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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

AFFINOR STRUCTURES IN WALKER MANIFOLDS AND THEIR BUNDLE SPACES

Specialty: 1204.01– Geometry

Field of science: Mathematics

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GENERAL CHARACTERISTICS OF THE WORK

Rationale and development degree of the topic. The study of various differential-geometrical structures on differentiable manifolds is one of the interesting topic of modern differential geometry. Some of them are complex, almost complex, paracomplex structures defined by affinors.

In recent years, various manifolds have been considered in differential geometry and topology. Among these manifolds, Norden-Walker manifolds attract more attention. Such manifolds are determined in terms of complex structures. Norden structures can be shown as an example of such complex structures. Such structures transform a polynomial into a holomorphic polynomial in suitable algebras. Algebraic tensor fields on a holomorphic manifold correspond to pure tensor fields on a real manifold. Tensor fields are used in differential geometry, algebraic geometry, general relativity, stress analysis of materials and physics. For this reason, the investigation of tensor fields on manifolds is one of the actual issues.

Differential-geometrical structures defined as lifts of given differential geometrical structures on the base of tangent and cotangent bundle spaces are of more interest. The vertical, complete and horizontal lifts of vector fields to the tangent bundle space were investigated by S.Ishihara, K.Yano, Sh.Kobayashi and the issue of constructing lifts of affine connections in the tangent bundle space was also considered in their articles. The construction of lifts allowed to study almost complex, paracomplex, almost product structures in the tangent bundle space. The existence of a dual structure in tangent bundle space allowed A.Shirokov to interprete this bundle space as a manifold structured over the algebra of dual members. Based on this, it becomes much easier to construct lifts of tensor fields and affine connections in the tangent bundle space. This idea was developed during the study of semitangent spaces. Similar results to the results obtained as a result of the study of tangent bundle spaces were obtained during the study of cotangent bundle spaces. Some interesting results on the study of differential-geometrical structures, also tensor fields, affine connection lifts and various metrics in different type tensor bundle spaces on differentiable manifolds were obtained. In spite of the fact that the issue of construction of different new types of natural Riemannian metrics in tangent and cotangent bundle spaces, study of Levi-Civita connection are the problems distinguished by their actuality. The topic of the represented dissertation work is related to the solution of this problem. In this sense the topic of dissertation work is actual.

Object and subject of the study. Study of properties of Walker metrics, affinor structures, Levi-Civita connection on Walker manifolds and their tangent bundle spaces.

Goal and objectives of the study. The purpose of the work is to construct almost complex structures and study their properties in Walker manifolds and their tangent bundle spaces.

Research methods. In the research of the problems under consideration the methods of tensor calculus on differentiable manifolds were used.

The main points of the study. The following main scientific results were obtained:

1. Integrability conditions of generalized Norden-Walker structures, holomorphic conditions of Walker metrics with respect to this structure are given;
2. It has been shown that the complete lift of a symplectic structure from its base manifold to its tangent bundle is a closed 2-form, and consequently its image by symplectic isomorphism is a 2-form.
3. It has been shown that in the tangent bundle of a Walker manifold, the vertical and horizontal lifts of a vector field are always zero, and the complete lift is zero only if and only if the vector field has a constant length.

4. The components of the deformed complete lifts of (1,1)-type tensor fields and connections in the tangent bundles of degree 2 are given.
5. The integrability conditions of the almost product Norden-Walker structure on a 3-dimensional Walker manifold, the Kahler conditions of the Walker metric with respect to this structure, and the equations of geodesics are given.

Scientific novelty of the study. The main results obtained in the study are new and there are the followings:

1. Integrability conditions of generalized Norden-Walker structures, holomorphic conditions of Walker metrics with respect to this structure are given;
2. It has been shown that the complete lift of a symplectic structure from its base manifold to its tangent bundle is a closed 2-form, and consequently its image by symplectic isomorphism is a 2-form.
3. It has been shown that in the tangent bundle of a Walker manifold, the vertical and horizontal lifts of a vector field are always zero, and the complete lift is zero only if and only if the vector field has a constant length.
4. The components of the deformed complete lifts of (1,1)-type tensor fields and connections in the tangent bundles of degree 2 are given.
5. The integrability conditions of the almost product Norden-Walker structure on a 3-dimensional Walker manifold, the Kahler conditions of the Walker metric with respect to this structure, and the equations of geodesics are given.

Theoretical and practical value of the study. The main results obtained in the dissertation work are mainly of theoretical character. The results obtained in the dissertation work and the used methods can be used in teaching of specialty courses.

Approbation and application. The results of the dissertation were presented at the Republican Scientific Conference "Mathematics, Mechanics and Their Applications" dedicated to the 98th anniversary of the birth of the National Leader of Azerbaijan Heydar Aliyev, at the Republican Scientific Conference "Actual Problems of Mathematics and Mechanics" dedicated to the 99th anniversary of the birth of the National Leader of the Azerbaijani People Heydar Aliyev, as well as at the "XIII Annual International Conference of the Georgian Mathematical Union" held abroad in Georgia, and at the "2nd International Symposium on Current Developments in Fundamental and Applied Mathematics Sciences" held in Turkey.

The organization where the work was executed: Chair of “Algebra and geometry” of department “Mechanics and Mathematics” of Baku State University.

Authors personal contribution. The results obtained in the dissertation belong to the applicant.

Published scientific works. The main results of the dissertation work were published in applicants 5 scinetific works, including 3 in scientific editions included Web of Science and Scopus databases. Furthermore, the results obtained in the dissertation were reported at international level 2 and republican level 2 scientific conferences, two of them were published abroad.

Total volume of the dissertation work indicating separate structural units of the work in signs. The dissertation work consists of introduction, four chapters, conclusions (chapter I – 34821 signs, chapter II – 14490 signs, chapter III – 31810 signs, chapters IV – 10061 signs) and list of references consisting of 81 names. The total volume of the dissertation work is 136522 signs.

CONTENT OF THE DISSERTATION WORK

Let us give brief review of the dissertation work consisting of 4 chapters.

In the introduction we give brief review of works related to the dissertation work, substantiate actuality of the dissertation work, give main results obtained in the work and compare them with the results of another works.

Chapter I consists of three subchapters. The first chapter provides concepts about Walker metrics and almost complex structures generalized over Walker manifolds.

In the first subchapter of the first chapter, an almost complex structure, the Norden metric is defined. let M_{2n} be an almost complex manifold, i.e., we assume that φ is an almost complex structure satisfying $\varphi^2 = -I$. An almost complex structure φ is said to be integrable if φ is reduced to the constant form in a collection of holonomic coordinates on M_{2n}. Also, an almost complex structure φ is integrable if and only if the Nijenhuis tensor $N_\varphi \in \Im_2^1(M_{2n})$ vanishes. The triple (M_{2n}, φ, g) is called complex manifold if φ is integrable. We say that a neutral metric g is a Norden metric if

$$g(\varphi X, \varphi Y) = -g(X, Y)$$

or equivalently

$$g(\varphi X, Y) = g(X, \varphi Y).$$

where $X, Y \in \Im_0^1(M_{2n})$. An almost Norden manifold is a triple (M_{2n}, φ, g) with the Norden metric g. The triple is called Norden manifold if φ is integrable.

We say that a Norden metric g on a Norden manifold (M_{2n}, φ, g) is holomorphic if

$$(\Phi_\varphi g)(X, Y, Z) = 0$$

for any vector fields X, Y, Z on M_{2n}, where $\Phi_\varphi g$ is the Tachibana operator:

$$(\Phi_\varphi g)(X, Y, Z) = -g((\nabla_X \varphi)Y, Z) + g((\nabla_Y \varphi)Z, X) + \\ +g((\nabla_Z \varphi)X, Y) = 0$$

In the second subchapter of the first chapter, the four-dimensional Walker manifold is defined and the integrability conditions of the generalized Norden-Walker structure built on it are stated. Let M_4 be a 4-dimensional C^∞ −manifold. A neutral metric g on a manifold M_4 is said to be Walker metric if there is a totally isotropic parallel 2-dimensional null distribution D on M_4. By a result of Walker theorem, for every Walker metric g on a 4-

manifold M_4, there exist a system of coordinates which the matrix of $g = (g_{ij})$ in these coordinates has following form:

$$g = (g_{ij}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a & c \\ 0 & 1 & c & b \end{pmatrix}$$

Let F be an almost complex structure on a Walker 4-manifold M_4, which satisfies

1) $F^2 = -I$,
2) $g(FX,Y) = g(X,FY)$, (Nordenian property),
3) $F\partial_x = \partial_y, F\partial_y = -\partial_x$, ($F$ induces a positive $\frac{\pi}{2}$ rotation on the degenerate parallel field D).

φ has the local components with respect to the natural frame $\{\partial_x, \partial_y, \partial_z, \partial_t\}$

$$\varphi = (\varphi_j^i) \begin{pmatrix} 0 & -1 & d & -\frac{1}{2}(a+b) \\ 1 & 0 & \frac{1}{2}(a+b) & d \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Where $d = d(x,y,z,t)$ is an arbitrary function.

The triple (M_4, φ, g) is called generalized almost Norden-Walker manifold. In some literature φ with $d = c$ is called the proper almost complex structure. Our purpose here is to investigate integrability and holomorphic (Kähler) conditions of a generalized almost complex structure φ.

An almost complex structure φ is integrable if the Nijenhuis tensor N_φ with the coordinates

$$(N_\varphi)_{jk}^i = \varphi_j^m \partial_m \varphi_k^i - \varphi_k^m \partial_m \varphi_j^i - \varphi_m^i \partial_j \varphi_k^m + \varphi_m^i \partial_k \varphi_j^m = 0,$$

vanishes.

Theorem 1: An almost complex structure φ on a generalized almost Norden-Walker manifold is integrable if and only if

$$\begin{cases} a_x + b_x + 2d_y = 0, \\ a_y + b_y - 2d_x = 0. \end{cases}$$

Theorem 2: If the triple (M_4, φ, g) is a generalized Norden-Walker manifold, then d is harmonic with respect to the arguments x and y.

In the third subchapter of the first chapter, a holomorphic or Kahler-Norden-Walker manifold is defined. If

$$\left(\Phi_\varphi g\right)_{kij} = \varphi_k^m \partial_m g_{ij} - \varphi_i^m \partial_k g_{mj} + g_{mj}(\partial_i \varphi_k^m - \partial_k \varphi_i^m) + g_{im} \partial_j \varphi_k^m = 0,$$

then φ is integrable and the manifold (M_4, φ, g) is called a holomorphic Norden-Walker or a Kähler-Norden-Walker manifold.

Theorem 3. The generalized Norden-Walker manifold (M_4, φ, g) is holomorphic if and only if the following PDEs hold:

$$a_x = -b_x = c_y, a_y = -b_y = -c_x, d_x = d_y = 0,$$

$$d a_x - a_t + c_z + d_z + \frac{1}{2}(a+b)a_y = 0,$$

$$d a_y - b_z + c_t - d_t - \frac{1}{2}(a+b)a_x = 0.$$

The fourth subchapter of the first chapter defines isotropic Kahler-Norden-Walker structures.

Theorem 4. (M_4, φ, g) is a generalized quasi-complex structure on a quasi-Norden-Walker manifold that is isotropic Kahler.

The fifth subchapter of the first chapter discusses the curvature properties of the Walker manifold.

In the sixth subchapter of the first chapter, quasi-Kahler-Norden-Walker structures are defined.

Theorem 5. Suppose that (M_4, φ, g) is a quasi-Norden manifold. Its metric is quasi-Kahler-Norden if and only if for every $X, Y, Z \in \Im_0^1(M_4)$ the condition

$$\left(\Phi_\varphi g\right)(X,Y,Z) + \left(\Phi_\varphi g\right)(Y,Z,X) + \left(\Phi_\varphi g\right)(Z,Y,X) = 0$$

is satisfied.

The second chapter consists of 6 subchapters. The second chapter considers symplectic 2-forms, lifts of Norden-Walker structures, and Norden-Walker metrics in tangent bundle spaces.

The first subchapter of the second chapter defines tangent and cotangent bundle spaces. Suppose that M_4 is a 4-dimensional Walker manifold of class $C^\infty -$ and $T_P(M_4)$ the tangent space at point $P \in M_4$ is the set of all tangent vectors of M_4 at point P. Then the set

$$T(M_4) = \bigcup_{P \in M_4} T_P(M_4)$$

by definition, a tangent bundle on a manifold M_4 is called a bundle.

The second subchapter of the second chapter provides information about the 2-form derived from the Walker metric in the tangent bundle space. A manifold M_4 is symplectic if it possesses a nondegenerate 2-form ε which is closed (i.e. $d\varepsilon = 0$). For any manifold M_4 of dimension 4, the cotangent bundle $T^*(M_4)$ is a symplectic 8-manifold with symplectic 2-form $\varepsilon = -dp = dx^i \wedge dp_i$, where $p = p_i dx^i$ is the Liouville form (basic 1-form) in $T^*(M_4)$. the musical isomorphisms $g^b: T(M_4) \to T^*(M_4)$ and $g^\sharp: T^*(M_4) \to T(M_4)$ are given by

$$g^b: x^I = \left(x^i, x^{\bar{\imath}}\right) = \left(x^i, v^i\right) \to \tilde{x}^K = \left(x^k, \tilde{x}^{\bar{k}}\right) =$$
$$= \left(x^k = \delta_i^k x^i, p_k = g_{ki} v^i\right)$$

and

$$g^\sharp: \tilde{x}^K = \left(x^k, \tilde{x}^{\bar{k}}\right) = \left(x^k, p_k\right) \to x^I = \left(x^i, x^{\bar{\imath}}\right) =$$
$$= \left(x^i = \delta_k^i x^k, v^i = g^{ik} p_k\right).$$

In the third subchapter of the second chapter, symplectic vector fields and some properties of symplectic 2-forms on the Walker manifold are studied.

$$\omega = \iota g = g_{ji} y^j dx^i$$

is a new 1-form in $T(M_4)$. An anti-symmetric (0,2)-tensor $d\omega$ has the following components:

$$d\omega = \left((d\omega)_{JI}\right) = \begin{pmatrix} (d\omega)_{ji} & (d\omega)_{j\bar{\imath}} \\ (d\omega)_{\bar{\jmath}i} & (d\omega)_{\bar{\jmath}\bar{\imath}} \end{pmatrix} =$$

$$= \begin{pmatrix} (\partial_j g_{is} - \partial_i g_{js}) y^s & -g_{ji} \\ g_{ji} & 0 \end{pmatrix}.$$

On the other hand $d(d\omega) = d^2\omega = 0$, i.e. $d\omega$ defines a symplectic 2-form in the tangent bundle $T(M_4)$.

Theorem 6. A vector field $T(M_4)$ on $T(M_4)$ is a symplectic vector field with respect to $d\omega$, if

$$d(\iota_{\tilde{X}} d\omega) = 0,$$

i.e. if the interior product $\iota_{\tilde{X}} d\omega$ is closed.

In the fourth subchapter of the second chapter, a theorem is given on the pullback of a 2-form in the cotangent bundle space of a Walker manifold. the pullback of $d\omega$ by $g^\sharp$ is a 2-form $(g^\sharp)^* d\omega$ on $T^*(M_4)$ and has components

$$(g^\sharp)^* d\omega = (((g^\sharp)^* d\omega\ )_{KL}) = \begin{pmatrix} 0 & -\delta_k^l \\ \delta_l^k & 0 \end{pmatrix}.$$

From here follows that the pullback $(g^\sharp)^* d\omega$ coincides with the symplectic form $\widetilde{\omega} = -dp = -dp_i \wedge dx^i = dx^i \wedge dp_i$. Thus we have

Theorem 7. Let (M_4, g) be a Walker manifold. *The natural symplectic structure $-dp = dx^i \wedge dp_i$ on cotangent bundle $T^*(M_4)$ is a pullback by $g^\sharp$ of exterior derivative $d\omega$, i.e. $(g^\sharp)^* d\omega = -dp$.*

In the fifth subchapter of the second chapter, the lifts of Norden-Walker structures in tangent bundle spaces are investigated.

Let (M_4, g) be a Walker manifold of class C^∞. The complete lift ${}^C g$ of Walker metric has components in the form:

$${}^C g = ({}^C g_{IJ}) = \begin{pmatrix} t^s \partial_s g_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}.$$

Theorem 8. Let (M_4, g) be a Walker manifold and $(T(M_4), {}^C g)$ its tangent bundle with complete lift of Walker metric. Then the vertical ${}^V X$ and horizontal lift ${}^H X$ of any vector field X to tangent bundle $T(M_4)$ are always null vector with respect to the metric ${}^{C}g$. The complete lift ${}^{C}X$ is a null vector if and only if the vector field X is of constant length, i.e. $g_{ij}X^iX^j = g(X,X) = const$.

The sixth subchapter of the second chapter provides information on the complete lift of Norden-Walker metrics over the tangent bundle of a Walker manifold. Let (M_4, φ, g) be a Norden-Walker manifold i.e.

$$G(X,Y) = (g^\circ\varphi)(X,Y) = g(\varphi X, Y) = g(X, \varphi Y) = g(\varphi Y, X) = \\ = G(X,Y)$$

where G is the twin Norden metric.

It is well known that the complete lift ${}^{C}\varphi$ of almost complex structure ${}^{C}\varphi$ has components

$$ {}^{C}\varphi = \begin{pmatrix} \varphi_i^j & 0 \\ t^s\partial_s\varphi_i^j & \varphi_i^j \end{pmatrix}.$$

For any vector fields $X, Y \in \Im_0^1(M_4)$, we have.

Theorem 9. Let (M_4, φ, g) be a Norden-Walker manifold. Then the triple $\left(T(M_4), {}^{C}\varphi, {}^{C}g\right)$ is a Norden manifold, but not a Walker-Norden manifold.

Theorem 10. *L*et (M_4, φ, g) be a Norden-Walker-Kahler manifold. Then the triple $\left(T(M_4), {}^{C}\varphi, {}^{C}g\right)$ is a Norden-Kahler manifold, but not a Walker-Norden-Kahler manifold.

The third chapter consists of 4 subchapters. The third chapter considers deformed complete lifts of (1,1)-type tensor fields and connections on a 3-dimensional nilpotent algebra.

In the first subchapter of the third chapter, the conditions for the holomorphism of the tangent bundle space to the Walker manifold of degree 2 with respect to a 3-dimensional nilpotent algebra are considered. Let $\Pi = \{J_{\alpha j}^i\}, \alpha = 1,2; i,j = 1, \dots, n$ be a $\Pi-$structure on a smooth manifold M_n. If there exists a frame $\left\{\partial_i = \frac{\partial}{\partial x^i}\right\}, i = 1, \dots, n, x = (x^i) \in M_n$ such that $\partial_i J_{\alpha j}^k = 0$, then the $\Pi-$structure is said to be integrable. An algebraic structure is said to be an r −regular Π −structure if the matrices $\left(J_{\alpha j}^{i}\right)$, $\alpha = 1{,}2$ of order $n \times n$, simultaneously reduced to the form

$$\left(J_{\alpha j}^{i}\right) = \begin{pmatrix} C_\alpha & 0 & \dots & 0 \\ 0 & C_\alpha & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & C_\alpha \end{pmatrix}, \alpha = 1{,}2; i, j = 1, \dots, n$$

with respect to the adapted frame $\{\partial_i\}$, where $C_\alpha = \left(C_{\alpha\beta}^{\gamma}\right)$ is the regular representation of $\mathfrak{A}_2$ and r is a number of C_α −blocks. We note that the r −regular Π −structure is integrable if a structure-preserving connection with free-torsion exists on M_n. Let $R(\varepsilon^2)$ be an algebra of order 3 with a canonical basis $\{e_1, e_2, e_3\} = \{1, \varepsilon, \varepsilon^2\}, \varepsilon^3 = 0$. It is clear that there exists an affinor field (a tensor field of type (1,1)) γ on $T^2(M_4)$ which has components of the form

$$\gamma = \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}$$

with respect to the natural frame $\{\partial_i, \partial_{\bar{\imath}}, \partial_{\bar{\bar{\imath}}}\} = \left\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{\imath}}}, \frac{\partial}{\partial x^{\bar{\bar{\imath}}}}\right\}$, $i = 1, \dots, 4$, where I denotes the 4×4 identity matrix. From here, we have

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{pmatrix}, \gamma^3 = 0,$$

i.e. $T^2(M_4)$ has a natural integrable structure $\Pi = \{I, \gamma, \gamma^2\}, I = id_{T^2(M_4)}$, which is an isomorphic representation of the algebra $R(\varepsilon^2), \varepsilon^3 = 0$. Using

$$\gamma\partial_i = \partial_{\bar{\imath}}, \gamma^2\partial_i = \gamma\partial_{\bar{\imath}} = \partial_{\bar{\bar{\imath}}}$$

we have $\{\partial_i, \partial_{\bar{\imath}}, \partial_{\bar{\bar{\imath}}}\} = \{\partial_i, \gamma\partial_i, \gamma^2\partial_i\}$. Also, using a frame $\{\partial_1, \gamma\partial_1, \gamma^2\partial_1, \partial_2, \gamma\partial_2, \gamma^2\partial_2, \dots, \partial_4, \gamma\partial_4, \gamma^2\partial_4\}$ which is obtained from $\{\partial_i, \partial_{\bar{\imath}}, \partial_{\bar{\bar{\imath}}}\} = \{\partial_i, \gamma\partial_i, \gamma^2\partial_i\}$ by changing of numbers of frame elements.

The second subchapter of the third chapter provides information on deformed complete lifts of (1,1)-type tensor fields in tangent bundle spaces of order 2. Let

$$\tilde{t} = \tilde{t}_J^I(x^1, \dots, x^{4m})\partial_I \otimes dx^J$$

be a pure (1.1)-tensor field on $T^2(M_4)$ with respect to $\Pi = \{I, \gamma, \gamma^2\}, I = id_{T^2(M_4)}$, and Π be the regular structure naturally existing on $T^2(M_4)$. it follows that the pure tensor field $\tilde{t} = (\tilde{t}_J^I)$ on $T^2(M_4)$ which is corresponding to the $R(\varepsilon^2)$ −holomorphic (1,1)-tensor field t^* on $X_4(R(\varepsilon^2))$ has components

$$\tilde{t} = \\ = \begin{pmatrix} t_j^i & 0 & 0 \\ x^{4+k}\partial_k t_j^i + H_j^i & t_j^i & 0 \\ x^{8+s}\partial_s t_j^i + \frac{1}{2}x^{4+k}x^{4+s}\partial_k\partial_s t_j^i + x^{4+k}\partial_k H_j^i + K_j^i & x^{4+k}\partial_k t_j^i + H_j^i & t_j^i \end{pmatrix}$$

We have

$$^D(t^{II}) = t^{II} + H^I + K^0.$$

In the third subchapter of the third chapter, theorems about deformed second lifts of almost complex structures in tangent bundle spaces of the second order are proved.

Theorem 11. Let H be an almost tangent structure ($H^2 = 0$) and H, K be the hybrid tensor fields with respect to t on M_4, then the deformed lift $^D(t^{II})$ is an almost complex structure on $T^2(M_4)$ if t is so on M_4.

The fourth subchapter of the third chapter provides information about deformed lifts of connections in tangent bundle spaces of the second degree.

The pure connection $\tilde{\nabla}$ with components $\tilde{\Gamma}_{\alpha\beta}^{\gamma}$ is called a holomorphic connection, if

$$\left(\Phi_{\varphi}\Gamma\right)_{\tau\alpha\beta}^{\gamma} = \varphi_{\tau}^{\sigma}\partial_{\sigma}\tilde{\Gamma}_{\alpha\beta}^{\gamma} - \varphi_{\alpha}^{\sigma}\partial_{\tau}\tilde{\Gamma}_{\sigma\beta}^{\gamma} = 0,$$

$$\left(\Phi_{\varphi^2}\Gamma\right)_{\tau\alpha\beta}^{\gamma} = (\varphi^2)_{\tau}^{\sigma}\partial_{\sigma}\tilde{\Gamma}_{\alpha\beta}^{\gamma} - (\varphi^2)_{\alpha}^{\sigma}\partial_{\tau}\tilde{\Gamma}_{\sigma\beta}^{\gamma} = 0$$

Let $\gamma = k$. Since $\sigma = (m, \overline{m}, \overline{\overline{m}})$, we have

$$\left(\tilde{\Gamma}^{k}_{\alpha\beta}\right)=\begin{pmatrix}\Gamma^{k}_{ij}(x^m) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}=\left({}^{C}\Gamma^{k}_{\alpha\beta}\right), x^m=(x^1,\dots,x^4)$$

Let $\gamma=\bar{k}$. In this case, we have

$$\left(\Phi_{\varphi}\Gamma\right)^{\bar{k}}_{\tau\alpha\beta}=0\Longrightarrow\varphi^{\bar{m}}_{\tau}\partial_{\bar{m}}\tilde{\Gamma}^{\bar{k}}_{\alpha\beta}+\varphi^{\bar{\bar{m}}}_{\tau}\partial_{\bar{\bar{m}}}\tilde{\Gamma}^{\bar{k}}_{\alpha\beta}-\varphi^{\bar{m}}_{\alpha}\partial_{\tau}\tilde{\Gamma}^{\bar{k}}_{\bar{m}\beta}=0.$$

From here it follows that

$$\left(\tilde{\Gamma}^{\bar{k}}_{\alpha\beta}\right)=\left({}^{C}\Gamma^{\bar{k}}_{\alpha\beta}\right)+\left({}^{C}G^{k}_{\alpha\beta}\right), x^m=(x^1,\dots,x^4).$$

Let $\gamma=\bar{\bar{k}}$.. In this case we have

$$\left(\Phi_{\varphi}\Gamma\right)^{\bar{\bar{k}}}_{\tau\alpha\beta}=\varphi^{m}_{\tau}\partial_{m}\tilde{\Gamma}^{\bar{\bar{k}}}_{\alpha\beta}+\varphi^{\bar{m}}_{\tau}\partial_{\bar{m}}\tilde{\Gamma}^{\bar{\bar{k}}}_{\alpha\beta}+\varphi^{\bar{\bar{m}}}_{\tau}\partial_{\bar{\bar{m}}}\tilde{\Gamma}^{\bar{\bar{k}}}_{\alpha\beta}-\varphi^{m}_{\alpha}\partial_{\tau}\tilde{\Gamma}^{\bar{\bar{k}}}_{m\beta}$$
$$-\varphi^{\bar{m}}_{\alpha}\partial_{\tau}\tilde{\Gamma}^{\bar{\bar{k}}}_{\bar{m}\beta}-\varphi^{\bar{\bar{m}}}_{\alpha}\partial_{\tau}\tilde{\Gamma}^{\bar{\bar{k}}}_{\bar{\bar{m}}\beta}=0.$$

From here, it follows that

$$\left(\tilde{\Gamma}^{\bar{\bar{k}}}_{\alpha\beta}\right)=\left({}^{C}\Gamma^{\bar{\bar{k}}}_{\alpha\beta}\right)+\left({}^{C}G^{\bar{k}}_{\alpha\beta}\right)+\left({}^{C}H^{k}_{\alpha\beta}\right).$$

Theorem 12. Let ∇ be a connection in the manifold M_4. Then the deformed complete lift $\tilde{\nabla}$ of ∇ to tangent bundle of 2-jets $T^2(M_4)$ has components

$$\tilde{\nabla}=\left(\tilde{\Gamma}^{k}_{\alpha\beta},\tilde{\Gamma}^{\bar{k}}_{\alpha\beta},\tilde{\Gamma}^{\bar{\bar{k}}}_{\alpha\beta}\right)=\left({}^{C}\Gamma^{k}_{\alpha\beta},\ {}^{C}\Gamma^{\bar{k}}_{\alpha\beta}+{}^{C}G^{k}_{\alpha\beta},\ {}^{C}\Gamma^{\bar{\bar{k}}}_{\alpha\beta}+{}^{C}G^{\bar{k}}_{\alpha\beta}+{}^{C}H^{k}_{\alpha\beta}\right)$$

where ${}^{C}G^{k}_{\alpha\beta}$ and ${}^{C}H^{k}_{\alpha\beta}$ are first components of complete lift of (1,2)-tensor fields G and H respectively, ${}^{C}G^{\bar{k}}_{\alpha\beta}$ are second components of complete lift of G and ${}^{C}\Gamma^{k}_{\alpha\beta}$, ${}^{C}\Gamma^{\bar{k}}_{\alpha\beta}$, ${}^{C}\Gamma^{\bar{\bar{k}}}_{\alpha\beta}$ denote the all components of complete lift ${}^{C}\nabla$ of connection $\nabla=\left(\Gamma^{k}_{ij}\right)$.

The fourth chapter consists of 5 subchapters.

The fourth chapter provides information about almost product Walker structures on 3-dimensional product Norden-Walker manifolds.

The first subchapter of the fourth chapter seems to be looking at the product manifolds. Let (M_n, g) be a Lorentzian manifold (pseudo-Riemannian manifold), with a pseudo-Riemannian metric g of signature $(1, n-1)$ (equivalently $(n-1,1)$). Let (M_n,φ,g) be an almost product manifold, i.e., we assume that φ is an almost product structure satisfying $\varphi^2 = I$.

In the second subchapter of the fourth chapter, a three-dimensional Walker manifold is defined, and under certain conditions, an almost product Walker structure is constructed. Let M_3 be a 3-dimensional C^∞ $-$manifold. A metric G_f on a manifold M_3 is said to be Walker metric if there is a parallel 1-dimensional null distribution D on M_3. By a result of Walker theorem, for every Walker metric G_f on a 3-manifold M_3, there exist a system of coordinates which the matrix of $G_f = \left(\left(G_f\right)_{ij}\right)$ in these coordinates has following form:

$$G_f = \left(\left(G_f\right)_{ij}\right) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f \end{pmatrix},$$

where f is an arbitrary, differentiable function depending on the coordinates (x, y, z) and $Det\left(\left(G_f\right)_{ij}\right) = \varepsilon = \pm 1 \neq 0$.

Let φ be an almost product structure on a Walker 3-manifold M_3, which satisfies

1) $\varphi^2 = I$,
2) $\left(G_f\right)_{im}\varphi_j^m = \left(G_f\right)_{mi}\varphi_i^m$ or $G_f \cdot \varphi = {}^T\varphi \cdot G_f$ (property of purity).

We choose the almost product structure φ, which has the local components with respect to the natural frame $\{\partial_x, \partial_y, \partial_z\}$:

$$\varphi = \left(\varphi_j^i\right) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\sqrt{f\varepsilon}} & 0 & \sqrt{\frac{f}{\varepsilon}} \\ -\frac{1}{f} & \sqrt{\frac{\varepsilon}{f}} & 0 \end{pmatrix}.$$

The triple $\left(M_3, \varphi, G_f\right)$ is called almost product Norden-Walker manifold.

In the third subchapter of the fourth chapter, the conditions for the integration of the almost product structure on the almost product Norden-Walker manifold are shown. An almost product structure φ is integrable if the Nijenhuis tensor N_φ with the coordinates

$$\left(N_\varphi\right)_{jk}^{i} = \varphi_j^m \partial_m \varphi_k^i - \varphi_k^m \partial_m \varphi_j^i - \varphi_m^i \partial_j \varphi_k^m + \varphi_m^i \partial_k \varphi_j^m = 0$$

vanishes.

So we have obtained the following theorem:

Theorem 13: An almost product structure φ on an almost product Norden-Walker manifold is integrable if and only if

$$\begin{cases} \sqrt{f\varepsilon} f_x + f_y = 0, \\ f_z = 0, \end{cases}$$

where $f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}, f_z = \frac{\partial f}{\partial z}$.

In the fourth subchapter of the fourth chapter, the conditions for a Walker metric to be Kahler with respect to an almost product structure on the almost product Norden-Walker manifold are considered. Now let $\left(M_3, \varphi, G_f\right)$ be an almost product Norden-Walker manifold. First, we note that if

$$\left(\Phi_\varphi G_f\right)_{kij} = \varphi_k^m \partial_m \left(G_f\right)_{ij} - \varphi_i^m \partial_k \left(G_f\right)_{mj} + \\ + \left(G_f\right)_{mj} (\partial_i \varphi_k^m - \partial_k \varphi_i^m) + \left(G_f\right)_{im} \partial_j \varphi_k^m = 0,$$

then φ is integrable and the manifold $\left(M_3, \varphi, G_f\right)$ is called a product anti-Kähler-Walker manifold. So, we have the following theorem:

Theorem 14: The product Walker manifold $\left(M_3, \varphi, G_f\right)$ is Kahler if and only if the following PDEs hold:

$$f_x = f_y = f_z = 0,$$

i.e., $f = const.$

In the fifth subchapter of the fourth chapter, geodesics on the three-dimensional Walker manifold are investigated and their equations are given. A curve $x = x(t)$ is geodesic in $\left(M_3, \varphi, G_f\right)$ with respect to ∇ if and only if it satisfy the differential equations

$$\frac{d^2 x^i}{dt^2} + \Gamma_{kl}^i \frac{dx^k}{dt} \cdot \frac{dx^l}{dt} = 0, i, j, k = 1,2,3.$$

We can write the last equation as follows:

$$\frac{d^2x^1}{dt^2} + f_x \frac{dx^1}{dt} \cdot \frac{dx^3}{dt} + f_y \frac{dx^2}{dt} \cdot \frac{dx^3}{dt} + \frac{1}{2}(f \cdot f_x + f_z)\frac{dx^3}{dt} \cdot \frac{dx^3}{dt} = 0,$$

$$\frac{d^2x^2}{dt^2} - \frac{\varepsilon}{2} f_y \frac{dx^3}{dt} \cdot \frac{dx^3}{dt} = 0, \frac{d^2x^3}{dt^2} - \frac{1}{2} f_x \frac{dx^3}{dt} \cdot \frac{dx^3}{dt} = 0.$$

Consclusion

1. The integration conditions of generalized Norden-Walker structures and the holomorphic conditions of Walker metrics with respect to this structure are given;
2. It has been shown that the complete lift of a symplectic structure from its base manifold to its tangent bundle is a closed 2-form, and consequently, its image by symplectic isomorphism is a 2-form;
3. It has been shown that in the tangent bundle of a Walker manifold, the vertical and horizontal lifts of a vector field are always zero, and the total lift is zero only if and only if the vector field has a constant length;
4. The components of the deformed complete lifts of (1,1)-type tensor fields and connections in the tangent bundles of degree 2 are given;
5. The integration conditions of the almost product Norden-Walker structure on a 3-dimensional Walker manifold, the Kahler conditions of the Walker metric with respect to this structure, and the equations of geodesics are given.

The main results of the presented thesis has been published in following works:

1. Qurbanova, N.E. Walker çoxobrazlısı üzərində sanki kompleks struktur // Azərbaycan Xalqının Ümummilli Lideri Heydər Əliyevin anadan olmasının 98-ci ildönümünə həsr olunmuş Elmi Konfransın materialları. -Bakı, -2021, -s.89.
2. Qurbanova, N.E. Riman genişlənməsinin sanki kompleks struktura nəzərən təmiz olması haqqında// Azərbaycan Xalqının Ümummilli Lideri Heydər Əliyevin anadan olmasının 99-cu ildönümünə həsr olunmuş Elmi Konfransın materialları. -Bakı, -2022, -s.181-182.
3. Gurbanova, N.E. On a generalized Norden-Walker 4-manifold // Transactions of NAS of Azerbaijan, Issue Mathematics, Series of Physical-Technical and Mathematical Sciences. – 2022, 42(2), -p. 127-131 **(Scopus)**.
4. Gurbanova, N.E. On a 2-form derived by Riemannian metric in the tangent bundle // International Electronic Journal of Geometry. – 2022, 15(2), - p. 225-228 **(Web of Science, ESCI)**.
5. Gurbanova, N.E. Lift of Norden-Walker structures // News of Baku University series of physico-mathematical sciences. – 2023, 2, - p. 58-65.
6. Gurbanova, N.E., Salimov, A.A. Holomorphic connections and problems of lifts // Chinnese Annals of Mathematics, Series B. -2021, 42(1), -p. 121-134. // Chinnese Annals of Mathematics, Series B. – 2024, 45(5), - p. 1-8 **(Web of Science, SCIE)**.
7. Gurbanova, N.E. A product Walker 3-manifolds // News of Baku University series of physico-mathematical sciences. – 2023, 4, - p. 53-58.
8. Gurbanova, N.E., Salimov, A.A. Lifting of endomorphism fields // Book of abstracts of the 13-rd International Conference of the Georgian Mathematical Union. – Batumi, Georgia: September 4-9, - 2023, -p. 113.

9. Salimov, A.A., Gurbanova, N.E. A new lift construction of connections in the bundle of 2-jets // Abstract and Full Text Symposium Book of 2-nd International Symposium on Current Developments in Fundamental and Applied Mathematics Sciences. – Ankara, Turkey: November 14-17, - 2023, -p. 16-17.

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