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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

EXİSTENCE AND NON-EXİSTENCE OF GLOBAL SOLUTİONS TO THE MİXED PROBLEMS WİTH HOMOGENEOUS AND NON-HOMOGENEOUS BOUNDARY CONDİTİONS FOR A SYSTEM OF SEMİ-LİNEAR HYPERBOLİC EQUATİONS

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GENERAL CHARACTERISTICS OF WORK

Relevance and degree of development of the topic. The study of various problems of natural history is reduced to partial differential equations. Among the partial differential equations, of particular interest is the study of nonlinear hyperbolic equations. Despite numerous studies in this area, many unresolved issues remain. The main problem for nonlinear hyperbolic equations is to show the existence of a solution to a suitable mixed problem and to study the behavior of these solutions.

The main research on the study of mixed problems for nonlinear hyperbolic equations and systems began in the middle of the last century. The main results of research in this area until 1969 are reflected in the monograph by J.L. Lyons. A number of monographs have been published on research in the field of nonlinear equations. After the publication of the monograph by J.L. Lyons, research on nonlinear hyperbolic equations was intensified and carried out more intensively. Among them, one can note the studies carried out in the works of L. Boggio, I. Lasiechk, I. Cheushov, M. Eller, M. Ramaha, K. Agre, K. Khudaverdiev, V. Kalantarov, A.B. Aliev and many other researchers. Among these studies, we note several articles that are directly related to the topic of the dissertation.

In 1994 V. Georgiev and G. Todorova investigated the existence of local and global solutions to the initial boundary value problem

$$u_{tt} - \Delta u + |u_t|^{m-1} u_t = |u|^{p-1} u,$$
$$u\big|_{\partial\Omega} = 0,$$
$$u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x), x \in \Omega$$

in area $[0,T]\times\Omega$. Here Ω bounded area with a smooth border. İn case $n=1,2$ the condition $1<p<m$ is satisfied, and in the case of $n\geq 3$ additionally, condition $p<\dfrac{n}{n-2}$ it was shown that for any $\phi \in H_0^1, \psi \in L_2(\Omega)$ the problem under consideration has a unique solution $u \in C\left(\left[0,T\right];H_0^1\right)\cap C^1\left(\left[0,T\right];L_2\left(\Omega\right)\right)$. In the case $p > m$ under given conditions, there are such initial conditions according to which local solutions undergo a jump in a finite time (blow-up) (Journal of Differential Equations 1994, 109, p.295-308).

Later, various studies were carried out in this area, and similar results were obtained for a wider class of equations. For example, L. Boggio, I. Lasiechka in their work investigated a bounded domain with a smooth boundary the mixed problem with the initial condition

$$u_{tt} - \Delta u + g_0(u_t) = f(u),$$

$$\partial_v u + u + g(u_t) = h(u), \Gamma \times [0,\infty),$$

$$u(0,x) = \varphi(x),\ u_t(0,x) = \psi(x), x \in \Omega$$

and showed the existence of local solutions (here ∂_v, Γ derivative concerning the unit normal to the surface). They investigated the existence of global solutions to this problem. Here
$g_0(s)s \approx |s|^{m+1}, m > 0,\ \ g(s)s \approx |s|^{q+1}, q > 0$, $|f'(s)| \le C|s|^{p-1}$, $1 \le p \le 3$, for $p > 3$ it is assumed that $|f''(s)| \le C(1+|s|^{p-2})$,
$3 < p < \dfrac{6m}{m+1}$, for $0 < q < 1$, $|h'(s)| \le C(1+|s|^{k-1})$, $1 \le k \le \dfrac{4q}{q+1}$, $q \ge 1$, $|h''(s)| \le C(1+|s|^{k-2})$.

One of the main studies in this area was carried out by E. Vitalaro. E. Vitalaro developed a new proof method for determining the blow-up of a solution that has positive initial energy (Archive for Rational Mechanics and Analysis,-1999.149,-p.155-182).

I. Cheushov, M. Eller, I. Lasiechka investigated the properties of global solutions to the initial-boundary value problem in the framework of the nonlinear dissipated Robin boundary condition for nonlinear dissipated wave equations and showed the existence of a global minimal attractor.

Note that V. Kamornik and E. Zuazua developed new methods for determining the order of decay of solutions of nonlinear wave equations (Journal de Mathematiques Pures et Appliques, -1990, 69, -p.33-54).

Serious research has been carried out to study systems consisting of nonlinear wave equations.

For example, Segal's 1964 paper considered the Cauchy problem for a semilinear hyperbolic system. In this paper, the Cauchy problem for the Klein-Gordon system was considered in the form

$$\left.\begin{array}{r} u_{1tt} - \Delta u + u_1 u_2^2 = f_1(t,x) \\ u_{2tt} - \Delta u_2 + u_1^2 u_2 = f_2(t,x) \end{array}\right\}$$

and the solutions of this problem were investigated. Then this system was studied in more general terms by L.Medeiros and M. Miranda. In this work, they investigated in the domain $t \in [0,T],\ x \in R^n$ a potential hole that does not have an unfocused source function for a nonlinear system of the problem in the form of

$$\left.\begin{array}{r} u_{1tt} - \Delta u_1 + |u_1|^{\rho} |u_2|^{\rho+2} u_1 = f_1(t,x) \\ u_{2tt} - \Delta u_2 + |u_1|^{\rho+2} |u_2|^{\rho} u_2 = f_2(t,x) \end{array}\right\}, \quad t \in [0,T],\ x \in R^n$$

and proved the theorem on the existence of a global solution using the obtained results (Funkialaj Exvacioi, 1987,30, p. 147-161).

Many studies have been carried out in this area. Among these studies are articles by W. Liu, M.M. Miranda, A.T. Lawrendo, A.T. Clark, H.R. Clark, J. Wang, Yu. Ye, Z. Young, G.V. Chen, A. B. Aliev, A. Kazimov and others.

Many articles have investigated mixed problems with a nonlinear operator in the boundary condition for the nonlinear wave equation. For example, in the articles of N.T. Long, L.V. Ut, N.T. Trak, N.T. Long, and others, the existence of local and global solutions of a mixed problem for a one-dimensional nonlinear wave equation with the participation of a nonlinear operator in the boundary condition. In these works, the Galerkin method was used to solve the mixed problem.

For a nonlinear wave equation, mixed problems of the dynamic boundary condition, which are of practical importance, have not been sufficiently studied. Research in this area includes articles by I. Lasiechka, L. Boggio, I. Cheushov, M. Eller, M. Ramakh, E. Zuazua, E. A. Vitalaro, M. Pulkina.

The dissertation investigates the existence or non-existence of a solution to the mixed boundary value Riquier problem for a class of systems consisting of high-order semilinear hyperbolic equations. In addition, we studied the existence of a solution to a semilinear dynamic boundary value mixed problem and the asymptotics of solutions.

All results obtained are based on rigorous mathematical proofs.

Based on the foregoing, the topic of the thesis is relevant, is the focus of attention of mathematicians working in this area, and research in this area continues.

Object and subject of research.

Mixed problems for a system of equations consisting of semilinear hyperbolic equations. Investigation of the existence and uniqueness of local solutions to the issues under consideration, determination of the existence or absence of global solutions.

The goal and objectives of the study.

The aim of the thesis is to study the existence of global solutions of the boundary conditional initial boundary value Riquier problem for a system of equations of semilinear hyperbolic type of a certain class and a dynamic boundary value conditional initial boundary value problem for a one-dimensional nonlinear wave equation.

The main goal of the work is to find sufficient conditions for determining the existence and absence of global solutions to the problems under consideration.

The general technique of studies.

- Riquier problems are modeled in certain function spaces for a system of semilinear hyperbolic equations with boundary initial-boundary conditions, and the following methods are used to study them;
- Galerkin method;
- Apriori estimation method;

- Evaluation of the energy functional;
- Theory of nonlinear semigroups for the study of a dynamic boundary value problem with an initial boundary condition for a one-dimensional wave equation.

Main provisions of dissertation:

Within the Riquier boundary conditions for the system of semilinear hyperbolic equations:

- proof of theorems on the existence of a local solution to the initial-boundary value problem;
- a judgment that local solutions can be extended to the entire region if the nonlinear function of the source is not focused;
- the theorem that local solutions can continue throughout the entire domain if the nonlinear source function is focused and the growth order of the source function is less than the growth order of the nonlinear dissipative term;
- exponential decrease of the energy function according to global solutions, if the nonlinear function of the source is not focused;
- the theorem that local solutions undergo a jump in finite time if the nonlinear source function is focused and the growth order of the source function exceeds the growth order of the nonlinear dissipation term.

For a one-dimensional nonlinear wave equation

- modeling for the operator equation of a nonlinear boundary value mixed problem in the form of the Cauchy problem and conclusions about the existence of solutions;
- a theorem on the existence and uniqueness of a solution to a problem obtained in the limit when the coefficient at the highest derivative in the boundary condition is close to zero in a nonlinear mixed boundary value problem.

Scientific novelty. It was found that if the nonlinear source function is out of focus, then the local solutions continue globally regardless of the order of growth of the source function in the system of high-order semilinear hyperbolic equations under the conditions of local solutions. If the nonlinear source function is focused, local solutions continue globally if the sum of the increments for each of the sought functions does not exceed the order of increment of the nonlinear dissipative term, otherwise, it cannot continue.

It was also found that if the function of a nonlinear source is out of focus, an exponential decrease in the energy function is determined, corresponding to a homogeneous system with linear dissipation.

For the one-dimensional wave equation, the result of a complete solution is obtained by applying the nonlinear theory of semigroups for the dynamic boundary-value conditional initial-boundary value problem.

Sufficient conditions for the existence and uniqueness of local and global solutions of the semilinear dynamic boundary value and quasistatic mixed problem for the one-dimensional semilinear wave equation are determined.

The theoretical and practical significance of the study.

The results of the dissertation are theoretical. The results of the dissertation can be compared with recent results for nonlinear hyperbolic equations and systems of equations, and the main results are the latest results obtained in this area. Nevertheless, the results obtained can be used to solve some problems in the theory of elasticity, physics, relativistic quantum mechanics, and mathematical physics.

Approbation and application. The main results of the dissertation were presented and discussed at the following seminars, republican and international scientific seminars: At the seminars of Department “Differential equations” of IMM of ANAS (the head of the department, prof. A.B.Aliev), at the seminars of department of "Mathematical Analysis" of the Azerbaijan State Pedagogical University (the head of the department, prof. B.A.Aliev) and were reported at the international workshop "Non-Harmonic Analysis and Differential Operators" (Baku, 2016), at the VII international conference "Functional differential equations and their applications" dedicated to the 80th anniversary of Professor G.A. Magamedov (Makhachkala, 2015), at the international conference "Mathematical Analysis, Differential Equations and their Applications", held jointly by Azerbaijani, Turkish and Ukrainian mathematicians (Baku, 2016), in the spring mathematical school "Modern methods of the theory of boundary value problems" - "XXXI Pontryagin readings" (Voronezh, 3- May 9, 2020).

The personal contribution of the author is to state the purpose of the study and to choose a direction. In addition, all the obtained results and research methods belong to the author himself.

Publications of the author. On the topic of the dissertation, the author published 10 scientific works, of which 6 articles and 4 abstracts were published in publications recommended by the EAC under the President of the Republic of Azerbaijan.

The name of the institution where the dissertation was completed. The work was performed at the Differential equations department of The Institute of Mathematics and Mechanics of Azerbaijan National Academy of Sciences.

The total volume of the dissertation with a sign indicating the volume of the structural sections of the dissertation separately.

The dissertation consists of title page-449 characters, table of contents-2218 characters, introduction-35037 characters, III chapters (Chapter I-76000 characters, II chapter-50000 characters, III chapter-40000 characters), conclusion, and literature list. The total volume of work is 203704 characters.

CONTENT OF THE DISSERTATION

The introduction substantiates the relevance of the dissertation topic and outlines the purpose and a summary of the research work.

The first chapter of the dissertation is devoted to the mixed problem for the system of nonlinear hyperbolic equations with dissipation.

We denote the set of measurable and square-summable functions in Ω by $L_2(\Omega)$. The scalar product in $L_2(\Omega)$ is denoted as

$\langle u,v\rangle = \int\limits_{\Omega} u(x)v(x)dx$. The norm in this space is denoted by $\|u\| = \sqrt{\langle u,u\rangle}$.

We denote by $W_2^k(\Omega)$ the space of functions occurring in $L_2(\Omega)$ with generalized derivatives up to order k:

$$\|u\|_{W_2^k(\Omega)} = \left[\sum_{|\alpha|\le k}\int\limits_{\Omega}\left|D^{\alpha}u(x)\right|^2 dx\right]^{\frac{1}{2}}.$$

We denote by $\hat{W}_2^k(\Omega)$ the following space in $W_2^k(\Omega)$:

$$\hat{W}_2^k(\Omega) = \left\{u : u \in W_2^k(\Omega), \Delta^r u(x) = 0, x \in \Gamma, r = 0,1,...,\left[\frac{k-1}{2}\right]\right\}.$$

X - arbitrary Banach space, $C([0,T];X)$ - set of continuous functions acting from $[0,T]$ to X:

$$\|u(t)\|_{C([0,T];X)} = \max_{0\le t\le T}\|u(t)\|_X.$$

$C^k([0,T];X)$ - the set of continuous and differentiable functions of order k from $[0,T]$ to X:

$$\|u(t)\|_{C^k([0,T];X)} = \sum_{i=0}^{k}\left\|u^{(i)}(t)\right\|_{C([0,T];X)}.$$

We introduce the following functional spaces

$$H_{T,\infty}^1 = \Big\{(u_1(\cdot),u_2(\cdot)) : u_i(\cdot) \in L_\infty\left(0,T;\hat{W}_2^{k_i}(\Omega)\right),$$
$$u_{i_t}(\cdot) \in L_\infty\left(0,T;L_2(\Omega)\right),\ i = 1,2\Big\}$$

and

$$H_{T,\infty}^2 = \Big\{(u_1(\cdot),u_2(\cdot)); u_i(\cdot), u_{i_t}(\cdot) \in L_\infty\left(0,T;\hat{W}_2^{k_i}(\Omega)\right),$$
$$u_{i_{tt}}(\cdot) \in L_\infty\left(0,T;L_2(\Omega)\right),\ i = 1,2\Big\}.$$

Chapter I considers the following mixed problem for a system of semilinear hyperbolic equations in the domain $R_+ \times \Omega$

$$\left.\begin{aligned} u_{1_{tt}} + (-1)^{k_1} \Delta^{k_1} u_1 + \alpha_1 \left|u_{1_t}\right|^{r_1 - 1} u_{1t} = g_1(u_1, u_2) \\ u_{2_{tt}} + (-1)^{k_2} \Delta^{k_2} u_2 + \alpha_2 \left|u_{2_t}\right|^{r_2 - 1} u_{2t} = g_2(u_1, u_2) \end{aligned}\right\}, \tag{1}$$

$$\Delta^s u_i(t,x) = 0,\ t > 0,\ x \in \Gamma,\ s = 0,1,2,\ldots,k_i - 1,\ i = 1,2, \tag{2}$$

$$u_i(0,x) = \varphi_i(x),\ u_{i_t}(0,x) = \psi_i(x),\ x \in \Omega,\ i = 1,2. \tag{3}$$

Here $\Omega \subset R^n$-domain with smooth boundary, $R_+ = [0,+\infty)$, $\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$- Laplace operator, Δ^s- the Laplace operator of order s, $s = 1,2,\ldots,$, $\alpha_1 > 0$, $\alpha_2 > 0$, $r_1 \geq 1$, $r_2 \geq 1$ and (u_1, u_2)-the real function defined in $R_+ \times \Omega$.

g_1 and g_2 nonlinear functions of the following form:

$$g_1(u_1,u_2) = a_1|u_1 + u_2|^{p_1+p_2}(u_1 + u_2) + b_1|u_1|^{p_1-1}|u_2|^{p_2+1}u_1,$$

$$g_2(u_1,u_2) = a_2|u_1 + u_2|^{p_1+p_2}(u_1 + u_2) + b_2|u_1|^{p_1+1}|u_2|^{p_2-1}u_2,$$

such, that $a_1, a_2, b_1, b_2, p_1, p_2$ real numbers (constants) and

$$p_1 > 0,\ \ p_2 > 0. \tag{4}$$

Without loss of generality, it is assumed for certainty that

$$k_1 \leq k_2. \tag{5}$$

Note that for $k_1 = k_2 = 1$ each of the equations that make up the system (1) is a nonlinear wave operator, and for $k_1 = k_2 = 2$ each of these equations in the literature is called the Petrovsky equation or the Euler equation.

In the work, it is considered the following general cases:

$$\frac{n}{2} \leq k_1 \tag{6}$$

or

$$k_1 < \frac{n}{2} \leq k_2,\ p_1 + p_2 \leq \frac{2k_1}{n - 2k_1}, r_j \geq \frac{n + 2k_1}{n - 2k_2},\ j = 1,2. \tag{7}$$

First, in Section 1.1, we prove the following theorem on the existence of a local solution to problems (1)-(3).

Theorem 1. Let conditions (4), (5) and one of conditions (6), (7) be satisfied. Then, for any initial data $\varphi_i \in \hat{W}_2^{2k_i}(\Omega)$, $\psi_i \in \hat{W}_2^{k_i}(\Omega) \cap L_{2r_i}(\Omega)$, $i=1,2$ there exists $T'>0$, such that problem (1)-(3) has a unique solution $(u_1(\cdot),u_2(\cdot)) \in H^2_{T',\infty}$.

So if

$$(-1)^{k_i}\Delta^{k_i}u_i(\cdot)+\left|u_{i_t}(\cdot)\right|^{r_i-1}u_{i_t}(\cdot)\in L_\infty\left(0,T';L_2(\Omega)\right),\ i=1,2,$$

$T_{\max}$ is the length of the maximum interval of existence of this solution, then one of the following:

1) $\lim\limits_{t\to T_{\max}-0}\sum\limits_{i=1}^{2}\left[\left\|u_{i_t}(t,\cdot)\right\|^2+\left\|\nabla^{k_i}u_i(t,\cdot)\right\|^2\right]=+\infty;$

2) $T_{\max}=+\infty$.

Using Theorem 1, we prove the following existence and uniqueness theorem for weak local solutions.

Theorem 2. Let conditions (4), (5), and one of conditions (6), (7) be satisfied. Then, for any initial data $\varphi_i \in \hat{W}_2^{k_i}(\Omega)$, $\psi_i \in L_2(\Omega), i=1,2$ there exists $T'>0$, such that problem (1)-(3) has a unique solution $(u_1,u_2)\in H^1_{T,\infty}$.

So,

$u_i \in C\left([0,T'];\hat{W}_2^{k_i}(\Omega)\right)$, $u_{i_t}\in C\left([0,T'];L_2(\Omega)\right)$, $u_{i_t}\in L_{r_i}(Q_{T'})$, $i=1,2$.

So, if $T_{\max}=T'$ is the length of the maximum interval for the existence of a given solution, then one of the following is true:

1) $\lim\limits_{t\to T_{\max}-0}\sum\limits_{i=1}^{2}\left[\left\|u_t(t,\cdot)\right\|^2+\left\|\nabla^{k_i}u(t,\cdot)\right\|^2\right]=+\infty;$

2) $T_{\max}=+\infty$.

In case $k_1=k_2=1,\ p_1=p_2$ the result is the same as in the article of B.Said-Horari (Differential Integral Equations, -2010. 23, №1(2) - p.79-92).

Note that for $a_1 = a_2 = 0$ and $k_1 = k_2 = 1$ for different p_1, p_2 problems (1)-(3) were considered in the article by Y.Wang. However, the article contains a mistake in proving the existence of a solution, and the conclusion is incorrect (IMA Journal of Applied Mathematics -2009. 74, -p. 392-415).

In the first chapter's first paragraph we study the existence and uniqueness of global solutions in the case of an unfocused nonlinear source function.

The problem is investigated under one of the following conditions:

$$p_1 > 0,\ p_2 > 0,\ \frac{n}{2} < k_1 \le k_2, \tag{8}$$

$$p_1 > 0,\ p_2 > 0,\ p_1 + p_2 \le \frac{2k_1}{n - 2k_1},\ k_1 < \frac{n}{2} < k_2. \tag{9}$$

Suppose that the coefficients of the nonlinear part satisfy the following conditions:

$$a_i < 0,\ b_i < 0,\ i = 1,2, \tag{10}$$

$$\lambda = \frac{a_1(p_1+1)}{b_1} = \frac{a_2(p_2+1)}{b_2}. \tag{11}$$

Theorem 3. Let the conditions (10),(11) and one of the conditions (8),(9) be satisfied, then for $T > 0$, $\varphi_i \in \hat{W}_2^{2k_i}(\Omega)$, $\psi_i \in \hat{W}_2^{k_i}(\Omega)$, problem (1)-(3) has a unique solution

$$(u_1(\cdot), u_2(\cdot)) \in C([0,T]; (\hat{W}_2^{2k_1}(\Omega) \times \hat{W}_2^{2k_2}(\Omega)) \cap$$
$$\cap C^1([0,T]; \hat{W}_2^{k_1}(\Omega) \times \hat{W}_2^{k_2}(\Omega)) \cap C^2([0,T], L_2(\Omega) \times L_2(\Omega)).$$

In 1.1.2 we consider the case of a focused nonlinear source function.

It is assumed that

$$a_i > 0,\ b_i > 0,\ i = 1,2, \tag{12}$$

$$p_1 + p_2 + 1 \le \min\{r_1, r_2\}. \tag{13}$$

Theorem 4. Let conditions (8), (9), (11), (12) and (13) be satisfied. Then for $\forall T>0$ the local solution defined in Theorem 1 can be extended to the domain $[0,T]\times\Omega$.

Chapter II investigates the nature of solutions to mixed problems for a system of semilinear hyperbolic equations at infinity and the non-existence of a global solution.

First, in the second chapter's first paragraph, we also studied the exponential rate of decrease of the total energy of a system of hyperbolic equations with an unfocused nonlinear source function and linear dissipation.

Here in the domain $Q=[0,\infty)\times\Omega$ the following mixed problem is considered:

$$\begin{cases} u_{1_{tt}}+(-1)^{k_1}\Delta^{k_1}u_1+u_{1_t}+|u_1|^{p_1-1}|u_2|^{p_2+1}u_1=0 \\ u_{2_{tt}}+(-1)^{k_2}\Delta^{k_2}u_2+u_{2_t}+|u_1|^{p_1+1}|u_2|^{p_2-1}u_2=0 \end{cases}, \tag{14}$$

$$\Delta^s u_i(t,x)=0,\ t\in(0,\infty),\ x\in\partial\Omega,\ s=0,1,\ldots,k_i-1,\ i=1,2, \tag{15}$$

$$u_i(0,x)=\varphi_i(x),\ u_{i_t}(0,x)=\psi_i(x),\ x\in\Omega,\ i=1,2. \tag{16}$$

Without loss of generality, for definiteness, it is assumed that $k_1\le k_2$ and the following conditions are allowed to be satisfied:

$$\frac{n}{2}\le k_1,\ p_1>0,\ p_2>0. \tag{17}$$

Under condition 17 according to Theorem 2, any $\varphi_i\in\hat{W}_2^{k_i}$, $\psi_i\in L_2(\Omega)$ problem (14)-(16) has a unique solution (u_1,u_2) in space

$$C\left([0;\infty);\hat{W}_2^{k_1}\times\hat{W}_2^{k_2}\right)\cap C^1\left([0,\infty);[L_2(\Omega)]^2\right).$$

Let denotes the total energy of the system by

$$E(t)=\sum_{i=1}^{2}\frac{p_i+1}{2}\left[\left|u_{i_t}(t,x)\right|^2\right]+\left|\nabla^{k_i}u_i(t,x)\right|^2+$$
$$+\int_\Omega |u_1(t,x)|^{p_1+1}|u_2(t,x)|^{p_2+1}dx.$$

Theorem 5. Let conditions (17) be satisfied. Then there are positive constants M and ω such that

$$E(t) \le Me^{-\omega t},\ t \ge 0.$$

In 2.2, the subject to a jump of solutions of the mixed problem of semilinear hyperbolic systems of equations with nonlinear dissipation is investigated in a finite time interval (blow-up).

Chapter I shows that under the condition

$$p_1 + p_2 + 1 \le \max\{r_1, r_2\}. \tag{18}$$

The problem

$$\left.\begin{aligned} u_{1_{tt}} - \Delta u_1 + \alpha_1 \left|u_{1_t}\right|^{r_1 - 1} u_{1_t} = g_1(u_1, u_2) \\ u_{2_{tt}} - \Delta u_2 + \alpha_2 \left|u_{2_t}\right|^{r_2 - 1} u_{2_t} = g_2(u_1, u_2) \end{aligned}\right\}, \tag{19}$$

$$u_i(t,x) = 0,\ t > 0,\ x \in \Gamma,\quad i = 1,2, \tag{20}$$

$$u_i(0,x) = \varphi_i(x),\ u_{i_t}(0,x) = \psi_i(x),\ x \in \Omega,\ i = 1,2, \tag{21}$$

has a global solution.

If condition (17) is not satisfied, that is, if

$$p_1 + p_2 + 1 > \max\{r_1, r_2\} \tag{22}$$

then the question of whether it is possible to globally extend the local solutions defined by Theorem 3.

Let, the condition is satisfied:

$$\begin{gathered} a_i < 0,\ b_i < 0,\ i = 1,2, \\ \frac{a_1(p_1+1)}{b_1} = \frac{a_2(p_2+1)}{b_2} = \lambda. \end{gathered} \tag{23}$$

Here λ arbitrary positive number. It is proved that there exist numbers $c_1 > 0$, $c_2 > 0$, such that

$$c_1(|u_1|^{p_1+p_2+2} + |u_2|^{p_1+p_2+2}) \le G(u_1,u_2) \le c_2(|u_1|^{p_1+p_2+2} + |u_2|^{p_1+p_2+2}).$$

Here

$$G(u_1,u_2) = \frac{p_1+1}{b_1} u_1 g_1(u_1,u_2) + \frac{p_2+1}{b_2} u_2 g_2(u_1,u_2) =$$

$$= \lambda |u_1 + u_2|^{p_1+p_2+2} + (p_1 + p_2 + 2)|u_1|^{p_1+1} \cdot |u_2|^{p_2+1}.$$

Let's take the following notation:

$$A = a^{\frac{p_1+p_2+2}{2}}, \; a = \max\{a_1, a_2\}, \; \Lambda = \lambda^{\frac{p_1+p_2+2}{2}}, \; \alpha_1 = \left(\frac{\Lambda}{c_2 A B^{p_1+p_2+2}} \right)^{\frac{1}{p_1+p_2}}.$$

So, B is the norm of the embedding operator. Let's define the following functionals corresponding to the $(u_1(\cdot), u_2(\cdot))$ solutions of the problem (18)-(20).

$$E_0(t) = \sum_{i=1}^{2} \frac{\lambda}{2a_i} \left[\|u_{it}(t)\|^2 + \|\nabla u_i(t)\|^2 \right],$$

$$E(t) = E_0(t) - \frac{1}{p_1 + p_2 + 1} \int_\Omega G(u_1, u_2) dx.$$

Theorem 6 Let conditions (22), (23) be satisfied and $\varphi_i(\cdot) \in \hat{W}_2^1(\Omega)$, $\psi_i(\cdot) \in L_2(\Omega)$, $i = 1, 2$. Suppose that

$$E(0) < E_1 = \left(\frac{1}{2} - \frac{1}{p_1 + p_2 + 2} \right) \alpha_1^2, \; \left[\sum_{i=1}^{m} \frac{\lambda}{a_i} \|\nabla \varphi_i(\cdot)\|^2 \right]^{\frac{1}{2}} > \alpha_1 \text{ and}$$

$$B \left[\sum_{i=1}^{2} \frac{\lambda}{a_i} \|\nabla \varphi_i(\cdot)\|^2 \right]^{\frac{p_1+p_2+2}{2}} > 1.$$

Then the solution of the problem (19)-(21) collapses in a finite time.

Chapter III investigates a mixed problem with a nonlinear boundary condition for a one-dimensional nonlinear wave equation on the boundary.

In 3.3, the mixed problem with a dynamic boundary condition is modeled in the form of the Cauchy problem for the operator equation, and the existence of a solution is proved.

First, we consider the following mixed problem for a one-dimensional wave equation with nonlinear dissipation and nonlinear source function:

$$u_{tt} - u_{xx} + B_1(u_t) + B_2(u) = f(t, x), \; x \in (0,1), \; t > 0, \tag{24}$$

$$\varepsilon u_{tt}(t,0)-u_x(t,0)+b_{10}(u_t(t,0))+b_{20}(u(t,0))=f_0(t),\ t>0, \tag{25}$$
$$\delta u_{tt}(t,1)+u_x(t,1)+b_{11}(u_t(t,1))+b_{21}(u(t,1))=f_1(t),\ t>0, \tag{26}$$
$$u(0,x)=\varphi(x),\ u_t(0,x)=\psi(x) \tag{27}$$

where f, f_0 and f_1 are real functions.

$B_1(s)=\mu|s|^{q-1}s,\ b_{10}(s)=\mu_0|s|^{q_0-1}s,\ b_{11}(s)=\mu_1|s|^{q_1-1}s$

so, $\mu,\mu_0,\mu_1,q,q_1,q_2$ real numbers satisfy the following conditions:

$$\mu\ge 0,\mu_0\ge 0,\mu_1\ge 0\ q>1,q_0>1 \text{ and } q_1>1 \tag{28}$$

that is B_2,b_{20} and b_{21} are functions satisfying the Lipschitz condition.

Let us denote by H the direct sum of spaces $L_2(0,1)$ and $R^2=R\oplus R$:
$H=L_2(0,1)\oplus R\oplus R=\{w: w=(u,\alpha,\beta),u\in L_2(0,1),\alpha,\beta\in R\}$.

So,

$$\langle w_1,w_2\rangle_H=\int_0^1 u_1(x)u_2(x)dx+\varepsilon\alpha_1\cdot\alpha_2+\delta\beta_1\cdot\beta_2,$$
$$w_k=(u_k,\alpha_k,\beta_k),u_k\in L_2(0,1),\alpha_k,\beta_k\in R,k=1,2.$$

Let denote by H_0 and H_1 respectively the following spaces:

$$H_0=\{\tilde u:\tilde u=(u,u(0),u(1)),u\in W_2^1(0,1)\},$$
$$H_1=\{\tilde u:\tilde u=(u,u(0),u(1)),u\in W_2^2(0,1)\}.$$

We define in space H a linear operator $A_{\varepsilon,\delta}$

$$\begin{cases} D(A_{\varepsilon,\delta})=H_1, \\ A_{\varepsilon,\delta}\tilde u=\left(-u_{xx}(x),-\frac{1}{\varepsilon}u_x(0),\frac{1}{\delta}u_x(1)\right),\tilde u=(u,u(0),u(1))\in D(A_{\varepsilon,\delta}) \end{cases}.$$

In addition, we introduce nonlinear operators as follows:

$$G_{\varepsilon,\delta}(\tilde{u})=\left(\mu|\vartheta(x)|^{q-1}u(x), \frac{\mu_{10}}{\varepsilon}|u(0)|^{q_{10}-1}u(0), \frac{\mu_{11}}{\delta}|u(1)|^{q_{11}-1}u(1)\right)$$

$$\Phi_{\varepsilon,\delta}(\tilde{u})=\left(B_2(x,u(x)),\frac{1}{\varepsilon}b_{20}(u(0)),\frac{1}{\delta}b_{20}(u(1))\right),$$

$$\tilde{u}=(u(x),u(0),u(1))\in H_1.$$

It is proved that for each $\varepsilon>0$, $\delta>0$ the operator $A_{\varepsilon,\delta}$ is a self-adjoint positive operator in the space $H=L_2(0,1)\oplus R\oplus R.$

For each $\varepsilon>0, \delta>0, \mu\geq 0, \mu_0\geq 0, \mu_1\geq 0$ the operator $G_{\varepsilon,\delta}$ a monotone operator from H_0 in H_0', where H_0' is a double (dual) space in H_0.

It is shown that the nonlinear operator $\Phi_{\varepsilon,\delta}(\cdot)$ acts from the space H_0 into H and satisfies the local Lipschitz condition.

Problems (25)-(28) can be written as the following Cauchy problem in the space $H=L_2(0,1)\oplus R\oplus R$:

$$w''(t)+A_{\varepsilon,\delta}w(t)+G_{\varepsilon,\delta}(w'(t))+\Phi_{\varepsilon,\delta}(w(t))=F_{\varepsilon,\delta}(t), \tag{29}$$

$$w(0)=w_0, w'(0)=w_1 \ . \tag{30}$$

Here $F_{\varepsilon,\delta}(t,x)=\begin{pmatrix} f(t,x) \\ \frac{1}{\varepsilon}f_0(t) \\ \frac{1}{\delta}f_1(t)\end{pmatrix}$, $w_0=\begin{pmatrix}\varphi(x)\\ \varphi(0)\\ \varphi(1)\end{pmatrix}$, $w_1=\begin{pmatrix}\psi(x)\\ \psi(0)\\ \psi(1)\end{pmatrix}$.

Problems (29)-(30) are reduced to a first-order operator-differential equation in a certain Hilbert space. At first, we consider the linearly dissipative case. For this equation, existence and uniqueness theorems are proved for problems (29)-(30) in the case of linear dissipation, using already known theorems (Theorem 3.3.1). Applying these results, we obtained a solvability result for the following problem:

$$u_{tt} - u_{xx} + B_1(u_t) = f(t,x),\ t>0, x\in(0,1), \tag{31}$$

$$\varepsilon u_{tt}(t,0) - u_x(t,0) + b_{10}(u_t(t,0)) = f_0(t),\ t>0, \tag{32}$$

$$\delta u_{tt}(t,1) + u_x(t,1) + b_1(u_t(t,1)) = f_1(t),\ t>0, \tag{33}$$

$$u(0,x) = \varphi(x), u_t(0,x) = \psi(x),\ x\in(0,1). \tag{34}$$

Theorem 7. Let $\varepsilon > 0$, $\delta > 0$, $\mu_{10} \geq 0$, $\mu_{11} \geq 0$ $f(\cdot) \in W_1^1(0,T;W_2^1(0,1),L(0,1))$, $f_0(t)$ и $f_1(t) \in W_1^1(0,T)$. Then for any $\varphi \in W_2^2(0,1)$, $\psi \in W_2^1(0,1)$ and $T>0$ problem (31)-(34) has a unique solution $u_{\varepsilon\delta}(\cdot) \in W_\infty^2(0,T;W_2^2(0,1),W_2^1(0,1),L_2(0,1))$.

So $u_{\varepsilon\delta}(0,\cdot), u_{\varepsilon\delta}(1,\cdot) \in W_\infty^2(0,T;R)$.

Further, problem (29)-(30) was investigated in the case of nonlinear dissipation, and a result was obtained on the existence of a local solution (Theorem 3.3.3).

Applying this result, a theorem was proved on the existence of a local solution to a problem (24)-(27) (Theorem 3.3.4).

Section 3.2 investigates the "complete solution" of a mixed problem with a dynamic boundary condition for the one-dimensional nonlinear wave equation.

Section 3.3 investigates problems (24)-(27) under the following conditions:

$$B_2(u) = \eta|u|^{p-1}u,\ \eta \geq 0,\ p \geq 1,$$

$$b_{20}(\xi) = \eta_{20}|\xi|^{p_{20}-1}\xi,\ \eta_{20} \geq 0,\ p_{20} \geq 1,$$

$$b_{21}(\xi) = \eta_{21}|\xi|^{p_{21}-1}\xi,\ \eta_{21} \geq 0,\ p_{21} \geq 1.$$

In this case, the existence and uniqueness of a global solution to the problem (24)-(27) is proved (Theorem 3.4.1).

In subsection 3.3 problem (24)-(27) is considered in the case of linear dissipation:

$$u_{tt}-u_{xx}+u_t+B_2(u)=f(t,x),\ x\in(0,1),\ t>0, 0\le t\le T, \quad (35)$$

$$\varepsilon u_{tt}(t,0)-u_x(t,0)+u_t(t,0)+b_{20}(u(t,0))=f_0(t), 0\le t\le T, \quad (36)$$

$$\delta u_{tt}(t,1)-u_x(t,1)+u_t(t,1)+b_{21}(u(t,1))=f_1(t),\ 0\le t\le T, \quad (37)$$

$$u(0,x)=\varphi(x),\ u_t(0,x)=\psi(x),\ \ x\in(0,1). \quad (38)$$

where, $B_2(s)=\eta|s|^{p-1}s,\ b_{20}(s)=\eta_0|s|^{p_0-1}s,\ b_{21}(s)=\eta_1|s|^{p_1-1}s,$
$f(t,x)\in W_2^1([0,T]\times(0,1)),\ \ f_0(t),\ f_1(t)\in W_2^1(0,T).$

Again, the problem was reduced to the Cauchy problem for an operator-differential equation in space $H=L_2(0,1)\oplus R\oplus R$ and after studying this problem, the following result was obtained for the problem (35) - (38)

Theorem 8. Suppose that $\varepsilon>0,\ f(t,x)\in W_2^1([0,T]\times(0,1)),$ $f_0(\cdot), f_1(\cdot)\in W_2^1(0,T).$ Then for arbitrary $\varphi\in W_2^2(0,1),\ \ \psi\in W_2^1(0,1)$ there exists $T'>0$ such that the problem (35)-(38) has the only solution $u_\varepsilon(\cdot)\in C^2([0,T'];W_2^2(0,1),W_2^1(0,1),L_2(0,1)).$

So $u_\varepsilon(0,t),\ u_\varepsilon(1,t)\in C^2([0,T'];R).$

If $T_{\max}$ is the length of the maximum interval of existence of the solution, then one of the following alternatives is fulfilled:

1. $\lim\limits_{t\to T_{\max}}\left[\|\tilde{u}_{\varepsilon t}(t)\|_H^2+\|\tilde{u}_\varepsilon(t)\|_{H_1}\right]=+\infty;$
2. $T_{\max}=+\infty.$

Theorem 9. Suppose that $\varepsilon>0,\eta_1\ge 0,\eta_2\ge 0,$
$f(t,x)\in W_2^1([0,T]\times(0,1)),\ f_0(\cdot), f_1(\cdot)\in W_2^1(0,T).$ Then for arbitrary $\varphi\in W_2^2(0,1),\ \ \psi\in W_2^1(0,1)$, the problem (35)-(38) has a unique solution

$$u_\varepsilon(\cdot)\in C^2([0,T'];W_2^2(0,1),W_2^1(0,1),L_2(0,1)).$$

So, $u_\varepsilon(0,t),\ u_\varepsilon(1,t)\in C^2([0,T];R).$

In the end, we consider the following mixed problem on $[0,T]\times(0,1)$:

$$u_{tt}-u_{xx}+u_t+B_2(u)=f(t,x),x\in(0,1),t\in[0,T], \tag{39}$$

$$-u_x(t,0)+u_t(t,0)+b_{20}(u(t,0))=f_0(t),t\in[0,T], \tag{40}$$

$$u_x(t,1)+u_t(t,1)+b_{21}(u(t,1))=f_1(t),t\in[0,T], \tag{41}$$

$$u(0,x)=\varphi(x),u_t(0,x)=\psi(x),\quad x\in(0,1). \tag{42}$$

Here $B_2(s)=\eta|s|^{p-1}s,b_{20}(s)=\eta_0|s|^{p_0-1}s,\ b_{21}(s)=\eta_1|s|^{p_1-1}s.$

Theorem 10. Let $\eta\geq 0,\ \eta_0\geq 0,\ \eta_1\geq 0,\ \ f(t,x)\in W_2^1([0,T]\times(0,1))$, $f_0(t),f_1(t)\in W_2^1(0,T)$. Then for arbitrary $\varphi\in W_2^2(0,1),\psi\in W_2^1(0,1)$ there exists a solution to the problem (39)-(42), so that

$$u(\cdot)\in L_\infty(0,T;W_2^2(0,1)),\ u_t(\cdot)\in L_\infty(0,T;W_2^1(0,1)),$$

$$u_{tt}(\cdot)\in L_\infty(0,T;L_2(0,1)),\ u_t(0,\cdot),u_t(1,\cdot),u_x(0,\cdot),u_t(1,\cdot)\in L_2(0,T).$$

CONCLUSIONS

For a system of semilinear hyperbolic equations, there is only one local solution to the initial-boundary value problem within the framework of the Riquier boundary conditions, and the existence of the solution can be solved by the Galerkin method.

For this problem, the following statements are true:

- for a system of semilinear hyperbolic equations, local solutions of the initial-boundary value problem within the framework of the Riquier boundary conditions can be continued in the entire domain if the function of the nonlinear source is not focused;
- local solutions can continue throughout the entire region if the nonlinear source function is focused and the order of growth of the source function is less than the order of growth of the nonlinear dissipation term;
- the energy function corresponding to global solutions decreases exponentially if the nonlinear source function is not focused;
- local solutions under certain conditions undergo a jump in a finite time interval if the nonlinear source function is focused and the order of growth of the source function exceeds the order of growth of the nonlinear dissipation term;
- For a one-dimensional nonlinear wave equation, the nonlinear mixed boundary problem can be modeled as the Cauchy problem for an operator equation in a given functional space, and this problem is solvable;
- the quasi-static problem, obtained in the limit when the coefficient of the higher-order derivative in the boundary condition of the nonlinear boundary value mixed problem is zeroed out, has a solution ,and is unique.

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