 \\ \frac{\delta^2 X^\gamma_k}{dt^2} = 0 \end{cases}$$

with respect to the induced local coordinates $(\pi^{-1}(U), x^i, X^\alpha_i)$

Theorem 10. *Assume that a torsion-free (symmetric) m-affine connection* ∇ *defined on a smooth manifold* M *and* $\overline{C}(t) = (C(t), X^\gamma_k(t))$ *is a curve on the coframe bundle c. In order for the curve* $\overline{C}(t)$ *to be a geodesic line of a complete lift* ${}^C\nabla$, *it is necessary and sufficient that the following conditions be satisfied:*

1) The curve $C(t)$ is a geodesic curve with respect to the affine connection ∇ on M;

2) Each covector field $X^\gamma_k(t)$ *along the curve* $C(t)$ *csatisfies the relation*

$$\frac{\delta}{dt}\left(\frac{\delta X^\gamma_k}{dt}\right) + R^m_{kji}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0..$$

In Section 2.8 of **Chapter II** we study the properties of horizontal lifts of derivations.

The horizontal lift ${}^H D$ of the derivation $D = L_X + i_\varphi$ of the algebra $\Im(M)$ to the coframe bundle m $F^*(M)$ is defined in the form

$${}^H D = L_{{}^H X} + i_{{}^H \varphi}.$$

Theorem 11. *(a) The mapping* $i_\varphi \to i_{{}^H\varphi}$ *is a homomorphism from the algebra* $E(M)$ *onto the algebra* $E(F^*(M))$*;*

(b) If ∇ *is a flat connection, then the mapping* $L_X \to L_{{}^H X}$ *is a homomorphism of the Lie algebra* $L(M)$ *onto the Lie algebra* $L(F^*(M))$.

Chapter III of the dissertation work is devoted to the study of the behavior of lifts of differential-geometric structures on sections of a bundle of coframes. The main results of this chapter were published in the author's papers [4, 5, 30].

In Section 3.1 of **Chapter III**, we study the behavior of complete and horizontal lifts of vector portleys to a coframe bundle on sections of this bundle. It is assumed that an M $n-$-dimensional parallelizable manifold. This means that a global section is defined on the coframe bundle $F^*(M)$. Then the section $\sigma : M \to F^*(M)$ on the manifold M defines a global field of coframes such that at every point $x \in M$, $\sigma(x) = (\sigma^1(x),...,\sigma^n(x))$ is a coframe at x. If we denote $\sigma = (\sigma^1,...,\sigma^n)$, then each σ^α is a covector field (1-form) globally defined on a smooth manifold M. A covector field σ^α with respect to the local coordinate system (U, x^i) has a decomposition $\sigma^\alpha = \sigma^\alpha_h(x)dx^h$. Therefore the section $\sigma(M)$ defined by σ is a $n-$ dimensional submanifold of the coframe bundle $F^*(M)$ and is defined in the form

$$\begin{cases} x^h = x^h, \\ X_{\alpha h} = \sigma_{\alpha h}(x). \end{cases}$$

Vector fields B_i and C_{i_α} with components

$$(B_i^H) = \begin{pmatrix} \delta_i^h \\ \partial_i \sigma_h^\alpha \end{pmatrix}$$

and

$$(C^H_{i^\alpha}) = (\partial_{i_\alpha} x_H) = \begin{pmatrix} 0 \\ \delta_h^i \delta_\beta^\alpha \end{pmatrix},$$

respectively, along the section $\sigma(M)$ form the frame $E_I = \left(B_i, C_{i_\alpha}\right)$.

It is proved that the completew lift ${}^CV$ and the horizontal lift ${}^HV$ of an arbitrary vector field $V \in \Im_0^1(M)$ to the coframe bundle $F^*(M)$, along the section $\sigma(M)$ with respect to the frame (B, C), respectively, have expansions

$${}^CV = V^h B_h + (-L_V \sigma_h^\alpha) C_{h_\alpha}$$

and

$$ {}^{H}V = V^{h}B_{h} + (-\nabla_{V}\sigma^{\alpha}_{h})C_{h_\alpha}\,. $$

In Section 3.2 we study the question of the behavior of the complete and horizontal lifts of tensor fields of type $(1, p)$ to the coframe bundle on sections.

Theorem 12. *The complete lift* ${}^{C}\varphi$ *of an arbitrary tensor field* $\varphi \in \Im^1_1(M)$ *along the section* $\sigma(M)$ *with respect to the frame* (B,C) *has nonzero components*

$$ {}^{C}\tilde{\varphi}^{i}_{j} = \varphi^{i}_{j},\ {}^{C}\tilde{\varphi}^{i_\gamma}_{j} = -(\Phi_{\varphi}\sigma^{\gamma})_{ji}\,,\ \ {}^{C}\tilde{\varphi}^{i_\gamma}_{j_\beta} = \delta^{\gamma}_{\beta}\varphi^{j}_{i} $$

where $\Phi_{\varphi}\sigma^{\gamma}$ *is the Tachibana operator applied to the covector field* σ^{γ}.

It is proved that the complete lift ${}^{C}S$ and horizontal lift ${}^{H}S$ of an arbitrary skew-symmetric tensor-field $S \in \Im^1_2(M)$ to the coframe bundle $F^{*}(M)$, along the section $\sigma(M)$ in the frame (B,C) have non-zero components

$$ {}^{C}S^{k}_{ij} = S^{k}_{ij},\quad {}^{C}S^{k_\gamma}_{ij} = -(\Phi_{S}\sigma^{\gamma})_{ijk}\,, $$
$$ {}^{C}\tilde{S}^{k}_{i_\alpha j} = \delta^{\gamma}_{\alpha}S^{i}_{kj},\quad {}^{C}\tilde{S}^{k_\gamma}_{ij_\beta} = \delta^{\gamma}_{\beta}S^{j}_{ik} $$

and

$$ {}^{H}\tilde{S}^{k}_{ij} = S^{k}_{ij},\quad {}^{H}\tilde{S}^{k_\gamma}_{ij} = -(\tilde{\Phi}_{S}\sigma^{\gamma})_{ijk}, $$
$$ {}^{H}\tilde{S}^{k}_{i_\alpha j} = \delta^{\gamma}_{\alpha}S^{i}_{kj},\quad {}^{H}\tilde{S}^{k_\gamma}_{ij_\beta} = \delta^{\gamma}_{\beta}S^{j}_{ik}, $$

where $\Phi_{S}\sigma^{\gamma}$ and $\tilde{\Phi}_{S}\sigma^{\gamma}$ the generalized Yano-Ako operator of the I kind and Vishnevskii operator applied to a covector field.

In Section 3.3 of **Chapter III**, we study the problem of the behavior of the complete lift of a symmetric affine connection on the sections of coframe bundle. For an affine connection $\tilde{\nabla}$ generated by the complete lift ${}^{C}\nabla$ on cross-section $\sigma(M)$ with respect to the adapted frame (B,C), the Gauss formula

$$ {}^{C}\nabla_{B_i}B_{j} = \tilde{\nabla}_{B_i}B_{j} + (\nabla_{i}\nabla_{j}\sigma^{\beta}_{m} + \sigma^{\beta}_{l}R^{l}_{mji})C_{m_\beta} $$

is determined. In the Gauss formula, term

$$(\nabla_i \nabla_j \sigma^{\beta}_m + \sigma^{\beta}_l R^l_{mji}) C_{m_\beta}$$

is the second main form of section $\sigma(M)$.

. In section 3.4 of Chapter III we study the behavior of lifts of almost complex structures on sections of the coframe bundle.

Chapter IV of the dissertation work is devoted to the study of some important differential-geometric structures on the coframe bundle of a Riemannian manifold. The main results of this chapter were published in the author's papers [2, 7, 8, 12, 14, 16, 19, 20, 25, 27, 28, 32].

In Section 4.1 of **Chapter IV**, the diagonal lift of the Riemannian metric (Sasaki metric) in the coframe bundle is defined.

Definition 4. *The Sasaki metric, or diagonal lift of a Riemannian metric* g *in a coframe bundle* $F^*(M)$ *over a Riemannian manifold* (M,g) *is a differential-geometric structure* ${}^D g$ *defined in the form*

$$^D g = g_{ij} \tilde{\eta}^i \otimes \tilde{\eta}^j + \delta_{\alpha\beta} \sum_{i,j=1}^{n} g^{ij} \tilde{\eta}^{i_\alpha} \otimes \tilde{\eta}^{j_\beta},$$

where $\{\tilde{\eta}^J\}$ *is the coframe conjugate to the adapted frame* $\{D_I\}$ *and* g^{ij} *are the contravariant components of the Rmanian metric* g*, i.e.* $g_{ij} g^{jk} = \delta_i^k$.

It is established that the diagonal lift ${}^D g$ in an invariant form can be defined as

$$^D g({}^H X, {}^H Y) = {}^V(g(X,Y)) = g(X,Y),$$

$$^D g({}^H X, {}^{V_\beta}\theta) = 0,$$

$$^D g({}^{V_\alpha}\omega, {}^{V_\beta}\theta) = \delta_{\alpha\beta} {}^V(g^{-1}(\omega,\theta)) = \delta_{\alpha\beta} g^{-1}(\omega,\theta).$$

for $\forall X,Y \in \Im_0^1(M)$ and $\forall \omega,\theta \in \Im_1^0(M)$, therefore the diagonal lift ${}^D g$ belongs to the class of natural metrics.

It is shown that the diagonal lift ${}^D g$ has a matrix of components in the form

$$\begin{pmatrix} g_{ij} & 0 & \cdot & 0 \\ 0 & g^{ij} & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & g^{ij} \end{pmatrix}$$

with respect to the adapted frame $\{D_I\}$..

In Section 4.2 of **Chapter IV**, the properties of Levi-Civita connection of the diagonal lift ${}^D g$ are studied.

Theorem 13. *Assume that* (M, g) *is a Riemannian manifold,* ${}^D\nabla$ *is the Levi-Civita connection of the coframe bundle* $F^*(M)$ *with the Sasaki metric* ${}^D g$. *Then the affine connection* ${}^D\nabla$ *satisfies the following conditions for arbitrary vector fields* $X, Y \in \Im^1_0(M)$, *and 1-forms (covector fields)* $\omega, \theta \in \Im^0_1(M)$ *:*

i) $${}^D\nabla_{{}^H X}{}^H Y = {}^H(\nabla_X Y) + \frac{1}{2}\sum_{\sigma=1}^{n} {}^{V\sigma}(X^\sigma \circ R(X, Y)),$$

ii) $${}^D\nabla_{{}^H X}{}^{V\beta}\theta = {}^{V\beta}(\nabla_X \theta) + \frac{1}{2}{}^H(X^\beta(g^{-1} \circ R(\quad, X)\tilde{\theta})),$$

iii) $${}^D\nabla_{{}^{V\alpha}\omega}{}^H Y = \frac{1}{2}{}^H(X^\alpha(g^{-1} \circ R(\quad, Y)\tilde{\omega})),$$

iv) $${}^D\nabla_{{}^{V\alpha}\omega}{}^{V\beta}\theta = 0,$$

where $\tilde{\omega} = g^{-1} \circ \omega \in \Im^1_0(M), \tilde{\theta} = g^{-1} \circ \theta \in \Im^1_0(M)$, R *is the curvature tensor.*

Telrema 14. *Let m be* (M, g) *Riemannian manifold,* ${}^D\nabla$ *be the Levi-Civita connection of the cjframe bundle* $F^*(M)$ *with the Sasaki metric* ${}^D g$. *Then the coefficients* ${}^D\Gamma^K_{iIJ}$ *of the Levi-Civita connection* ${}^D\nabla$ *have the following values for different indices:*

$${}^D\Gamma^k_{ij} = \Gamma^k_{ij}, \qquad {}^D\Gamma^{k\gamma}_{ij} = \frac{1}{2} X^\gamma_m R^m_{ijk}, \qquad {}^D\Gamma^k_{i_\alpha j_\tau} = 0, \qquad {}^D\Gamma^{k_\gamma}_{i_\alpha j_\beta} = 0.$$

$$ {}^D\Gamma^{k}_{ij_\beta} = \frac{1}{2} X^{\beta}_{m} R^{k}{}_{\cdot i \cdot}{}^{jm}, \qquad {}^D\Gamma^{k_\gamma}_{ij_\beta} = -\delta^{\beta}_{\gamma}\Gamma^{j}_{ik}, \qquad {}^D\Gamma^{k}_{i_\alpha j} = \frac{1}{2} X^{\alpha}_{m} R^{k}{}_{\cdot j \cdot}{}^{im}, $$

$$ {}^D\Gamma^{k_\gamma}_{i_\alpha j} = 0. $$

In Section 4.3 of **Chapter IV** we study the problem of the curvature properties of the diagonal lift ${}^D g$.

The values of the components

$$ {}^D R(D_I, D_J)D_K = {}^D\nabla_I {}^D\nabla_J D_K - {}^D\nabla_J {}^D\nabla_I D_K - \Omega_{IJ}{}^{L}\, {}^D\nabla_L D_K $$

of the curvature tensor field ${}^D R$ of Levi-Civita connection ${}^D\nabla$ of the diagonal lift (Sasaki metrics) ${}^D g$ relative to the adapted frame $\{D_I\}$ are calculated for different indices.

Theorem 15. *Suppose that* (M, g) *is a Riemannian manifold,* $F^*(M)$ *is a coframe bundle with the Sasaki metric* ${}^D g$ *, and* r *and* ${}^D r$ *are the scalar curvatures of the metrics* g *and* ${}^D g$ *, respectively. Then the equality*

$$ {}^D r = r - \tfrac{1}{4}\sum_{\delta=1}^{n} \left| X^{\delta} R \right|^2 $$

is true, where

$$ \left| X^{\delta} R \right|^2 = g^{ij} g^{kt} g^{pq} (X^{\delta} R)_{kip} (X^{\delta} R)_{tjq}. $$

In Section 4.4 of **Chapter IV**, the geodesic curves of the diagonal lift nt are studied.

Theorem 16. *Assume that* $\tilde{C}$ *is a geodesic of Levi-Civita connection* ${}^D\nabla$ *of the Sasaki metric* ${}^D g$ *on the coframe bundle* $F^*(M)$*..Then the covector fields* $X^{\beta}_{h}(t), \beta = 1,2,...,n,$ *defined along the line* $C = \pi \circ \tilde{C}$ *satisfy the equations*

$$ \frac{\delta^2 x^h}{dt^2} + \sum_{\gamma=1}^{n} X^{\gamma}_{m} R^{h}{}_{\cdot k \cdot}{}^{jm} \frac{\delta X^{\gamma}_{j}}{dt} \frac{dx^k}{dt} = 0 $$

and their second-order covariant derivatives are equal to zero.

Theorem 17. *Any horizontal lift of a geodesic line of a Riemannian manifold* (M, g) *is a geodesic line on the coframe* *bundle $F^*(M)$ with the Sasaki metric ${}^Dg$.*

In Section 4.5 of **Chapter IV** we consider f – structures of different ranks defined in a coframe bundle.

In the coframe bundle $F^*(M)$ of an $n=2m$-dimensional Riemannian manifold (M,g), with the help of the relations

$$F_\alpha(D_i)=-D_{i_\alpha}, \quad F_\alpha(D_{i_\beta})=\delta_\alpha^\beta D_i$$

n tensor fields of type $(1,1)$ (affinor fields) $F_\alpha, 1\le\alpha\le n,$ are defined, where $\{D_I\}$ is an adapted frame.

Theorem 18. *On the coframe bundle $F^*(M)$, tensor fields $F_\alpha, 1\le\alpha\le n,$ of type $(1,1)$ are f – structures of rank $2n$.*

With the help of the relations

$$\tilde F(D_i)=t_\beta D_{i_\beta}, \qquad \tilde F(D_{i_\beta})=-t_\beta D_i$$

on the coframe bundle $F^*(M)$, *f – structure* $\tilde F$ of rank $2n$ is defined, where $\sum_{\beta=1}^{n} t_\beta^2=1$ and $t_\beta D_{i_\beta}=t_1 D_{i_1}+t_2 D_{i_2}+\cdots+t_n D_{i_n}$.

With the help of the relations

$$\begin{cases} \bar F(D_i)=0, \\ \bar F(D_{i_\alpha})=D_{i_{2m+1-\alpha}}, \ if \quad \alpha=1,2,\dots,m \\ \bar F(D_{i_\alpha})=-D_{i_{2m+1-\alpha}}, \ if \quad \alpha=m+1,\dots,2m \end{cases}$$

on the coframe bundle $F^*(M)$, the f – structure $\bar F$ of rank n^2 is defined.

It is proved that the f – structures $F_\alpha, 1\le\alpha\le n,$ and $\tilde F$ of rank 2n, as well as the f – structure $\bar F$ of rank n^2, are integrable when the base manifold (M,g) is a Euclidean space.

. **In Section 4.6** of **Chapter IV** we study the relations between the diagonal lift m of the Riemannian metric and f – structures.

Theorem 19. *On the coframe bundle $F^*(M)$, the diagonal lift ${}^Dg$ is adapted with each of the f –-structures $F_\alpha, 1\le\alpha\le n,$ of rank*

$2n$ ..

It is established that in the general case the f – structure F of rank n^2, the f – structure $\tilde{F}$ of rank $2n$, and the f – structures $F_\alpha, 1 \leq \alpha \leq n,$ of rank $2n$, as its special cases, are not parallel with respect to the Levi-Civita connection ${}^D\nabla$.

In Section 4.7 of **Chapter IV,** on the coframe bundle of a pseudo-Riemannian manifold, g – lifts of certain differential-geometric structures are studied.

For a frame bundle $F(M)$ and a coframe bundle $F^*(M)$ of a pseudo-Riemannian manifold (M, g) , the canonical isomorphisms

$$g^1 : F(M) \to F^*(M)$$

and

$$g^2 : F^*(M) \to F(M)$$

are defined.

Theorem 20. *Assume that* (M, g) *is a pseudo-Riemannian manifold,* ${}^C V_{F(M)}$ *and* ${}^C V_{F^*(M)}$ *are complete lifts of the vector field* $V \in \Im_0^1(M)$, *respectively, to the frame bundle* $F(M)$ *and the coframe bundle* $F^*(M)$. *Then a necessary and sufficient condition for the coincidence under the mapping* g^1 *of the image of the complete lift* ${}^C V_{F(M)}$, *i.e.* (g, C) – *lift* $g^1_*({}^C V_{FM})$ *with completel lift* ${}^C V_{F^*(M)}$ *is that* V *Killing vector field.*

Theorem 21. *Suppose that* (M, g) *is a pseudo-Riemannian manifold,* ${}^C\underset{F(M)}{\tau}$ *is a complete lift of the* $1-$*form* $\tau \in \Im_1^0(M)$ *to the frame bundle* $F(M)$. *Then the* (g,C) – *lift of the* $1-$*form* τ *to the coreframe bundle* $F^*(M)$ *has the local expression*

$$g^2_*\left({}^C\underset{FM}{\tau}\right) = \underset{K}{\tilde{\tau}} = \sum_{\beta=1}^{n} \left\{ X^\beta_m (g^{jm} \partial_j \tau_k + \partial_k g^{im} \tau_i) dx^k + g^{ik} \tau_i dX^\beta_k \right\}.$$

In the case when the tensor field S of type $(0,2)$ is a pseudo-Riemannian metric g , then

$$g_*^2\left({}^C g_{FM}\right)=\begin{pmatrix}\sum_{\beta=1}^{n} X_m^{\beta}(-2\Gamma_{kl}^{m}) & \delta_k^l\sum_{\beta=1}^{n}\delta_{\sigma}^{\beta}\\ \delta_l^k\sum_{\beta=1}^{n}\delta_{\gamma}^{\beta} & 0\end{pmatrix},$$

(1) where $\overline{\overline{\Gamma}}_{ij}^{k}$ are the coefficients of the Levi-Civita connection of the pseudo-Riemannian metric S.

It is well known that in the cotangent bundle the Riemannian extension has components

$$\nabla_R = g = \begin{pmatrix} -2\rho_m\Gamma_{ji}^{m}) & \delta_j^i \\ \delta_i^j & 0\end{pmatrix} \qquad (2)$$

Comparison of equalities (1) and (2) leads to the conclusion that the $(g,C)-$lift of the pseudo-Riemannian metric g to the coframe bundle $F^*(M)$ is an analogue of the Riemannian extension ∇^R, but does not define the Riemannian metric (non-degeneracy is not satisfied).

In Section 4.8 of **Chapter IV**, a Sasaki lift of a homogeneous type of Riemannian metric is constructed in the coframe bundle of the Riemannian manifold. In the tangent bundle

$$\tilde{T}(M) = T(M)\backslash\{0\}$$

with dropped zeros, a homogeneous lift of the Riemannian metric is constructed by R. Miron, in the cotangent bundle

$$\tilde{T}^*(M) = T^*(M)\backslash\{0\}$$

with dropped zeros, by P.Stavre and L.Popescu, in the tensor bundle of type (1,1)

$$\tilde{T}_1^1(M) = T_1^1(M)\backslash\{0\}$$

with dropped zeros, by E.Peyghan, M.Nasrabadi and A.Tayebi. However, due to the fact that coframe bundle is the principal bundle, this bundle does not have zero sections. Taking advantage of this property, we define a Sasaki lift of homogeneous type in the coframe bundle itself.

The new homogeneous lift $\tilde{G}$ of the Riemannian metric g г in the bundle of corepers m , or the Sasaki lift of the homogeneous type is defined in the form

$$\tilde{G} = g_{ij} dx^i \otimes dx^j + \frac{1}{h} \delta_{\alpha\beta} g^{ij} \delta X_i^{\alpha} \otimes \delta X_j^{\beta}$$

where $\delta X_i^{\alpha} = dX_i^{\alpha} - \Gamma_{ki}^{m} X_m^{\alpha} dx^k$ and

$$h = \sum_{\alpha=1}^{n} \left\| X^{\alpha} \right\|^2 = \sum_{\alpha=1}^{n} g^{ij} X_i^{\alpha} X_j^{\alpha} = \sum_{\alpha=1}^{n} g^{-1}(X^{\alpha}, X^{\alpha}).$$

Theorem 22. *The following properties are true:*

1. The pair $(F^*(M), \tilde{G})$ is a Riemannian manifold that depends only on the metric g;

2. $\tilde{G}$ is homogeneous on the coframe bundle $F^*(M)$;

3. The distributions $\tilde{H}$ and $\tilde{V}$ are orthogonal with respect to the metric $\tilde{G}$, i.e. for $\forall \tilde{X}, \tilde{Y} \in \Im_0^1(F^*(M))$, $\tilde{G}(h\tilde{X}, v\tilde{Y}) = 0$, where $h\tilde{X} \in \tilde{H}, v\tilde{Y} \in \tilde{V}$.

On the coframe bundle $F^*(M)$, n tensor structures $\overset{*}{F}_{\beta}$, $\beta = 1,2,...,n,$ are defined:

$$\overset{*}{F}_{\beta}(D_i) = \sum_j \sqrt{h}\, g_{ij} \frac{\partial}{\partial X_j^{\beta}},$$

$$\overset{*}{F}_{\beta}(D_i^{\alpha}) = -\frac{1}{\sqrt{h}} \delta_{\alpha\beta}\, g^{ij} D_j.$$

Theorem 23. For each $\beta = 1,2,...,n,$ the tensor structure $\overset{*}{F}_{\beta}$ has the following properties:

1. $\overset{*}{F}_{\beta}$ is a tensor structure of type $(1,1)$ on the coframe bundle $F^*(M)$;

2. $\overset{*}{F}_{\beta}$ depends only on the metric g;

3. $\overset{*}{F}_{\beta}$ is 0–homogeneous on the fibers of the coframe bundle $F^*(M)$ with respect to vector fields.

Corollary 24. *For every $\beta = 1,2,...,n,$, the triple $(F^*(M_n), \tilde{G}, \overset{*}{F}_{\beta})$ is an almost Hermitian manifold.*

In Section 4.9 of **Chapter IV,** we study the Levi-Civita connection properties of a Sasaki lift of homogeneous type $\tilde{G}$. It is proved that the Levi-Civita connection $\tilde\nabla$ of the Sasaki lift $\tilde G$ relative to the adapted frame $\{D_I\}$ for $\forall X,Y\in\Im^1_0(M)$ and $\forall\omega,\theta\in\Im^0_1(M)$ has the properties

i) $\tilde\nabla_{{}^HX}{}^HY = {}^H(\nabla_X Y) + \frac{1}{2}\sum_{\sigma=1}^{n} {}^{V_\sigma}(X_\sigma \circ R(X,Y)),$

ii) $\tilde\nabla_{{}^HX}{}^{V_\beta}\theta = {}^{V_\alpha}(\nabla_X\omega) + \frac{1}{2h}\sum_{\sigma=1}^{n}\delta_{\beta\sigma}\,{}^H(X^\sigma \circ R(\ ,X)\tilde\theta),$

iii) $\tilde\nabla_{{}^{V_\alpha}\omega}{}^HY = \frac{1}{2h}\sum_{\sigma=1}^{n}\delta_{\alpha\sigma}\,{}^H(X^\sigma \circ R(\ ,Y)\tilde\omega),$

iv) $\tilde\nabla_{{}^{V_\alpha}\omega}{}^{V_\beta}\theta = -\tilde G({}^{V_\alpha}\omega,\sum_{\sigma=1}^{n}{}^{V_\sigma}X_\sigma)\,{}^{V_\beta}\theta - \tilde G({}^{V_\beta}\theta,\sum_{\sigma=1}^{n}{}^{V_\sigma}X_\sigma)\,{}^{V_\alpha}\omega +$

$+\tilde G({}^{V_\alpha}\omega,{}^{V_\beta}\theta)\sum_{\sigma=1}^{n}{}^{V_\sigma}X^\sigma,$

where $\tilde\omega = g^{-1}\circ\omega\in\Im^1_0(M_n)$.

It is established that coefficients $\tilde\Gamma^K_{IJ}$ of Levi-Civita connection $\tilde\nabla$ relative to the adapted frame $\{D_I\}$ for different indices have values of

$$\tilde\Gamma^k_{ij}=\Gamma^k_{ij},\qquad \tilde\Gamma^{k_\gamma}_{ij}=\frac{1}{2}X^\gamma_m R^m_{ijk},$$

$$\tilde\Gamma^k_{ij_\beta}=\frac{1}{2h}X^\beta_m R^{k\ \ jm}_{\cdot i\cdot},\qquad \tilde\Gamma^{k_\gamma}_{ij_\beta}=-\delta^\beta_\gamma\Gamma^j_{ik},$$

$$\tilde\Gamma^k_{i_\alpha j}=\frac{1}{2h}X^\alpha_m R^{k\ \ im}_{\cdot j\cdot},\qquad \tilde\Gamma^{k_\gamma}_{i_\alpha j}=\tilde\Gamma^k_{i_\alpha j_\beta}=0,$$

$$\tilde\Gamma^{k_\gamma}_{i_\alpha j_\beta}=-\frac{1}{h}(g^{rs}\omega_r X^\alpha_s\delta^\beta_\tau\theta_l+g^{rs}\theta_r X^\beta_s\delta^\alpha_\tau\omega_l-\delta_{\beta\sigma}g^{rs}\omega_r\theta_s X^\tau_l)D_{l_\tau}.$$

In Section 4.10 of **Chapter IV,** we consider the problem of the curvature properties of a Sasaki lift $\tilde G$ cof homogeneous type,

Including using the equality

$$\tilde{R}(D_I, D_J)D_K = \tilde{\nabla}_I \tilde{\nabla}_J D_K - \tilde{\nabla}_J \tilde{\nabla}_I D_K - \Omega_{IJ}{}^{\tilde{L}} \tilde{\nabla}_L D_K$$

which is satisfied by the tensor field of curvature $\tilde{R}$ of the Levi-Civita connection $\tilde{\nabla}$ of the Sasaki lift $\tilde{G}$ of homogeneous type of the Riemannian metric g to the coframe bundle $F^*(M)$, the components of this tensor field are calculated with respect to the adapted frame $\{D_I\}$.

Chapter V of the dissertation is devoted to the study of the Cheeger-Gromoll metric and some structures related to this metric in the coreper bundle. The main results of this chapter were published in the author's papers [21, 23, 33, 34].

The Riemannian metric of a special form, called the Cheeger-Gromall metric, was first introduced on the tangent bundle by J. Cheeger and D. Gromoll. On the tangent bundle, the curvature properties of the Cheeger-Gromall metric were studied by M. Sekizawa. The geodesic lines of this metric were studied by A. Salimov and S. Kazimova. Generalized Cheeger-Gromoll metrics (they are also called Cheeger-Gromoll-type metrics) on the tangent bundle investigated by M. Munteanu , Z. Hou and L.Sun.. The Cheeger-Gromoll metric on the cotangent bundle was introduced and studied by A.A. Salimov and F. Agca.

In Section 5.1 of **Chapter V**, in the coframe bundle of a Riemannian manifold, the Cheeger-Gromoll metric is defined, and the properties of the Levi-Civita connection of this metric are studied.

Definition 5. *Assume that g is a Riemannian metric on a manifold M. The Cheeger-Gromoll metric on the coframe bundle $F^*(M)$ is the Riemannian metric ${}^{CG}g$, which for $\forall X, Y \in \Im^1{}_0(M)$ and $\forall \omega, \theta \in \Im^0_1(M)$ satisfies the relations*

$${}^{CG}g({}^H X, {}^H Y) = g(X, Y)),$$

$${}^{CG}g({}^{V_\alpha}\omega, {}^H Y) = 0,$$

$${}^{CG}g({}^{V_\alpha}\omega, {}^{V_\beta}\theta) = 0, \quad for \quad \alpha \neq \beta,$$

$${}^{CG}g({}^{V_\alpha}\omega, {}^{V_\alpha}\theta) = \frac{1}{1+r_\alpha^2}(g^{-1}(\omega, \theta) + g^{-1}(\omega, X^\alpha) g^{-1}(\theta, X^\alpha)).$$

where

$$r_\alpha^2 = \left| X^\alpha \right|^2 = g^{-1}(X^\alpha, X^\alpha).$$

The properties of the Levi-Civita connection of the Cheeger-Gromoll metrics are investigated.

Theorem 25. *The connection* ${}^{CG}\nabla$ *for* $\forall X, Y \in \Im_0^1(M)$ *and* $\forall \omega, \theta \in \Im_1^0(M)$ *satisfies the relations*

i) $${}^{CG}\nabla_{{}^HX}{}^HY = {}^H(\nabla_X Y) + \frac{1}{2}\sum_{\sigma=1}^{n} {}^V(X^\sigma \circ R(X,Y)),$$

ii) $${}^{CG}\nabla_{{}^HX}{}^{V_\beta}\theta = {}^{V_\beta}(\nabla_X \theta) + \frac{1}{2h_\beta}{}^H(X^\beta(g^{-1} \circ R(\ ,X)\tilde{\theta})),$$

iii) $${}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^HY = \frac{1}{2h_\alpha}{}^H(X^\alpha(g^{-1} \circ R(\ ,Y)\tilde{\omega})),$$

iv) $${}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^{V_\beta}\theta = 0, \quad \textit{for} \quad \alpha \neq \beta$$

$${}^{CG}\nabla_{{}^{V_\alpha}\omega}{}^{V_\alpha}\theta = -\frac{1}{h_\alpha}({}^{CG}g({}^{V_\alpha}\omega, \gamma\delta)^{V_\alpha}\theta + {}^{CG}g({}^{V_\alpha}\theta, \gamma\delta)^{V_\alpha}\omega) +$$

$$+\frac{1+h_\alpha}{h_\alpha}{}^{CG}g({}^{V_\alpha}\omega, {}^{V_\alpha}\theta)\gamma\delta - \frac{1}{h_\alpha}{}^{CG}g({}^{V_\alpha}\theta, \gamma\delta){}^{CG}g({}^{V_\alpha}\omega, \gamma\delta)\gamma\delta,$$

where $\tilde{\omega} = g^{-1} \circ \omega$, $R(\ ,X)\tilde{\omega} \in \Im_1^1(M)$, $h_\alpha = 1 + r_\alpha^2$, R *and* $\gamma\delta$ *are, respectively, the curvature tensor field of the metric* g *and the canonical vertical vector field on the coframe bundle* $F^*(M)$ *with the local expression* $\gamma\delta = X_i^\sigma D_{i_\sigma}$.

In Section 5.2 of **Chapter V**, we study the curvature properties of the coframe bundle with the Cheeger-Gromoll metric,

Theorem 26. *Assume that* (M, g) *an* $n-$*dimensional Riemannian manifold and* ${}^{CG}\nabla$ *is the Levi-Civita connection of a coframe bundle* $F^*(M)$ *with the Cheeger-Gromoll metric* ${}^{CG}g$. *Then the connection coefficients* ${}^{CG}\Gamma_{IJ}^K$ *for different indices have the values*

$$ {}^{CG}\Gamma^k_{ij} = \Gamma^k_{ij}, \qquad {}^{CG}\Gamma^{k_\gamma}_{ij} = \frac{1}{2} X^{\gamma}_{m} R_{ijk}{}^{m}, $$

$$ {}^{CG}\Gamma^k_{ij_\beta} = \frac{1}{2h_\beta} X^{\beta}_{m} R^{k}_{\cdot\, i\, \cdot}{}^{jm}, \qquad {}^{CG}\Gamma^{k_\gamma}_{ij_\beta} = -\delta^{\gamma}_{\beta}\Gamma^{j}_{ik}, $$

$$ {}^{CG}\Gamma^k_{i_\alpha j} = \frac{1}{2h_\alpha} X^{\alpha}_{m} R^{k}_{\cdot\, j\, \cdot}{}^{im}, \qquad {}^{CG}\Gamma^{k_\gamma}_{i_\alpha j} = {}^{CG}\Gamma^{k}_{i_\alpha j_\beta} = 0, $$

$$ {}^{CG}\Gamma^{k_\gamma}_{i_\alpha j_\beta} = 0 \quad \mathit{for}\ \alpha \neq \beta \ . $$

$$ {}^{CG}\Gamma^{k_\gamma}_{i_\alpha j_\alpha} = -\frac{1}{h_\alpha}(\tilde{X}^{\alpha i}\delta^{\alpha}_{\gamma}\delta^{j}_{k} + \tilde{X}^{\alpha j}\delta^{\alpha}_{\gamma}\delta^{i}_{k}) + \frac{1+h_\alpha}{h_\alpha^2} g^{ij} X^{\gamma}_{k} + \frac{1}{h_\alpha^2}\tilde{X}^{\alpha i}\tilde{X}^{\alpha j} X^{\gamma}_{k}, $$

where $\tilde{X}^{\alpha i} = g^{is} X^{\alpha}_{s}$.

In this section, for various indices, the values of the components

$$ {}^{CG}R_{IJK}{}^{L} = D_I\, {}^{CG}\Gamma^L_{JK} - D_J\, {}^{CG}\Gamma^L_{IK} + {}^{CG}\Gamma^L_{IS}\, {}^{CG}\Gamma^S_{JK} - {}^{CG}\Gamma^L_{JS}\, {}^{CG}\Gamma^S_{IK} - \Omega_{IJ}{}^{S}\, {}^{CG}\Gamma^L_{SK} $$

of curvature tensor field ${}^{CG}R$ of the Cheeger-Gromoll metric ${}^{CG}g$ are also determined with respect to the adapted frame $\{D_I\}$, also the values ${}^{CG}K({}^{H}X, {}^{H}Y)$, ${}^{CG}K({}^{H}X, {}^{V_\beta}\theta)$, ${}^{CG}K({}^{V_\alpha}\omega, {}^{V_\alpha}\theta)$ and ${}^{CG}K({}^{V_\alpha}\omega, {}^{V_\beta}\theta)$ of the sectional curvature ${}^{CG}K$ of the Cheeger-Gromoll metric ${}^{CG}g$ are calculated, where $\Omega_{IJ}{}^{S}$ is the object of nonholonomy.

Theorem 27. *The horizontal lift of a geodesic curve of a Riemannian manifold* (M, g) *is a geodesic curve in the coframe bundle* $F^*(M)$*, where the Cheeger-Gromoll metric* ${}^{CG}g$ *is defined.*

In Section 5.3 of **Chapter V** we study almost complex structures on a coframe bundle with the Cheeger-Gromoll metric ${}^{CG}g$ ..

In the coframe bundle $F^*(M)$ for $\forall X \in \Im^1_0(M)$, $\forall \omega \in \Im^0_1(M)$

using the relations

$$J_\beta\, {}^H X = \sqrt{h_\beta}\, {}^{V_\beta}\tilde{X} - \frac{1}{\sqrt{h_\beta}+1} X^\beta (X)^{V_\beta} X^\beta,$$

$$\beta \neq \gamma \;\; J_\beta\, {}^{V_\gamma}\omega = 0, \quad \text{if} \;\; \beta \neq \gamma\,,$$

$$J_\beta\, {}^{V_\beta}\omega = -\frac{1}{\sqrt{h_\beta}}\left({}^H\tilde{\omega} + \frac{1}{(\sqrt{h_\beta}+1)} g^{-1}(X^\beta, \omega)\, {}^H\tilde{X}^\beta \right),$$

almost complex structures $J_\beta, \beta = 1,2,...,n,$ are determined.

Theorem 28. *The triple* $(F^*(M), {}^{CG}g, J_\beta)$ *is an almost Hermitian manifold. for eac* $\beta = 1,2,...,n$.

Lemma 29. *For each* $\beta = 1,2,...,n,$ *on the coframe bundle* $(F^*(M), {}^{CG}g)$, *the almost complex structure* J_β *is integrable if and only if the relation* $N_{J_\beta}({}^H X, {}^H Y) = 0$ *holds for* $\forall X, Y \in \Im_0^1(M)$.

Theorem 30. *On a coframe bundle* $(F^*(M), {}^{CG}g)$, *for each* $\beta = 1,2,...,n,$*, an almost complex structure* J_β *is integrable if and only if the relation*

$$\gamma R(X,Y) = \sum_{\sigma=1}^{n} {}^{V_\sigma}(X^\sigma \circ R(X,Y)) =$$

$$= \frac{1+\sqrt{h_\beta}+h_\beta}{\sqrt{h_\beta}(\sqrt{h_\beta}+1)}\, {}^{V_\beta}\left(g^{-1}(X^\beta, \tilde{X})\tilde{Y} - g^{-1}(X^\beta, \tilde{Y})\tilde{X}\right)$$

holds.

In the coframe bundle $(F^*(M), {}^{CG}g)$ with the Cheeger-Gromoll metric for $\forall X \in \Im_0^1(M)$, $\forall \omega \in \Im_1^0(M)$, tensor fields ${}^{CG}F_\alpha$, $\alpha = 1,2,...,n,$ of type $(1,1)$ are defined using the relations

$${}^{CG}F_\alpha({}^H X) = \sqrt{h_\alpha}\, {}^{V_\alpha}\tilde{X} - \frac{1}{\sqrt{h_\alpha}+1} X^\alpha (X)^{V_\alpha} X^\alpha,$$

$${}^{CG}F_\alpha({}^{V_\beta}\omega) = 0, \quad \beta \neq \alpha,$$

$$ {}^{CG}F_\alpha({}^{V_\alpha}\omega) = \frac{1}{\sqrt{h_\alpha}}\left({}^{H}\omega + \frac{1}{\sqrt{h_\alpha}+1} g^{-1}(X^\alpha,\omega)\, {}^{H}X^\alpha \right), $$

where $\tilde{X} = g \circ X \in \Im^0_1(M)$, $\tilde{\omega} = g^{-1} \circ \omega \in \Im^1_0(M)$.

Lemma 31. *On a coframe bundle* $(F^*(M), {}^{CG}g)$, *for each* $\alpha = 1,2,...,n,$ *an almost paracomplex structure* ${}^{CG}F_\alpha$ *is integrable if and only if the relation* $N_{{}^{CG}F_\alpha}({}^{H}X, {}^{H}Y) = 0$ *for* $\forall X, Y \in \Im^1_0(M)$ *holds.*

It is proved that in the coframe bundle $(F^*(M), {}^{CG}g)$ for each $\alpha = 1, 2,..., n,$ bat olunthe almost paracomplex structure ${}^{CG}F_\alpha$ is integrable if and only if for $\forall X, Y \in \Im^1_0(M)$ the relation

$$ \gamma R(X,Y)) = \sum_{\sigma=1}^{n} {}^{V_\sigma}(X^\sigma \circ R(X,Y)) = $$

$$ = \frac{1+\sqrt{h_\alpha}+h_\alpha}{\sqrt{h_\alpha}(\sqrt{h_\alpha}+1)} {}^{V_\alpha}\left(g^{-1}(X^\alpha, \tilde{Y})\tilde{X} - g^{-1}(X^\alpha, \tilde{X})\tilde{Y} \right) $$

is satisfied.

Chapter VI of the dissertation is devoted to the study of some important structures on a bundle of tensor frames of type (1,1) . The main results of this chapter were published in the author's papers [6, 9, 24, 26].

In Section 6.1 of **Chapter VI**, the bundle of tensor frames of type (1,1) (or affinor frames), the vertical lift of a function, the horizontal lifts of vector fields, and the vertical lifts of affinor fields are defined, the concept of an adapted frame is introduced, and a diagonal lift of the Riemannian metric (Sasaki metric) is constructed.

Definition 6. *The Sasaki metric* ${}^{D}g$ *(or the diagonal lift of the Riemannian metric* g*) in the bundle of affinor frames* $L^1_1(M)$ *for* $\forall X, Y \in \Im^1_0(M)$ *and* $\forall B, C \in \Im^1_1(M)$ *is defined by the following three relations:*

$$ {}^{D}g({}^{V_{\alpha\beta}}B, {}^{V_{\gamma\sigma}}C) = \delta^{\alpha\gamma}\delta_{\beta\sigma}\, {}^{V}(G(B,C)), $$

$$ {}^{D}g({}^{V_{\alpha\beta}}B, {}^{H}Y) = 0 \quad , $$
$$ {}^{D}g({}^{H}X, {}^{H}Y) = {}^{V}(g(X,Y)), $$

where $G(B,C) = g_{pq} g^{ij} B_i^p C_j^q$ *for* $\forall B, C \in \Im_1^1(M)$.

Theorem 32. *Assume that* (M, g) *is a Riemannian manifold and* ${}^{D}\nabla$ *is a Levi-Civita connection of the affinor frame bundle* $L_1^1(M)$*, on which the Sasaki metric* ${}^{D}g$ *is defined. Then the connection coefficients* ${}^{D}\Gamma_{IJ}^{K}$ *for different indices with respect to the adapted frame* $\{D_K\}$ *have the following values:*

$$ {}^{D}\Gamma_{ij}^{k} = \Gamma_{ij}^{k}, \quad {}^{S}\Gamma_{ij}^{k_{\tau\lambda}} = \frac{1}{2}\left(X^{\tau r}_{\lambda m} R^{m}_{ijk} - X^{\tau m}_{\lambda k} R^{r}_{ijm}\right), $$
$$ {}^{D}\Gamma_{i_{\alpha\gamma} j_{\beta\sigma}}^{k} = {}^{D}\Gamma_{i_{\alpha\beta} j}^{k_{\tau\lambda}} = {}^{D}\Gamma_{i_{\alpha\gamma} j_{\beta\sigma}}^{k_{\tau\lambda}} = 0, $$
$$ {}^{D}\Gamma_{ij_{\beta\sigma}}^{k} = \frac{1}{2}\left(g_{la} X^{\beta a}_{\sigma m} R^{mj\ \ k}_{\cdot\cdot i} - g^{jb} X^{\beta m}_{\sigma b} R^{k}_{lmi}\right), $$
$$ {}^{D}\Gamma_{ij_{\beta\sigma}}^{k_{\tau\lambda}} = \delta^{\tau}_{\beta}\delta^{\sigma}_{\lambda}(\delta^{j}_{k}\Gamma^{r}_{il} - \delta^{r}_{l}\Gamma^{j}_{ik}), $$
$$ {}^{D}\Gamma_{i_{\alpha\gamma} j}^{k} = \frac{1}{2}(g_{ha} X^{\alpha a}_{\gamma m} R^{mi\ \ k}_{\cdot\cdot j} - g^{ib} X^{\alpha m}_{\gamma b} R^{k}_{hmj}) $$

where $R^{mj\ \ k}_{\cdot\cdot i} = g^{ml} g^{js} R_{lsi}^{\ \ \ k}$.

In Section 6.2 of **Chapter VI,** we study the problem of constructing complete and horizontal lifts of an affine (linear) connection in a bundle of tensor frames of type (1,1) .

Definition 7. *A horizontal lift of a symmetric affine (linear) connection* ∇ *defined on a smooth manifold* M *to a bundle of affinor frames* $L_1^1(M)$ *is an affine connection* ${}^{H}\nabla$ *defined by the relations*

$$ {}^{H}\nabla_{{}^{H}X} {}^{H}Y = {}^{H}(\nabla_X Y), \qquad {}^{H}\nabla_{{}^{H}X} {}^{V_{\beta\sigma}}B = {}^{V_{\beta\sigma}}(\nabla_X B), $$
$$ {}^{H}\nabla_{{}^{V_{\alpha\gamma}}A} {}^{H}Y = 0 \quad , \qquad {}^{H}\nabla_{{}^{V_{\alpha\gamma}}A} {}^{V_{\beta\sigma}}B = 0, $$

for $\forall X, Y \in \Im_0^1(M)$ *and* $\forall A, B \in \Im_1^1(M)$.

Theorem 33. *With respect to the adapted frame* $\{D_I\}$*, the coefficients of the horizontal lift* ${}^{H}\nabla$ *of a symmetric affine connection* ∇ *defined on a smooth manifold* M *to the bundle of affinor frames* $L_1^1(M)$ *for different indices have the values*

$$^H\Gamma^{p}_{i_{\alpha\gamma}k_{\beta\sigma}} = {}^H\Gamma^{p_{\eta\varepsilon}}_{i_{\alpha\gamma}k_{\beta\sigma}} = {}^H\Gamma^{p}_{i_{\alpha\gamma}k} = {}^H\Gamma^{p_{\eta\varepsilon}}_{i_{\tau\gamma}k} = 0\,,$$
$$^H\Gamma^{p}_{ik} = \Gamma^{p}_{ik}\,, \qquad {}^H\Gamma^{p_{\eta\varepsilon}}_{ik} = {}^H\Gamma^{p}_{ik_{\beta\sigma}} = 0,$$
$$^H\Gamma^{p_{\eta\varepsilon}}_{ik_{\beta\sigma}} = \delta^{\eta}_{\beta}\delta^{\sigma}_{\varepsilon}\delta^{k}_{p}\Gamma^{q}_{il} - \delta^{\eta}_{\beta}\delta^{\sigma}_{\varepsilon}\delta^{q}_{l}\Gamma^{k}_{ip}\ .$$

It is established that the coefficients of the horizontal lift $^H\nabla$ of a symmetric affine (or linear) connection ∇ given on a smooth manifold M, to the bundle of affinor frames $L_1^1(M)$ with respect to the natural frame $\{\partial_I\}$ for different indices have the values

$$^H\Gamma^{s}_{jl} = \Gamma^{s}_{jl}\,,$$
$$^H\Gamma^{s_{\varphi\psi}}_{j_{\tau\nu}l} = \delta^{\phi}_{\tau}\delta^{\nu}_{\psi}\delta^{j}_{s}\Gamma^{r}_{lq} - \delta^{\phi}_{\tau}\delta^{\nu}_{\psi}\delta^{r}_{q}\Gamma^{j}_{ls}\,,$$
$$^H\Gamma^{s_{\varphi\psi}}_{jl_{\omega\sigma}} = \delta^{\varphi}_{\omega}\delta^{\sigma}_{\psi}\delta^{l}_{s}\Gamma^{r}_{js} - \delta^{\varphi}_{\omega}\delta^{\sigma}_{\psi}\delta^{r}_{s}\Gamma^{l}_{js}\,,$$
$$^H\Gamma^{s_{\varphi\psi}}_{jl} = X^{\varphi b}_{\psi s}(\partial_j\Gamma^{r}_{lb} - \Gamma^{r}_{pb}\Gamma^{p}_{jl} + \Gamma^{r}_{jp}\Gamma^{p}_{lb}) +$$
$$+X^{\varphi r}_{\psi a}(-\partial_j\Gamma^{a}_{ls} + \Gamma^{p}_{js}\Gamma^{a}_{lp} + \Gamma^{a}_{ps}\Gamma^{p}_{jl}) - X^{\varphi b}_{\psi a}(\Gamma^{r}_{lb}\Gamma^{a}_{sj} - \Gamma^{a}_{ls}\Gamma^{r}_{bj}),$$
$$^H\overline{\Gamma}^{s}_{j_{\tau\nu}l} = {}^H\overline{\Gamma}^{s}_{jl_{\omega\sigma}} = {}^H\overline{\Gamma}^{s}_{j_{\tau\nu}l_{\omega\sigma}} = {}^H\Gamma^{s_{\varphi\psi}}_{j_{\tau\nu}l_{\omega\sigma}} = 0.$$

It is proved that the complete lift $^C\nabla$ of a symmetric affine (linear) connection ∇ to the bundle of tensor frames of type (1,1) for arbitrary vector fields $X, Y \in \Im^1_0(M)$ is determined using the relation

$$^C\nabla_{^CX}{}^CY = {}^C(\nabla_X Y) + \gamma\,(Q(X,Y))$$

where $\gamma\,(Q(X,Y)) = \begin{pmatrix} 0 \\ F^{k_{\beta\sigma}} \end{pmatrix}$ is a vertical vector field on $\overset{1}{L_1}(M)$ with non-zero components

$$F^{k_{\beta\sigma}} = X^{\beta l}_{\sigma a}(\nabla_k X^m \nabla_m Y^a + \nabla_k Y^m \nabla_m X^a -$$
$$- X^i Y^j (R^a_{jki} + R^a_{ikj})) - X^{\beta b}_{\sigma a}(\nabla_b X^l \nabla_k Y^a + \nabla_k X^a \nabla_b Y^l)\,.$$

It is established that with respect to the natural frame $\left\{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial X^{\beta q}_{\sigma j}}\right\}$, the coefficients of the complete lift $^C\nabla$ for various indices have non-zero values

$$ {}^C\Gamma^p_{jl} = \Gamma^p_{jl}, $$
$$ {}^C\Gamma^{p_{\varphi\psi}}_{j_{\tau\nu} l} = \delta^\varphi_\tau \delta^\nu_\psi \delta^j_p \Gamma^r_{lq} - \delta^\varphi_\tau \delta^\nu_\psi \delta^r_q \Gamma^j_{lp}, $$
$$ {}^C\Gamma^{p_{\varphi\psi}}_{j l_{\omega\sigma}} = \delta^\varphi_\omega \delta^\sigma_\psi \delta^l_p \Gamma^r_{js} - \delta^\varphi_\omega \delta^\sigma_\psi \delta^r_s \Gamma^l_{jp}, $$
$$ {}^C\Gamma^{p_{\varphi\psi}}_{jl} = X^{\varphi b}_{\psi p}\partial_b \Gamma^r_{jl} + X^{\varphi r}_{\psi a}(\partial_p \Gamma^a_{lj} - \partial_l \Gamma^a_{pj} - \partial_j \Gamma^a_{lp} + 2\Gamma^a_{mp}\Gamma^m_{jl}) - $$
$$ - X^{\varphi b}_{\psi a}(\Gamma^r_{lb}\Gamma^a_{pj} + \Gamma^a_{lp}\Gamma^r_{bj}) $$

In Section 6.3 of **Chapter VI** investigates the relation between the horizontal lift of a symmetric affine connection and the Sasaki metric in a bundle of tensor frames of type (1,1) . It is proved that the horizontal lift ${}^H\nabla$ has a nonzero torsion tensor even when the affine connection ∇ is the Levi-Civita connection of the Riemannian metric.

Theorem 34. *Let g be a Riemannian metric on a manifold M and let ∇ be its metric connection. Then the horizontal lift ${}^H\nabla$ of the connection ∇ is a metric connection of the Sasaki metric ${}^D g$ on the bundle $L^1_1(M)$.*

In conclusion, the author expresses his deep gratitude to his scientific adviser, doctor of physical and mathematical sciences, professor A.A. Salimov for his constant attention to the work and valuable advice.

Conclusions

The dissertation work is devoted to the study of the main differential-geometric structures, including vertical, complete, horizontal and g -lifts of vector and tensor fields, horizontal and complete lifts of affine connections and derivations, Sasaki and Cheeger-Gromoll metrics, geodesic curves of their Levi-Civita connections , almost complex, paracomplex, and f -structures on a coframe bundle of a smooth manifold, in particular, a Riemannian manifold, as well as a confirmation of the generalization of the results obtained in the study of bundles of tensor frames, the study of some structures on a bundle of tensor frames of type (1,1).

The main results of the dissertation are as follows:

1. Complete and horizontal lifts of tensor fields of type $(1, p)$, in particular of type $(1,1)$, and affine connections to the coframe bundle are constructed, and on the basis of this, theorems related to lifts of almost complex structure and $f-$structures are proved.

2. In the case of a parallelizable base manifold, global sections of the bundle of coframes are determined, and along these sections, using $\Phi-$operator theory, the behavior of lifts of differential-geometric structures is investigated, and the corresponding theorems are proved.

3. In the coframe bundle of a Riemannian manifold, the Sasaki metric is defined, theorems related to the Levi-Civita connection, connection coefficients, and curvature properties of this metric are proved.

4. On a coframe bundle over a Riemannian manifold, a Sasaki metric (lift) of homogeneous type is defined, and theorems on the Levi-Civita connection and on the curvature properties of this metric are proved.

5. $f-$tructures of ranks $2n$ and n^2 are defined on the coframe bundle, and theorems on the adaptation of the Sasaki metric with these structures are proved.

6. On a Riemannian manifold, canonical isomorphisms between frame bundles and coreframe bundles are constructed, and

with the help of these isomorphisms theorems on $g-$lifts of vector, covector, and tensor fields of type $(0,2)$ to the coframe bundle are proved.

7. In a coframe bundle over a Riemannian manifold, the Cheeger-Gromoll metric is defined, the properties of the Levi-Civita connection of this metric are studied, the connection coefficients, the components of the curvature tensor field of this metric are calculated, and theorems on almost complex and paracomplex structures adapted with the Cheeger-Gromoll metric are proved.

8. A bundle of tensor frames of type $(1,1)$ is defined, lifts of functions, vector and affinor fields, affine connections, the Sasaki metric in this bundle are constructed, and theorems related to the study of the properties of these objects are proved.

The main results of the dissertation work were published in the following works:

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9. Фаттаев, Г.Д. О метрической связности метрики Сасаки в расслоении тензорных реперов типа (1,1) // Известия Педагогического Университета, серия математических и естественных наук, - 2019, № 3, - с. 37-47.
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The defense will be held on **4 june 2024** year at 14^{00} at the meeting of the one time Dissertation council BED 2.17 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at Baku State University.

Address: AZ 1148, Baku city, Z.Khalilov str., 23.

Dissertation is accessible at the library of the Baku State University.

Electronic versions of the dissertation and its abstract are available on the official website of the Baku State University.

Abstract was sent to the required addresses on **23 april 2024** year.

Signed for print: 27.03.2024
Paper format: 60x84 1/16
Volume: 74000
Number of hard copies: 20