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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

MANIFOLDS AND DIFFERENTIAL GEOMETRIC STUCTURES ON THEIR BUNDLES

Speciality: **1204.01– Geometry**

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GENERAL CHARACTERISTICS OF THE WORK

Rationale and development degree of the topic.

The study of differential geometry of tangent bundles was started in the early 1960s. A special metric on the tangent bundle of a Riemannian manifold was defined by Sasaki in 1958, and it was later realized that this special Sasakian metric can be seen as a diagonal lift from the base. Research about tangent bundle of first order was studied by K.Yano and Sh.Ishihara, A.Morimoto, I.Sato, V.Vishnevski, A.Shirokov, V.Shurıgin, A.Salimov and continued with others. In some of these works, algebraic structures in the tangent bundles of order I and II were found and new classes of lifts (synectic and modification lifts) were defined in the theory of lifts.

Another main stream is the study of the extension of differential geometric structures to tangent and cotangent bundles. In this context, Sasaki and Kenmotsu have very interesting studies on almost contact structures. We can also mention the works of D.Blair, K.Kenmotsu, J.Qubina, A.Salimov, S.Tanno and others on this subject.

One of the fundamental topics in modern differential geometry is the theory of lifts. Some studies have focused on lifting problems along pure subbundles. It is clear that tangent and cotangent bundles are pure subbundles. But the second-order tangent bundle is not a pure subbundle, and there are many standard methods that do not work here. Therefore the lift problem studies on these bundles are quite interesting and actual.

Object and subject of the study. The main subject of our study is lifting problems on tangent (I and II order) and cotangent bundles and properties of differential geometric structures.

Goal and objectives of the study. The aim of the study is to investigate the liftings from base manifolds to tangent and cotangent bundles. For this purpose, tensor operators (Lie and covariant derivative, Tachibana operator) and holomorphism in nilpotent algebras are used. Construction of deformed lifts on high order tangent bundle is antoher goal of the work.

Research methods. The following methods are used in our study:

1. Classic tensor method by index;
2. Invariant method for arbitrary coordinate system;
3. Real modeling of spaces constructed on spaces over algeras.

The main points of the study. The main points are:

1. Complete lifts of α-Sasaki, β-Kenmotsu and trans-Sasaki structures;

2. Integrability of almost complex structures on the cotangent bundle. Also the Tachibana operator associated with them;

3. Holomorphic functions over commutative and associative algebras which dimension more than 2;

4. Kahler type metrics on tangent bundle of order II;

5. Relation between higher order tangent bundles and deformed lifts of objects.

Scientific novelty of the study. The following new results are presented:

1. Obtained the explicit form of holomorphic functions in nilpotent algebras of order III and IV;
2. Kahler type metrics found in 2-jet tangent bundle;
3. On tangent bundle of order 2 were constructed lifts of α-Sasakian, β-Kenmotsu, trans-Sasakian structurs;
4. The integrability condition for almost complex structures in the cotangent bundle is found. Also, the Tachibana operator that relates these structures and applies to complete and horizontal lifts is found.

Theoretical and practical value of study. The main results obtained in the thesis are mostly of a theoretical nature. The results obtained and the methods used in the thesis can be used in the problems of extensions of differential geometric structures to higher order tangent bundles. In addition, it will be also applied to the properties ((holomorphism, curvature) of many differential geometric objects.

Aprobation and application. The most important achievements of this dissertation have been presented at local and international conferences: Republican scientific conference “Actual problems of Mathematics and Mechanics” devoted to 96 years of the National leader

Haydar Aliyev (Baku, 2019), at the International conference "Modern problems of innovative technologies in oil and and gas production and applied mathematics" dedicated to the 90th anniversary of Academician Azad Khalil oglu Mirzajanzade, at the International conference "Modern Problems of Mathematics and Mechanics" devoted to the 60th anniversary of the Institute of Mathematics and Mechanics of Azerbaijan National Academy of Sciences, XXIII republican scientific conference of doctoral students and young researchers kicked off at the Azerbaijan University of Architecture and Construction (AUAC) with the organization of the Ministry of Education, Republican scientific conference on "Mathematics, Mechanics and their applications" dedicated to the 97th anniversary of National Leader Heydar Aliyev (Baku, 2020), also 18th International Geometry Symposium in Malatya-TURKEY, Mathematical Analysis, Differential Equation & Applications - MADEA 9, Kyrgyz-Turkish Manas University, Bishkek, Kyrgyz Republic, Republican virtual scientific conference on "Mathematics, Mechanics and their applications" dedicated to the 98th and 99th anniversary of National Leader Heydar Aliyev, Republican virtual scientific conference on "Differensial and integral operators" dedicated to the 100th anniversary of National Leader Heydar Aliyev.

The organization where the work was executed: Dissertation work was executed in the chair of "Algebra and Geometry" of "Mechanics-Mathematics" department of Baku State University.

Authors personal contribution. The results obtained in the dissertation belong to the applicant.

Published scientific works. The main results of the dissertation work were published in journals recommended by HAC under President of the Republic of Azerbaijan – 5 scientific works: 3 in scientific journals incuded in "Web of Science" database. All the conclusions and results obtained in the dissertation are published in given scientific editions. Furthermore, the results obtained in the dissertation were reported and published at international level 2 and republican level 8 scientific conferences.

Total volume of the dissertaton work indicating separate structural units of the work in signs.

The dissertation work consists of introduction, three chapters, conclusions (title page– 424 signs, table of contents – 2875 signs, introduction – 24000 signs, chapter I 40000 signs, chapter II–66000 signs, chapter III -76000 signs, conclusions–602 signs) and list of references consisting of 106 names. The total volume of the dissertation work is 209901 signs.

CONTENT OF THE DISSERTATION WORK

Let us give brief review of the dissertation work consisting of 3 chapters.

In the Introduction we give the brief review of work related to the dissertation work, substantlate actuality of the dissertation work, give main results obtained in the work.

Chapter I consists of five subchapters. The tangent bundle of I and II order of differentiable manifold are given in Chapter I.

In subchapter I of chapter I introduces the concept of tangent bundle of the I order. There are vertical, complete and horizontal lifts of functions, vector fields, tensor fields and 1-forms are considered on $T(M)$ in invariant and coordinate form.

Let M be an $n-$ dimensional differentiable manifold of class C^∞ and $T_P(M)$ the tangent space at a point P of $M,$ that is, the set of all tangent vectors of M at P. Then the set

$$T(M) = \bigcup_{P \in M} T_P(M)$$

is, by definition, the tangent bundle over the manifold M.

For any point $\widetilde{P}$ of $T(M)$ such that $\tilde{P} \in T_P(M),$ the correspondence $\tilde{P} \to P$ determines the bundle projection $\pi: T(M) \to M$ that is, $\pi\big(\tilde{P}\big) = P,$ where $\pi: T(M) \to M$ defines the natural bundle structure of $T(M)$ over M. The set $\pi^{-1}(P)$ that is, $T_P(M)$, is called the fibre over $P \in M$ and M the base space.

If f is a function in $M,$ we write f^V for the function in $T(M)$

obtained by forming the composition of $\pi: T(M) \to M$ and $f: M \to R$, so that

$$f^V = f \circ \pi.$$

If f is a function in M, we write f^C for the function in $T(M)$ defined by

$$f^C = \iota(df)$$

and call f^C the complete lift of the function f in M to the tangent bundle $T(M)$.

We assume that there is given an affine connection ∇ in a differentiable manifold M. If f is a function in M, we write ∇f for the gradient of f in M. We apply the operation γ to the gradient ∇f and get $\gamma(\nabla f)$. We put

$$\nabla_\gamma f = \gamma(\nabla f) = y^s \partial_s f.$$

We now define the horizontal lift f^H of f in M to the tangent bundle $T(M)$ by

$$f^H = f^C - \nabla_\gamma f = 0.$$

Let, $\tilde{X} \in \Im_0^1(T(M))$ be such that $\tilde{X} f^V = 0$ for all $f \in \Im_0^0(M)$. Then we say that $\tilde{X}$ is a vertical vector field.

$$\tilde{X} f^V = 0.$$

Suppose that, $X \in \Im_0^1(M)$. We define a vector field X^C in $T(M)$ by:

$$X^C f^C = (Xf)^C,$$

f being an arbitrary function in M and call X^C the complete lift of X to $T(M)$.

Let there be given an element X of $\Im_0^1(M)$. Then we define the horizontal lift X^H of X by $X^H = X^C - \nabla_\gamma X$.

$$\nabla_\gamma X = \gamma(\nabla X) = (0, y^s \partial_s X^h) = (0, y^s(\partial_s X^h + \Gamma_{sm}^h X^m)).$$

Let $\tilde{\omega} \in \Im_1^0(T(M))$ be such that $\tilde{\omega}(X^V)$ for all $\tilde{X} \in \Im_0^1(M)$.

$$\tilde{\omega}(X^V) = 0$$

Then we say that $\widetilde{\omega}$ is a vertical 1-form.
Vertical lift ω^V of ω with local expression $\omega = \omega_i dx^i$ has components of the form $\omega^V = (\omega_i, 0)$.

Suppose that $\tilde{\omega} \in \Im_1^0(T(M))$ a 1-form. Then, we can define a 1-form ω^C in M and we call ω^C the complete lift of ω in M to $T(M)$ by

$$\omega^C\left(X^C\right) = (\omega(X))^C,$$

X being an arbitrary vector field in M, and we call ω^C the complete lift of ω in M to $T(M)$.
Thus the complete lift ω^C of ω with components ω_i in M has components of the form $\omega^C = (\partial\omega_i, \omega_i)$.

Let ω be a 1-form in a manifold M with affine connection ∇. Then we define the horizontal lift ω^H of ω by

$$\omega^H = \omega^C - \nabla_\gamma \omega$$

in $T(M)$.

In second subchapter first chapter we give induced coordinates of lifts over cotangent bundle.

Let M be an $n-$dimensional differentiable manifold and $T_P^*(M)$ the cotangent space at a point $P \in M$, that is, the dual space to the tangent space $T_P(M)$ at P. Any element of $T_P^*(M)$ is called a covector at $P \in M$. Then the set

$$T^*(M) = \bigcup_{P \in M} T_P^*$$

is, by definition, the cotangent bundle over the manifold M.

Let us consider a $1-$form p in $\pi^{-1}(U) \in T^*(M)$ whose components are $(p_i, 0)$, that is $p = p_i dx^i$. We call the 1-form p the basic $1-$form in $T^*(M)$. The exterior dfferential dp of the basic $1-$form p is the $2-$ form given by in $\pi^{-1}(U)$

$$dp = dp_i \wedge dx^i.$$

In third subchapter of first chapter we give definition tangent bundle of order 2 and defined here the 0-th, the 1-st and the 2-nd lifts edirik.

Let f be a function in M and suppose that f is locally expressed by $f = f(x)$ in a coordinate neighborhood $\{U, x^h\}$. We now intoduce in each $\pi^{-1}(U)$ three functions f^0, f^I and f^{II} with local expressions

$$f^0: f(x),$$
$$f^I: y^i \partial_i f(x),$$
$$f^{II} = z^i \partial_i f(x) + \frac{1}{2} y^j y^i \partial_j \partial_i f(x).$$

These functions f^0, f^I and f^{II} are called respectively the 0-th, the 1-st and 2-nd lifts of the function f in M to $T_2(M)$.

Let $X \in \Im_0^1(M)$ with the local components X^h in U coodinate neighborhod. We then consider in $\pi^{-1}(U)$ three local vector fields X^0, X^I and X^{II} respectively with coordinates

$$X^0 : \begin{pmatrix} 0 \\ 0 \\ X^h \end{pmatrix}, X^I : \begin{pmatrix} 0 \\ X^h \\ y^i \partial_i X^h \end{pmatrix}, X^{II} : \begin{pmatrix} X^h \\ y^i \partial_i X^h \\ z^i \partial_i X^h + \frac{1}{2} y^j y^i \partial_j \partial_i X^h \end{pmatrix}.$$

Local vector fields X^0, X^I and X^{II} determine vector fields in $T_2(M)$, which are called respectively 0-th lift X^0, the 1-st lift X^I and 2-nd lift X^{II} of X vector field to $T_2(M)$.

In fourth subchapter of first chapter we consider geometrical structures defined by tensor of type $(1,1)$. Introduce notion of $\Pi -$structure and demonstrated connection with algebra U_m.

If a set of $(1,1) -$ tensor (affinor) fields $\underset{1}{J}, \underset{2}{J}, \ldots, \underset{m}{J}$ are given on a smooth manifold M_n, then one says that a multi – affinor structure (or $\Pi -$structure) is given on M_n:

$$\Pi = \left\{ \underset{\alpha}{J} \right\} = \left\{ \left(\underset{\alpha}{J}{}^i_j \right) \right\}, \alpha = 1,2,..., m; i, j = 1,..., n.$$

If there exists a frame $\{X_i\}, i = 1,..., n$ such that each structure affinors $\underset{\alpha}{J}$ of $\prod -$ structure has constant components $\underset{\alpha}{J}{}^{i}_{j}$ with respect to this frame, then $\prod -$structure is called rigid structure. In particular case, the frame $\{X_i\} = \{\partial_i\}, \partial_i J^k_l = 0,$ then the $\prod -$structure is said to be integrable. It is clear that the integrable $\prod -$ structure is rigid, the contrary statement is true only under some additional conditions on $\prod -$structure. For example, if $\prod = J$, i.e. the $\prod -$structure contains only one affinor J, and if there exists a torsion – free connection ∇ on M_n preserving the rigid structure $J -$structure, then the $J -$structure is integrable. It is well known that for simplest rigid $J -$ structures (almost complex and almost paracomplex structures, etc.) the integrability is equivalent to the vanishing of the Nijenhuis tensor.

Let U_m be a commutative algebra. An almost algebraic structure on M_n is a multi – affinore $\prod -$ structure such that

$$J_\alpha \circ J_\beta = C^\gamma_{\alpha\beta} J_\gamma$$

i.e. if there exists an isomorphism $U_m \leftrightarrow \Pi,$ where $\underset{\alpha}{J}$ are structure affinors corresponding to the base elements $e_\alpha \in U_m, \alpha = 1,..., m$.

An almost algebraic structure on M_n is said to be an $r -$ regular $\prod -$ structure (or regular $\prod -$structure) if the matrices $\left(\underset{\alpha}{J}{}^{i}_{j}\right) n \times n, \alpha = 1,\ldots,m$ simultaneously reduce to the form

$$\left(\underset{\alpha}{J}{}^{i}_{j}\right) = \begin{pmatrix} C_\alpha & 0 & ... & 0 \\ 0 & C_\alpha & ... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & C_\alpha \end{pmatrix}, \alpha = 1,..., m; i, j = 1,..., n$$

with repect to the adaped frame $\{X_i\}$, where $C_\alpha = \left(C^\gamma_{\alpha\beta}\right)$ is regular representation of U_m and r is a number of $C_\alpha -$ blocks. $r -$ regular Π structures are rigid structures. Almost complex and paracomplex structres on M_n ($\dim M_n = 2r$) automatically are $r -$regular.

An $U -$holomorphic manifold $X_r(U)$ of dimension r is a

Hausdorff space with a fixed complete atlas compatible with a group of $U-$ holomorphic transformations of space U_m^r, where $U_m^r = U_m \times \ldots \times U_m$ is the space of $r-$tuples of algebraic numbers

$$z^u = x^{u\alpha} e_\alpha \in U_m, x^{u\alpha} = x^i \in R, i = 1,\ldots, n; u = 1,\ldots, r; \alpha = 1,\ldots, m.$$

Therefore, the topological manifold $X_r(U)$ of dimension r also has the structure of a real smooth manifold M_{rm} of dimension rm.

In the second chapter we consider lift prolems on tangent bundle of second order.

In the first subchapter of second chapter give information about contact, almost contact and $\alpha-$Sasaki, $\beta-$ Kenmotsu structures.

$\alpha-$ Sasaki və $\beta-$Kenmotsi strukturlar haqqında bəhs olunur.

A $(2n+1)-$ dimensional manifold M is said to have a contact structure and is called a contact manifold if it carries a global 1-form η such that

$$\eta \wedge (d\eta)^n \neq 0$$

everywhere on M. We call η a contact form on M, where $\wedge$ denoted exteriror product.

Let M be a $(2n+1)-$ dimensional manifold and ϕ, ξ, η be a tensor field of type $(1,1)$, vector field and a $1-$form on M respectively. If ϕ, ξ, η satisfy the conditions

$$\eta(\xi) = 1,$$
$$\phi^2 X = -X + \eta(X)\xi, \forall X \in \Im_0^1(M)$$

then M is said to have an almost contact structure (ϕ, ξ, η).

Every almost contact manifold M admits Riemannian metric tensor field g such that

$$\eta(X) = g(X, \xi),$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Sasakian manifold is a normal contact metric manifold. Well known, condition of Sasakian structure is

$$(\nabla_\sigma \phi)\theta = g(\sigma, \theta)\xi - \eta(\theta)\sigma$$

almost contact metric structure.

A $\alpha -$ Sasakian strukture is defined by the requirement

$$(\nabla_\sigma \phi)\theta = \alpha(g(\sigma,\theta) - \eta(\theta)\sigma)$$

where α is a non - zero constant. Setting $\theta = \xi$ in this formula, one redily obtains

$$\nabla_\sigma \xi = -\alpha\phi\sigma .$$

Theorem 2.1.2 Let $\xi -$a vector field, $\phi -$ be a tensor field of type $(1,1)$ and 1-form η satisfying condition: $\phi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1, \phi\xi = 0$, $\eta \circ \phi = 0$. A $\alpha -$ Sasakian structure on tangent bundle defined by:

$$(\nabla^C{}_{\sigma^C} \phi^C)\theta^C = \alpha((g(\sigma,\theta))^V \xi^C + (g(\sigma,\theta))^C \xi^V - (\eta(\theta))^C \sigma^V - (\eta(\theta))^V \sigma^C),$$

where $g -$is a Riemannian metric, $\alpha -$is a non – zero constant. In addition, if we put $\theta = \xi$, we get

$$\nabla^C_{\sigma^C} \xi^C = -\alpha\phi^C \sigma^C .$$

In particular the almost contact metric structure in this case satisfies

$$(\nabla_\sigma \phi)\theta = g(\phi\sigma ,\theta)\xi - \eta(\theta)\phi\sigma$$

and an almost contact metric manifold satisfying this condition is called a Kenmotsu manifold. Again one has the more general notion of a $\beta -$Kenmotsu strukture which may be defined by

$$(\nabla_\sigma \phi)\theta = \beta(g(\phi\sigma ,\theta) - \eta(\theta)\phi\sigma),$$

where $\beta -$is a non – zero constant. From the condition one may readil reduce that

$$\nabla_\sigma \xi = \beta(\sigma - \eta(\sigma)\xi).$$

Teorem 2.1.3 Let $\phi -$be a tensor field of type $(1,1)$, a vector field ξ and $\eta -$ 1 form satisfying $\phi^2 = -1 + \eta \otimes \xi = 0$ and $\eta \circ \phi = 0$. A $\beta -$Kenmotsu strukture on tangent bundle defined by

$$((\nabla_\sigma \varphi)\theta)^C = \beta((g(\varphi\sigma ,\theta))^V \xi^C + (g(\varphi\sigma ,\theta))^C - (\eta(\theta))^C (\varphi\sigma)^V (\varphi\sigma)^C),$$

where $g -$Riemannian metric, $\beta -$is non – zero constant. In addition, if we put $\theta = \xi$, we get

$$\nabla^C{}_{\sigma^C}\xi^C = \beta\left(\xi^C - \left((\eta(\sigma))\xi\right)^C\right).$$

In second subchapter of second chapter we looking at Trans-sasakian structures.

An almost contact metric structure (ϕ, ξ, η, g) on M^n is trans-Sasakian if $(M^n \times R, J, G)$ belongs to the class W_4.

Teorem 2.2.1 Let ϕ be a tensor field of type (1,1), a vector field ξ and η 1-form satisfying $\phi^2 = -1 + \eta \otimes \xi$ and

$$\eta(\xi) = 1, \varphi\xi = 0, \eta \circ \phi = 0.$$

A trans-Sasakian structure on tangent bundle defined by

$$\left(\nabla^C{}_{\sigma^C}\phi^C\right)\theta^C = \alpha\left((g(\sigma,\theta))^I \xi^C + (g(\sigma,\theta))^C \xi^V - (\eta(\theta))^V \sigma^V - (\eta(\theta))^V \sigma^C\right) +$$
$$+ \beta\left((g(\phi\sigma,\theta))^V \xi^C + (g(\phi\sigma,\theta))^C \xi^V - (\eta(\theta))^C (\phi\sigma)^V - (\eta(\theta))^V (\phi\sigma)^C\right)$$

where $g -$ is a Riemannian metric, and $\beta -$ are non-zero constants.

In third chapter we are research lift problems on tangent bundles of second order.

In first subspace of third chapter we give definitions of algebraic structure constnats, holomorphic function and apply general Cauchy-Riemann conditions.

We consider a m dimensional associative and commutative algebra U_m over R, with canonical basis $\{e_\alpha\}, \alpha = 1,..., m$ and $C^\gamma_{\alpha\beta}$ structure constants:

$$e_\alpha e_\beta = C^\gamma_{\alpha\beta}.e_\gamma.$$

Denoted that, $C^\gamma_{\alpha\beta}$ - are components of tensor of type (1,2). Unit element of algebra U_m is $e_1 = 1$.

Let $z = x^\alpha e_\alpha \in U_m$ be an algebraic variable, $x_\alpha \in R$. We introduce for variable $z \in U_m$ an algebraic function $w = w(z) \in U_4$ in the following form

$$w = y^\beta(x)e_\beta.$$

It is clear that, $w = w(z)$ an algebraic function. Let $dz = dx^\alpha e_\alpha$ and

$dy = dy^{\beta}(x)e_{\beta}$, be the differentials of z and w respectively. We shall say that the function $w = w(z)$ is an $U-$holomorphic if there exists a function $w^{'}(z)$ such that $dw = w^{'}(z)dz$. We shall call $w^{'}(z)$ the derivative of $w(z)$.

We now construct a matrix of order $m \times m$ in the following form: $C_{\alpha} = \left(C_{\alpha\beta}^{\gamma}\right), \alpha = 1,..., m$ where $C_{\alpha\beta}^{\gamma}$ are structure constants of algebra U_m. γ means rows and denoted columns.

The function $w = w(z)$ is $U-$ holomorphic if and only if the generalized Cauchy-Riemann conditions hold:

$$C_{\sigma}D = DC_{\sigma} \left(C_{\sigma\beta}^{\alpha} \frac{\partial y^{\beta}}{\partial x^{\gamma}} = \frac{\partial y^{\alpha}}{\partial x^{\beta}} C_{\sigma\gamma}^{\beta} \right),$$

burada $C_{\sigma} = \left(C_{\sigma\beta}^{\alpha}\right)$ və $D = \left(\frac{\partial y^{\beta}}{\partial x^{\gamma}} \right)$ Yakobi matrisidir.

We know, consider algebra U_4 with canonical basis

$$\{e_1, e_2, e_3, e_4\} = \{1, \varepsilon, \varepsilon^2, \varepsilon^3\}, \varepsilon^4 = 0$$

$$e_{\alpha}e_{\beta} = C_{\alpha\beta}^{\gamma}e_{\gamma}, \quad \alpha, \beta, \gamma = 1,2,3,4.$$

Let $z_{\alpha} = x^{\alpha}e_{\alpha} \in U_4$ be an algebraic variable, where $x^{\alpha} \in R$ $(\alpha = 1,2,3,4)$ are real variables. We introduce an algebraic function $z \in U_4$ as following form:

$$w = y^{\beta}(x)e_{\beta},$$

where $y^{\beta}(x) = y^{\beta}\left(x^1, x^2, x^3, x^4\right), \beta = 1,2,3,4$ are real-valued C^{∞} functions.

$$C_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$C_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, C_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

$C_\sigma = \left(C_{\alpha\beta}^{\gamma}\right),\ \sigma = 1,2,3,4$ matrices is called the regular representation of algebra U_4.

U_4 holomorphic function for $w = w(z)$ is a form

$$w(z) = y^\alpha e_\alpha = y^1\left(x^1, x^2, x^3, x^4\right) + \varepsilon y^2\left(x^1, x^2, x^3, x^4\right) + \\ + \varepsilon^2 y^3\left(x^1, x^2, x^3, x^4\right) + \varepsilon^3 y^4\left(x^1, x^2, x^3, x^4\right),$$

where $y^4 = x^1 + \varepsilon x^2 + \varepsilon^2 x^3 + \varepsilon^4 x^4$.

In second subchapter of third chapter we are considering holomorphic and vanish metrics. Using tensor of torsion for a linear connection ∇, about $\prod -$strucurure and its integrability, $\prod -$connection, holomorohic manifold and were obtained.manifolds of Kahler type.

Let know ∇ be a linear connection on M_{rm} with components Γ_{ij}^{k}. Using the covariant derivatives of tensors J_α, ω and the torsion tensor apply $\Phi_{\underset{\alpha}{J}}\omega$ Tachibana operator.

A linear connection ∇ on M_{rm} is called $\Pi -$connection with respect to the $r -$regular $\prod -$ structure if $\nabla J = 0$ for each $J \in \Pi$. The $r -$regular $\prod -$ structure on M_{rm} is called almost integrable, if there exists a torsion-free $\Pi -$connection. For some simple $\Pi -$ structures (almost complex structure, $r -$regular $\prod -$structure etc.) the notions of integrability and almost integrability are equivalent.

Theorem 3.2.1 If the $\prod -$structure on M_{rm} is an almost integrable structure, then

$$\left(\Phi_J \omega\right)_{kj_1j_2} = \underset{\sigma\ k}{J^{\,h}} \nabla_h \omega_{j_1j_2} - \underset{\sigma\ j_1}{J^{\,h}} \nabla_k \omega_{hj_2}$$

Let M_{rm} be a Riemannian manifold with a pure metric g with respect to the r regular $\prod -$ structure. If the $\prod -$structure is integrable, we say that the triple (M_{rm}, Π, g) is an U $-$holomorphic manifold with a pure metric g.

Theorem 3.2.2 The pure Riemannian metric g on the U-holomorphic manifold (M_{rm}, Π, g), $\Pi = \{\underset{\sigma}{J}\}$ is an U-holomorphic metric if and only if the triple (M_{rm}, Π, g) is a Kahler-type manifold, i.e. ${}^{g}\nabla \underset{\alpha}{J} = 0,\ \alpha = 1,\ldots,m,$ where ${}^{g}\nabla$ - is Levi – Civita connection of g.

Theorem 3.2.3 Let (M_{rm}, Π, g) be U - holomorphic manifold with a pure metric g. If torsion tensor of a metric-preserving $\prod -$ connection is pure with respect to the r $-$regular $\prod$ $-$structure, then the pure metric g is an U $-$holomorphic metric.

In third subchapter of third chapter we are consider Kahler type metrics in the bundle of 2-jets. On the tangent bundle of order 2 we introduce components of affine fields and research pure Riemannian metrics.

Let $U_3 = R(\varepsilon^2)$ be an algebra of order 3 with a canonical basis $\{e_1, e_2, e_3\} = \{1, \varepsilon, \varepsilon^2\}, \varepsilon^3 = 0.$ We see that the matrices $C_\sigma = \left(C_{\alpha\beta}^{\gamma}\right), \sigma = 1,2,3$ of regular representation of algebra $R(\varepsilon^2)$ have the following forms:

$$C_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}, C_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{pmatrix}.$$

It is clear that there exists an affinor fiel $T^2(V_r), \dim\left(T^2(V_r)\right) = 3r$ which has components of the form

$$\gamma = \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}, \gamma^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \end{pmatrix}, \gamma^3 = 0.$$

The Scheffers conditions is satisfy: $C_{\sigma\beta}^{\alpha}\dfrac{\partial y^{\beta}}{\partial x^{\gamma}}=\dfrac{\partial y^{\alpha}}{\partial x^{\beta}}C_{\sigma\gamma}^{\beta}$.

On $R(\varepsilon^2)$ holomorphic function:
$w=w(z)=y^1(x^1,x^2,x^3)+\varepsilon y^2(x^1,x^2,x^3)+\varepsilon^2 y^3(x^1,x^2,x^3)$, where $z=x^1+\varepsilon x^2+\varepsilon^2 x^3$ reduce to the following equations:

$$\frac{\partial y^1}{\partial x^2}=\frac{\partial y^1}{\partial x^3}=\frac{\partial y^2}{\partial x^3}=0,$$

$$\frac{\partial y^2}{\partial x^2}=\frac{\partial y^1}{\partial x^1}=\frac{\partial y^3}{\partial x^3},$$

$$\frac{\partial y^3}{\partial x^2}=\frac{\partial y^2}{\partial x^1}.$$

After some calculations, we see that the 2nd lift ${}^{II}g$ a Riemannian metric g to $T^2(V_r)$, i.e.

$${}^{II}g=\begin{pmatrix} \overset{=}{x}{}^{s}\partial_s g_{ji}+\dfrac{1}{2}\bar{x}^{t}\bar{x}^{s}\partial_t\partial_s g_{ji} & \bar{x}^{s}\partial_s g_{ji} & g_{ji} \\ \bar{x}^{s}\partial_s g_{ji} & g_{ji} & 0 \\ g_{ji} & 0 & 0 \end{pmatrix}$$

is a pure Riemannian metric with respect to the structure

$$\Pi=\{I,\gamma,\gamma^2\} \text{ and } \Phi_{\gamma}\,{}^{II}g=\Phi_{\gamma^2}\,{}^{II}g=0.$$

On $T^2(V_r)$ deformed lift of ${}^{II}g$ metric tensor is tensor ${}^{II}\tilde{g}$.

$${}^{II}\tilde{g}=\begin{pmatrix} \overset{=}{x}{}^{s}\partial_s g_{ji}+\dfrac{1}{2}\bar{x}^{t}\bar{x}^{s}\partial_t\partial_s g_{ji}+\bar{x}^{s}\partial_s G_{ji}+H_{ji} & \bar{x}^{s}\partial_s g_{ji}+G_{ji} & g_{ji} \\ \bar{x}^{s}\partial_s g_{ji}+G_{ji} & g_{ji} & 0 \\ g_{ji} & 0 & 0 \end{pmatrix}$$

and denoted as ${}^{def}({}^{II}g)$.

Theorem 3.3.2 If V_r is a Riemannian manifold with metric g, then the triple $\left(T^2(V_r),\Pi,{}^{def}\left({}^{II}g\right)\right)$ is a Kahler-type manifold.

We see that a general form of the deformed complete lift ${}^{def}\left({}^{II}g\right)$ is

$${}^{def}\left({}^{II}g\right)={}^{II}g+{}^{I}G+{}^{0}H,$$

where

$${}^{I}G=\begin{pmatrix} \bar{x}^{s}\partial_{s}G_{ji} & G_{ji} & 0 \\ G_{ji} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ və } {}^{0}H=\begin{pmatrix} H_{ji} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are the 1st and 0th lifts of G and H, respectively.

Conclusion

The thesis work is devoted to the study of liftings of differential geometric structures on higher order tangent and cotangent bundles. The main results of the thesis work are following:

1. Obtained the explicit expression of holomorphic functions in nilpotent algebras of order III and IV;
2. Kahler type metrics are found on the tangent bundle of 2-jets using holomorphic functions;
3. The deformed lift of the vector fields in the tangent bundle of 2-jets are found using holomorphic functions;
4. On tangent bundle of order 2 were constructed lifts of -Sasakian, $\beta-$Kenmotsu, trans-Sasakian structurs;
5. On cotangent bundle were found condition of integrability for almost complex structure and apply Tachibana operator to the complete and horizontal lifts.

The main results of the presented thesis has been published in following works:

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   pp. 292-294
2. Sultanova T.T., Salimov A.A. "Some notes on C manifolds", Azərbaycan Xalqının Ümummilli Lideri Heydər Əliyevin anadan olmasının 96-cı ildönümünə həsr olunmuş Elmi Konfransın materialları, Bakı 2020 səh. 173
3. Arif Salimov, Tarana Sultanova "On holomorphic pseudo-Riemannian manifolds" International Conference "Modern Problems of Mathematics and Mechanics" devoted to the 60th anniversary of the İnstitute of Mathematics and Mechanics , 23-25 October, 2019, Baku Azerbaijan, pp.447-448
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