REPUBLIC OF AZERBAIJAN

On the rights of the manuscript

ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

CONVERGENCE OF SPECTRAL EXPANSIONS CORRESPONDING TO THIRD ORDER ORDINARY DIFFERENTIAL OPERATORS

Specialty: 1211.01 – Differential Equations

Field of science: Mathematics

Applicant: **Elnara Bukhsay kizi Akhundova**

Baku – 2021

The work was performed at the department of "Functional analysis" of the Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan.

Scientific supervisor: doctor of phys.-math. sc., prof.
Vali Maqerram oglu Kurbanov

Official opponents: doctor of phys.-math. sc., prof.
Mamed Bayramoglu

doctor of sciences of math., assoc. prof.
Yashar Topush oglu Mehraliyev

cand. of phys.-math. sc., assoc. prof.
Elvin İbragim oglu Azizbayov

Dissertation council ED 1.04 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan.

Chairman of the Dissertation council:
corr.-member of the NASA, doctor of phys.-math. sc., prof.
Misir Jumail oglu Mardanov

Scientific secretary of the Dissertation council:
candidate of phys.-math. sc., assoc. prof.
Tamilla Khaveran gizi Hasanova

Chairman of the scientific seminar:
academician of NASA,doctor of phys.-math. sc., prof.
Yusif Abulfat oglu Mamedov

GENERAL CHARACTERISTICS OF THE WORK

Rationale of the topic and degree of development. The dissertation work was devoted to the study of convergence of spectral expansions in eigen and associated functions of a third order ordinary differential operator.

It is known that researches on spectral theory of ordinary differential operators take their start from the classic works of J.Liouville, Sh.Sturm and in later works of V.A.Steklov, Ya.D.Tamarkin, D.Birkhoff and other authors who studied asymptotics of eigenvalues and convergence of spectral expansions for various classes of boundary value problems.

During a long time, the main object of study were spectral properties of self-adjoint differential operators. However, in the last fifty years new problems of mathematical physics reducing to the study of spectral properties of not self-adjoint differential operators have arisen. The Bitzadze-Samarsky problem with nonlocal boundary conditions for a heat equation may serve as an example of such problems.

When studying not selfadjoint problems it was noticed that the system of eigen functions of such operators, generally speaking not only does not form a basis in the class L_2, but also is not complete in L_2. Therefore such systems should be supplemented with associated functions. In these problems, eigen and associated functions (root functions), generally speaking, are not orthogonal in L_2, and neither their closeness nor their minimality imply their basicity in this space.

Thus, the study of not selfadjoint problems reguire new approaches. M.V.Keldysh established the fact of completeness in L_2 of specially constructed system of root functions for a wide class of boundary value problems. Further, the completeness for a wide class of boundary value problems was studied in the papers of V.B.Lidsky, M.A.Naimark, V.N.Vizitey, A.S.Markus, J.E.Allahverdiyev, M.G.Gasimov, A.P.Kostyuchenko, A.P.Khromov, V.P.Mikhaylov, G.M.Keselman, A.M.Kroll, A.A.Shkalikov and others.

The last years, the method developed by V.A.Il'in is successfully used for studying differential operators. He has noticed that in the presence of infinitely many associated functions, the basicity and equiconvergence properties unlike the completeness property substantially depend on the choice of root functions and also are not determined only by the concrete from of boundary conditions. The values of the coefficients of a differential operator also affect on these properties and these properties change for whatever small change of the value of coefficients in the metrics of the classes where these coefficients are given. Therefore, in this situation, the basicity and equiconvergence conditions in the terms of boundary conditions can not be formulated. In this connection, V.A.Il'in offered a new treatment of root functions that are understood as regular solutions of the corresponding equation with a spectral parameter regardless of the form of boundary condition. It allows to consider arbitrary boundary conditions (both local and nonlocal) the systems of functions not associated with any boundary conditions and also some systems obtained by combining the subsets of the root functions of two various boundary value problems.

In his papers V.I.Il'in considered a system of root functions of an ordinary differential operator and under some natural conditions he established theorems on uniform equiconvergence and basicity on a compact.

Further, the study of these and other problems of spectral theory of differential operators was papers of V.A.Il'in and his followers: V.V.Tikhomirov, I.Io, I.S.Lomov, N.B.Kerimov, V.D.Budaev, V.Komornik, L.V.Kritskov, L.Lazhetich, V.M.Kurbanov and others.

Notice that componentwise equiconvergence for Schrodinger's operator was studied in the paper of V.A.Il'in and in the papers of V.M.Kurbanov. Componentwise equiconvergence rate in the metrics C and L_p was studied by V.M.Kurbanov.

Asbosute and uniform convergence and rate for Schrodinger's operator were studied in the papers of N.Lazhetich, V.M.Kurbanov, R.A.Safarov, A.T.Garayeva, for Dirac's operator in the papers of V.M.Kurbanov and A.I.Ismayilova.

Recently, dependence of convergence and equiconvergence rate on various characteristics is intensively investigated and a number of important results were established by V.M.Kurbanov, R.A.Safarov, L.S.Lomov, A.S.Markov, A.T.Garayeva.

In spite of the above studies, uniform equiconvergence rate on a compact and uniform convergence rate on a segment for higher order differential operators were not studied enough. Therefore, the further study of these and other issues for differential operators by the V.A.Il'in method is of interest.

In this dissertation work we study absolute and uniform convergence, uniform convergence rate of orthogonal expansion of a function from the class $W_{\rho}^{1}(G),\ p \geq 1$, in eigen function of a third order ordinary differential operator with summable coefficients; study dependence of component wise uniform equiconvergence rate of a biorthogonal expansion with trigonometric Fourier series of the expanded function on the continuity modulus of the coefficient $P_2(x)$, establish uniform equiconvergence rate for vector functions from various functional spaces $\left(H_{\rho}^{\omega}(G), B_{\rho,\theta}^{\alpha}(G)\right)$

We study absolute and uniform convergence of biorthogonal expansions of the function $W_2^1(G)$ in root functions of a third order differential operator with smooth coefficients, establish uniform convergence rate of these biorthogonal expansions.

Goal and tasks of the research. To research absolute and uniform convergence and uniform equiconvergence rate on a compact of spectral expansions in root functions of a third order ordinary differential equation.

Investigation methods. In the work, the methods of theory of differential operators, theory of functional analysis and theory of harmonic analysis are used.

The key points to be defended. The following key points are defended:

• The results of studies on absolute and uniform convergence on the segment $\overline{G} = [0,1]$ of spectral expansion of the function from

the Sobolev class $W_{\rho}^{1}(G),\ p>1,\ G=(0,1)$, in eigen functions of a third order ordinary differential operator with summable coefficients and the estimates of uniform convergence of this expansion.

• The result of studies on absolute and uniform convergence of orthogonal expansion of the function $f(x)\in W_{1}^{1}(G),\ G=(0,1)$, in eigen functions of a third order ordinary differential operator with summable coefficients and estimates of convergence of this expansion.

• The results of studies of uniform equiconvergence on a compact with trigonometric series of expansions in root functions of a third order differential operator with summable coefficients from the class $L_p(G),\ p\geq 1$, and estimates of uniform equiconvergence rate for a function from the class $H_{p}^{\omega}(G),\ B_{\rho,\theta}^{\alpha}(G)$.

• The results of studies on absolute and uniform convergence of biorthogonal expansions of a function from the class $W_{2}^{1}(G), G=(0,1)$, in the system of root functions of a third order ordinary differential operator and estimates of uniform convergence rate of these biorthogonal expansions.

Scientific novelty of the research. In the dissertation the following main results were obtained:

• Absolute and uniform convergence on the segment $\overline{G}=[0,1]$of spectral expansion of a function from the Sobolev class $W_{\rho}^{1}(G),\ p\geq 1,\ G=(0,1)$, in eigen functions of a third order ordinary differential operator with summable coefficients was studied and uniform convergence rate on this segment was estimated.

• Absolute and uniform convergence of orthogonal expansion of the function $f(x)\in W_{1}^{1}(G),\ G=(0,1)$, in eigen functions of a third order ordinary differential operator with summable coefficients is proved and the residue of the given expansion in the metric was $C(\overline{G})$ estimated.

• Theorems on uniform equiconvergence on a compact with trigonometric series of expansions in root functions of a third order differential operator with summable coefficients for a function from the class $L_p(G),\ p \geq 1$ were proved. Uniform equiconvergence rate for a function from the class $H_p^{\omega}(G),\ B_{\rho,\theta}^{\alpha}(G)$ was estimated.

• Theorem on absolute and uniform convergence of biorthogonal expansions of a function from the class $W_2^1(G), G = (0,1)$, in the system of root functions of a third order ordinary differential operator were proved and uniform convergence rate on $\overline{G} = [0,1]$. was set up.

Theoretical and practical value of the research. Its results may be used in spectral theory of differential operators; when justifying the solution of mathematical physics problems by the Fourier method and in theory of approximation of function.

Approbation and usu. The main results of the dissertation were repeatedly reported: at the International conference dedicated to 90-th anniversary of Heydar Aliyev (Baku 2013), at the International conference dedicated to 55 years of IMM (Baku 2014); at the International conference of Azerbaijan-Turkey-Ukraine MADEA-7 (Baku 2015), at the Republican conference dedicated to 100 years of honoured scientist prof. A.Sh.Habibzade (Baku 2016), at the seminar of the departments "Functional analysis" (the head of department prof. G.I.Aslanov), "Differential Equations" (the head of department prof. A.B.Aliyev) of IMM of ANAS.

Personal contribution of the author. All conclusions and obtained results belong to the author.

Author's publications. The basic results of the dissertation were published in 10 papers whose list is at the end of the abstract.

The organization where the work was executed. The work was executed at the department of "Functional analysis" of IMM of ANAS.

Structure and volume of the dissertation (in signs with indicating the volume of each structural subsection separately). The volume of the dissertation wirk consists of –207693 signs (the title page – 320 signs, the content 2173 signs, introduction – 50000 signs,

chapter I – 84000 signs, chapter II – 70000 signs, conclusions - 1200 signs). The list of references consists of 72 names.

THE CONTENT OF THE DISSERTATION

In the introduction the rationale of the work is justified, brief review of the results related to dissertations topic is given and the main results of the dissertation work are stated.

The dissertation work consists of introduction, two chapters, lists of references. Each chapter is divided into sections.

In chapter I the main results concerning orthogonal expansion in eigen functions of a third order differential operator with summable coefficients are stated. Absolute and uniform convergence of orthogonal expansion of an absolutely continuous function from the class $W_p^1(G)$, $p \geq 1$, in eigen functions of the given operator is proved and uniform convergence rate of this expansion is established, the influence of uniform convergence rate is studied; degree of summability of the coefficients of this operator and degree of summability of the derivative of the expanded function.

In section 1.1 we consider a formal differential operator $Lu = u^{(3)} + P_1(x)u^{(2)} + P_2(x)u^{(1)} + P_3(x)u$, $x \in G = (0,1)$ with summable complex valued coefficients

$$P_l(x) \in L_2(G), \quad P_l(x) \in L_1(G), \quad l = 2,3.$$

Denote by $D(G)$ a class of functions absolutely continuous with its second order derivatives inclusively, on a closed interval $\overline{G} = [0,1]$. Under the eigen-function of the operator L, responding to the eigen value λ, we understand any identically non zero function satisfying almost everywhere in G the equation $Lu + \lambda u = 0.\ u(x) \in D(G)$. Let $\{u_n(x)\}_{n=1}^{\infty}$ be a complex orthonormed in $L_2(G)$ system consisting of eigen functions of the operator L, and $\{\lambda_n\}_{n=1}^{\infty}$ be an appropriate system of eigenvalues and $Re\ \lambda_n = 0$ (it is suppose that the coefficients of the operator L allow the existence of such a system $\{u_n(x)\}_{n=1}^{\infty}$.

By μ_n the number $(\mp i\lambda_n)^{1/3}$ is denoted for $\pm \operatorname{Im} \lambda_n \geq 0$. We say that the function $f(x)$ belongs to $W_p^1(G)$, $1 \leq p \leq \infty$, if $f(x)$ is absolutely continuous on $\overline{G}$ and $f'(x) \in L_p(G)$.

Introduce a partial sum of spectral expansion of the function $f(x) \in W_p^1(G)$ in the system $\{u_n(x)\}_{n=1}^{\infty}$: $\sigma_\nu(x,f) = \sum_{\mu_n \leq \nu} f_n u_n(x)$, $\nu > 0$,

where $f_n = (f, u_n) = \int_0^1 f(x)\overline{u_n(x)}\, dx$. Denote $R_\nu(x,f) = f(x) - \sigma_\nu(x,f)$.

In the given section we prove the following theorems.

Theorem 0.0.1. *Let* $P_1(x) \equiv 0$, $P_l(x) \in L_1(G)$, $l = 2,3$; $f(x) \in W_p^1(G)$, $p > 1$, *and the following condition be fulfilled*

$$\left| f(x)\, \overline{u_n^{(2)}(x)}\Big|_0^1 \right| \leq C(f)\mu_n^\alpha \|u_n\|_\infty, \; 0 \leq \alpha < 2, \; \mu_n \geq 1, \qquad (0.0.1)$$

where $C(f) > 0$ *is a constant dependent on the function* $f(x)$.

Then spectral expansion of the function $f(x)$ *in the system* $\{u_n(x)\}_{n=1}^{\infty}$ *converges absolutely and uniformly on* $\overline{G} = [0,1]$ *and the following estimation is valid*

$$\sup_{x \in \overline{G}} |R_\nu(x,f)| \leq const \Big\{ C(f)\nu^{\alpha-2} + \nu^{-\beta}\|f'\|_p +$$

and

$$+ \nu^{-1}\left(\|f\|_\infty + \|f'\|_1\right)\sum_{r=2}^{3} \nu^{2-r}\|P_r\|_1 \Big\}, \qquad (0.0.2)$$

where $\beta = \min\left\{\frac{1}{2}, \frac{1}{q}\right\}$, $p^{-1} + q^{-1} = 1$, $\nu \geq 2$, *const* is independent of $f(x)$, $\|\cdot\|_p = \|\cdot\|_{L_p(G)}$.

Corollary 0.0.1. *If in theorem 0.0.1. the function* $f(x)$ *satisfies the condition* $f(0) = f(1) = 0$, *then condition (0.0.1) is fulfilled and the following estimation*

$$\sup_{x \in \overline{G}} |R_\nu(x,f)| \leq const\; \nu^{-\beta}\|f'\|_p, \quad \nu \geq 2;$$

is valid. But if $C(f)=0$ *on* $0\le\alpha<2-\beta$, *then the following estimation is valid*

$$\sup_{x\in\overline{G}}|R_\nu(x,f)|=o\left(\nu^{-\beta}\right),\quad \nu\to+\infty\cdot$$

Theorem 0.0.2. *Let* $P_1(x)\in L_2(G),\quad P_l(x)\in L_1(G),\quad l=2,3;$ $f(x)\in W_2^1(G)$ *and condition (0.0.1) be fulfilled. Then spectral expansion of the function* $f(x)$ *in the system* $\{u_n(x)\}_{n=1}^{\infty}$ *converges absolutely and uniformly on* $\overline{G}=[0,1]$ *and the following estimation is valid*

$$\sup_{x\in\overline{G}}|R_\nu(x,f)|\le const\Big\{\ C(f)\nu^{\alpha-2}+\nu^{-\frac{1}{2}}\left(\|f\overline{P_l}\|_2+\|f'\|_2\right)+ \tag{0.0.3}$$

$$+\nu^{-1}\|f\|_\infty\sum_{r=2}^{3}\nu^{2-r}\|P_r\|_1\ \Big\},\ \nu\ge 2.$$

Corollary 0.0.2. *If in theorem 0.0.2* $C(f)=0$ *or* $0\le\alpha<3/2$, *then the following estimation is valid*

$$\sup_{x\in\overline{G}}|R_\nu(x,f)|=o\left(\nu^{-\frac{1}{2}}\right),\quad \nu\to+\infty\cdot$$

Theorem 0.0.3. *Let* $P_l(x)\in L_2(G),\ P_l(x)\in L_1(G),\quad l=2,3;$ $f(x)\in W_p^1(G),\ 1<p<2,$ *condition (0.0.1) be fulfilled, and the system* $\{u_n(x)\}_{n=1}^{\infty}$ *be uniformly bounded. then spectral expansion of the function* $f(x)$ *in the system* $\{u_n(x)\}_{n=1}^{\infty}$ *converges absolutely and uniformly on* $\overline{G}$ *and the following estimation is valid*

$$\sup_{x\in\overline{G}}|R_\nu(x,f)|\le const\Big\{\ C(f)\nu^{\alpha-2}+\nu^{-\frac{1}{2}}\|f\overline{P_l}\|_2+\nu^{-\frac{1}{q}}\|f'\|_p+$$

$$+\nu^{-1}\|f\|_\infty\sum_{r=2}^{\infty}\nu^{2-r}\|P_r\|_1\ \Big\},\ \mu\ge 2,\ \ p^{-1}+q^{-1}=1.$$

Corollary 0.0.3. *If in theorem 0.0.3* $C(f)=0$ *or* $0\le\alpha<2-q^{-1}$, *then the following estimation is valid*

$$\sup_{x\in\overline{G}}\left|R_\nu(x,f)\right| = o\left(\nu^{-\frac{1}{q}}\right), \quad \nu\to+\infty.$$

Note that similar results for the Schrodinger operator $L_1 = -\frac{d^2}{dx^2}+q(x)$ were obtained in the paper of N.L.Lazhetich for $q(x)\in L_r(G),\ r\ge 1$ is a real potential, $f(x)\in W_p^1(G),\ p>1,\ f(0)=f(1)=0$; in the papers of V.M.Kurbanov and K.A.Safarov $q(x)\in L_1(G)$ is a real and complex potential, $f(x)\in W_p^1(G),\ p\ge 1,\ f(0)=f(1)=0$; in the papers of V.M.Kurbanov and A.T.Garayeva for summable matrix potential $Q(x)$ and $f(x)\in W_{p,m}^1(G),\ p\ge 1$.

In section 1.2 we consider the operator L for $P_1(x)\equiv 0$ and study absolute and uniform convergence of orthogonal expansion of the function $f(x)\in W_1^1(G),\ G=[0,1]$ in eigen function of this operator. To this end, the coefficients of the Fourier function $f(x)\in W_1^1(G)$ satisfying the condition are estimated

$$\left|f(x)u_n^{(2)}(x)\Big|_0^1\right| \le C(f)\mu_n^\alpha,\ \ 0\le\alpha<2,\ \ \mu_n\ge 4\pi; \qquad (0.0.4)$$

Based on this estimation in this section we prove the following theorem.

Theorem 0.0.4. *Let the function* $f(x)$ *belong to the class* $W_1^1(G)$, *the system* $\{u_n(x)\}_{n=1}^\infty$ *be uniformly bounded and the conditions (0.0.4) and*

$$\sum_{k=1}^{\infty} k^{-1}\omega_1\left(f',k^{-1}\right)<\infty \qquad .(0.0.5)$$

be fulfilled.

Then expansion of the function $f(x)$ *in the system* $\{u_n(x)\}_{n=1}^\infty$ *converges absolutely and uniformly on* $\overline{G}=[0,1]$ *and the following estimation is valid*

$$\sup_{x\in\overline{G}}|R_\nu(x,f)| \le const\Big\{ C(f)\nu^{\alpha-2} + \sum_{k=[\nu]}^{\infty}\omega_1\left(f',k^{-1}\right)k^{-1} + \tag{0.0.6}$$

$$\left(\|f'\|_\infty + \|f'\|_1\right)\sum_{l=2}^{3}\nu^{1-l}\|P_l\|_1 + \nu^{-1}\|f'\|_1\Big\},\quad \nu\ge 2,$$

where $\omega_1(g,\delta)$ *is an integral modulus of continuity of the function* $g(x)\in L_1(G)$; $\|P_l\|_l = \int_0^l |P_l(x)|dx,$ *const is independent of* $f(x)$.

Similar results for the Sturm-Liouville operator were earlier proved in the papers of N.Lazhetich, V.M.Kurbanov and R.A.Safarov and A.T.Karayeva. The following corollaries follow from theorem 0.0.4.

Corollary 0.0.4. *If the system* $\{u_n(x)\}_{n=1}^{\infty}$ *is uniformly bounded,* $f(x)\in W_1^1(G)$, $f(0)=f(1)=0$ *and* $f'(x)\in H_1^\alpha(G)$, $0<\alpha<1$, ($H_1^\alpha(G)$ - *is a Nikolsky class), then*

$$\sup_{x\in\overline{G}}|R_\nu(x,f)| \le const\,\nu^{-\alpha}\|f'\|_1^\alpha \text{ ,where } \|g\|_1^\alpha = \|g\|_1 + \sup_{\delta>0}\frac{\omega_1(g,\delta)}{\delta^\alpha}.$$

Corollary 0.0.5. *If the system* $\{u_n(x)\}_{n=1}^{\infty}$ *is uniformly bounded,* $f(x)\in W_1^1(G)$, $f(0)=f(1)=0$ *and for some* $\beta>0$ *the following estimation is fulfilled*

$$\omega_1(f',\delta) = O\left(\ln^{-(1+\beta)}\frac{1}{\delta}\right),\quad \delta\to+0.$$

then $\sup_{x\in\overline{G}}|R_\nu(x,f)| = O\left(\ln^{-\beta}\nu\right),\quad \nu\to\infty.$

In section 1.3. we study influence of the coefficient $P_1(x)$ on absolute and uniform convergence of spectral expansion of the function and $f(x)\in W_1^1(G)$ prove the following theorem.

Theorem 0.0.5. *Let the function* $f(x)$ *belong to the class,* $W_1^1(G)$, *the system* $\{u_n(x)\}_{n=1}^{\infty}$ *be uniformly bounded and conditions (0.0.4) and*

$$\sum_{k=1}^{\infty}k^{-1}\omega_1\left(f',k^{-1}\right)<\infty,\quad \sum_{k=1}^{\infty}k^{-1}\omega_1\left(f\overline{P}_1,k^{-1}\right)<\infty, \tag{0.0.7}$$

be fulfilled.

then expansion of the function $f(x)$ *in the system* $\{u_n(x)\}_{n=1}^{\infty}$ *converges absolutely and uniformly on* $\overline{G}=[0,1]$ *and the following estimatio is valid*

$$\begin{aligned}
&\sup_{x\in\overline{G}}|R_\nu(x,f)|\le const\Big\{\ C(f)\nu^{\alpha-2}+\left(1+\|P_1\|_1\right)\times\\
&\times\left[\sum_{k=[\nu]}^{\infty}k^{-1}\omega_1\left(f\overline{P}_1,k^{-1}\right)+\sum_{k=[\nu]}^{\infty}\omega_1\left(f',k^{-1}\right)k^{-1}+\nu^{-1}\left(\|f'\|_1+\|f\overline{P}_1\|_1\right)\right]+\\
&+\left(\|f'\|_1+\|f\overline{P}_1\|_1+\|f\|_\infty\right)\sum_{r=2}^{3}\nu^{1-r}\|P_r\|_1\Big\},\quad \nu\ge 2.
\end{aligned}\tag{0.0.8}$$

$\|P_1\|_1=\int_0^1|P_1(x)|dx,$ *const* is independent of $f(x)$.

The proof of theorem 0.0.5 is based on the following lemma.

Lemma 0.0.3. *Let* $\{u_n(x)\}_{n=1}^{\infty}$ *be uniformly bounded, the function* $f(x)\in W_1^1(G)$ *and the system* $\{u_n(x)\}_{n=1}^{\infty}$ *satisfy conditions (*0.0.10). *Then for the Fourier coefficients* f_n *of the function* $f(x)$ *the estimation* $(\mu_n\ge 4\pi)$ *is valid*

$$\begin{aligned}
&|f_n|\le const\Big\{\ C(f)\mu_n^{\alpha-3}+\mu_n^{-1}\left(1+\|P_1\|_1\right)\Big[\ \omega_1\left(f',\mu_n^{-1}\right)+\\
&+\omega_1\left(f\overline{P}_1,\mu_n^{-1}\right)+\mu_n^{-1}\|f'\|_1+\mu_n^{-1}\|f\overline{P}_1\|_1\Big]+\\
&+\mu_n^{-2}\left(\|f'\|_1+\|f\overline{P}_1\|_1+\|f\|_\infty\right)\sum_{r=2}^{3}\|P_r\|_1\mu_n^{2-r}\Big\}.
\end{aligned}\tag{0.0.9}$$

In chapter II of the dissertation on the interval $G=(0,1)$ we consider a third order ordinary differential operator with summable complex valued coefficients. We study equiconvergence of biorthogonal expansion of a function from the class $L_p(G), p\ge 1,$ with its trigonometric Fourier series. Uniform equiconvergence rate on a compact is estimated, influence of continuity modulus of the coefficients $P_2(x)$ on equiconvergence rate is studied. Absolute and uniform convergence on $\overline{G}=[0,1]$ of biorthogonal expansion of a func-

tion from the class $W_2^1(G)$ in eigen and associated functions of the given operator is also studied.

In section 2.1. we consider a third order ordinary differential operator

$$Lu = u^{(3)} + P_1(x)u^{(2)} + P_2(x)u^{(1)} + P_3(x)u, \qquad (0.0.10)$$

where $P_i(x) \in L_1^{loc}(G),\ i = \overline{1,3}$.

For the root functions of the given operator we derive a shift formula and a mean value formula. These formulas are the main tools in studying uniform equiconvergence and absolute and uniform convergence of biorthogonal expansions in root functions of the operator (0.0.10).

In sections 2.2 we consider an ordinary differential operator L with the coefficients $P_i(x) \equiv 0$. We study equiconvergence on a compact of spectral expansion in root functions of the given operator with trigonometric expansion. The influence of continuity modulus on the coefficient $P_1(x) \equiv 0$ on uniform equiconvergence rate on a compact of the interval $P_2(x)$ $G = (0,1)$ of biorthogonal expansion in root functions of the given operator with trigonometric expansion is studied. This time the V.A.Il'in spectral method is used.

Let us consider on the interval $G = (0,1)$ a formal differential operator $Lu = u^{(3)} + P_2(x)u^{(1)} + P_3(x)u$ with complex-valued coefficients $P_\ell(x) \in L_1(G)$, $\ell = \overline{2,3}$.

Under the eigen-function of the operator L, responding to the complex eigenvalue λ, we understand any identically non zero complex valued function $y_0(x) \in D(G)$, satisfying almost everywhere in G the equation $Ly_0 + \lambda y_0 = 0$. Similarly, under the associated function of this operator of order m $(m \geq 1)$, responding to the some eigenvalue λ and eigenvalue function $y_0(x)$ we understand any complex valued function $y_m(x) \in D(G)$, satisfying almost everywhere in G the equation $Ly_m + \lambda y_m = y_{m-1}$.

We will consider every eigen function as an associated function of order 0. The highest order of the root functions (associated) of func-

tion responding to the given eigen function we will call the rank of this eigen function.

Let us consider the arbitrary system $\{u_k(x)\}_{k=1}^{\infty}$, consisting of the root functions of the operator L, responding to the system of eigen values $\{\lambda_k\}_{k=1}^{\infty}$ and require that together with every root function of order $\ell \geq 1$, this system include corresponding root functions of order less than ℓ and the rank of eigen functions be uniformly bounded. This means that $u_k(x) \in D(G)$ and satisfies almost everywhere in G the equation $Lu_k + \lambda_k u_k = \theta_k u_{k-1}$, where θ_k equals either 0 (in this case $u_k(x)$ is an eigen-function), or 1 (in this case we require $\lambda_k = \lambda_{k-1}$ and call $u_k(x)$ an associated function) .

Denote $\mu_k = \begin{cases} (i\lambda_k)^{\frac{1}{3}} & for\ \operatorname{Im}\lambda_k < 0 \\ (-i\lambda_k)^{\frac{1}{3}} & for\ \operatorname{Im}\lambda_k \geq 0, \end{cases}$

where $\left(re^{i\varphi}\right)^{\frac{1}{3}} = r^{\frac{1}{3}} e^{\frac{i\varphi}{3}}$, $-\pi < \varphi \leq \pi$.

Let the system $\{u_k(x)\}_{k=1}^{\infty}$ satisfy conditions A_p (V.A.Il'in conditions):

1) The system $\{u_k(x)\}_{k=1}^{\infty}$ be closed and minimal in $L_p(G)$ for the fixed $p \geq 1$;

2) The Carleman and the "sum of units" conditions be fulfilled

$$|\operatorname{Im}\mu_k| \leq const,\ k = 1,2,...; \quad \sum_{\tau \leq \operatorname{Re}\mu_k \leq \tau+1} 1 \leq const, \quad \forall \tau \geq 0$$

For any compact $K \subset G$ there exist a constant $C_0(K)$ such that $\|u_k\|_{p,K} \|\vartheta_k\|_q \leq C_0(K)$, where $\|\cdot\|_{p,K} = \|\cdot\|_{L_p,(K)}$, $\|\cdot\|_q = \|\cdot\|_{L_q(G)}$, $p^{-1} + q^{-1} = 1$, $(p = 1,\ q = \infty)$ $\{\vartheta_k(x)\}_{k=1}^{\infty}$ is a system biorthogonally adjoined to the system $\{u_k(x)\}_{k=1}^{\infty}$.

By $S_\nu(x, f)$ we denote a partial sum of trigonometric series of the function $f(x) \in L_p(G)$ and introduce a partial sum of biorthogonal expansion of the function $f(x)$ in the system $\{u_k(x)\}_{k=1}^{\infty}$:

$$\sigma_\nu(x,f)=\sum_{\rho_k\le\nu} f_k u_k(x),\quad \rho_k=\operatorname{Re}\mu_k,\ \nu>0$$

where $f_k=\int_G f(x)\overline{\vartheta_k(x)}dx$.

We introduce the following denotation:

$$\Delta_\nu(K,f)=\|\sigma_\nu(\cdot,f)-S_\nu(\cdot,f)\|_{C(K)};$$

$$\hat{f}_k=f_k\|\vartheta_k\|_q^{-1},\quad \Omega\left(f,\frac{\nu}{2},\alpha\right)=\nu^{-1}\sum_{1\le\rho_k\le\frac{\nu}{2}}\left|\hat{f}_k\right|\rho_k^{-\alpha},\ 0\le\alpha\le1;$$

$$\Phi_p(f,\nu)=\nu^{-1}\|f\|_p+\max_{\rho_k\ge\frac{\nu}{2}}\left|\hat{f}_k\right|;$$

$Q_p(f,\nu)=\nu^{-1}\|f\|_p+\max_{2\pi k\ge\frac{\nu}{2}}\left|\tilde{f}_k\right|$, where $\tilde{f}_k$ are Fourier coefficients of the function $f(x)$ with respect to the trigonometric system normed in $L_q(G)$

$D(\nu)=\inf_{\substack{\alpha>1\\ n\ge2}}\left\{\omega_1(P_2,n^{-1})\Omega\left(f,\frac{\nu}{2},0\right)+n^{2(1-\alpha^{-1})}\|P_2\|_1\Omega\left(f,\frac{\nu}{2},1-\alpha^{-1}\right)\right\}+\omega_1(f,\delta)$ - is a

$+\|P_3\|_1\Omega\left(f,\frac{\nu}{2},1\right)$

is a continuity modulus of the function $f(x)$ in $L_1(G)$;

$$\Phi_p\left(f,\left[\frac{\nu}{2}\right],1-r^{-1}\right)=T\left(f,\left[\frac{\nu}{2}\right],1-r^{-1}\right)+\frac{1}{1-r^{-1}}\|f\|_p;$$

$$\varphi_p(f,\nu)=\omega_1\left(f,\nu^{-1}\right)+\nu^{-1}\|f\|_p;$$

$$E(P_2,\nu)=\inf_{\substack{\alpha>1\\ n\ge2}}\left\{\omega_1(P_2,n^{-1}\left(T\left(f,\left[\frac{\nu}{2}\right],0\right)+\ln\nu\|f\|_1\right)+\right.$$

$$\left.+\|P_2\|_1 n^{2(1-\alpha^{-1})}\left(T\left(f,\left[\frac{\nu}{2}\right],1-\alpha^{-1}\right)+\frac{1}{1-\alpha^{-1}}\|f\|_1\right)\right\};$$

$$T(f,\ell,\varepsilon)=\sum_{i=1}^{\ell} i^{-\varepsilon}\omega_1\left(f,i^{-1}\right),\ 0\le\varepsilon<1.$$

Let $\omega(t)$ be a non-decreasing continuous function on $[0,\infty)$ and satisfy the conditions a) $\omega(0)=0,\ \omega(t)>0$ for $t>0$; b) $t^{-1}\omega(t)$ does

not increase. By $H_p^{\omega}(G),\ p \ge 1$, we denote the set of functions from $L_p(G)$ satisfying the condition $\omega_p(f,\delta) \le C(f)\omega(\delta)$, where $C(f)$ is a constant independent of $f(x)$. The norm in $H_p^{\omega}(G)$ is determined by the equality $\|f\|_p^{\omega} = \|f\|_p + \sup\limits_{\delta>0} \dfrac{\omega_1(f,\delta)}{\omega(\delta)}$.

Denote by $B_{p,\theta}^{\alpha}(G),\ 0<\alpha<1,\ 1 \le \theta \le \infty$, the Besov class with the norm

$$\|f\|_{B_{p,\theta}^{\alpha}(G)} = \|f\|_p + \left(\int_0^{h_0} \left(t^{-\alpha-\frac{1}{\theta}} \omega_p(f,t) \right)^{\theta} dt \right)^{\frac{1}{\theta}},\ h_0 > 0.$$

Note that , $B_{p,\infty}^{\alpha}(G) \equiv H_p^{\alpha}(G);\quad H_p^{\omega}(G) \equiv H_p^{\alpha}(G)$ for $\omega(t) = t^{\alpha}$, $0<\alpha<1$ ($H_p^{\alpha}(G)$ is a Nikolsky class).

The main results of this section are the following theorems:

Theorem 0.0.6. *Let* $P_2(x) \in L_r(G),\ r \ge 1,\ P_3(x) \in L_1(G)$ *and the system* $\{u_k(x)\}_{k=1}^{\infty}$ *satisfy the conditions* A_p.

Then expansion of the arbitrary function $f(x) \in L_p(G)$ *in biorthogonal series in the system* $\{u_k(x)\}_{k=1}^{\infty}$ *and in trigonometric series uniformly equiconvergence on any compact* $K \subset G$, *i.e.*

$$\Delta_\nu(K,f) \to 0,\ \nu \to +\infty, \tag{0.0.11}$$

and the following estimations are valid:

$$\Delta_\nu(K,f) \le C_1(K)\Big\{ \|P_2\|_r \Omega\Big(f, \frac{\nu}{2},\ 1-r^{-1}\Big) + \|P_3\|_1 \Omega\Big(f, \frac{\nu}{2},\ 1\Big) +$$
$$+ \Phi_p(f,\nu) + Q_p(f,\nu)\Big\},\quad \textit{for } r>1; \tag{0.0.12}$$

$$\Delta_\nu(K,f) \le C_2(K)\{D(\nu) + \Phi_p(f,\nu) + Q_p(f,\nu)\},\ \text{for } r=1, \tag{0.0.13}$$

where $C_1(K)$, $C_2(K)$ *are constants independent of* ν *and* $f(x)$.

Theorem 0.0.7. *Let the conditions of theorem 0.0.6 for* $p=1$ *be fulfilled, and for the coefficients of the function* $\hat{f}_k$ $f(x) \in L_1(G)$ *the following estimation be fulfilled:*

$$\left|\hat{f}_k\right| \le const\left\{\omega_1(f,\rho_k^{-1}) + \rho_k^{-1}\|f\|_1\right\},\ \ \rho_k \ge 1\ . \quad (0.0.14)$$

Then for $r > 1$ *the estimation*

$$\Delta_\nu(K,f) \le C_3(K)\nu^{-1}\left\{\ \|P_2\|_r \Phi_1\left(f,\left[\frac{\nu}{2}\right],\ 1-r^{-1}\right) + \right. \quad (0.0.15)$$

$$\left. + \|P_3\|_1 \Phi_1\left(f,\left[\frac{\nu}{2}\right],\ 1\right) + \nu\varphi_1(f,\nu)\ \right\},$$

for $r = 1$ *the estimation*

$$\Delta_\nu(K,f) \le C_4(K)\nu^{-1}\left\{E(P_2,\nu) + \|P_3\|_1 \Phi_1\left(f,\left[\frac{\nu}{2}\right],\ 1\right) + \nu\varphi_1(f,\nu)\right\}, \quad (0.0.16)$$

are valid.

Here the constants $C_3(K)$, $C_4(K)$ *are independent of* ν *and* $f(x)$.

A number of corollaries follow from theorem 0.0.7:

Corollary 0.0.6. *Under the condition of theorem 0.0.7 we have the estimations*

$$\Delta_\nu(K,f) \le C_5(K)\nu^{-\alpha}\|f\|_{B^\alpha_{p,\theta}(G)} \quad \text{for} \quad r = 1,\ f \in B^\alpha_{p,\theta}(G); \quad (0.0.17)$$

$$\Delta_\nu(K,f) \le C_6(K)\omega\left(\nu^{-1}\right)\begin{cases}\|f\|_p^\omega & for\ r > 1 \\ (1+R(\nu))\|f\|_p^\omega & for\ r = 1,\end{cases} \quad (0.0.18)$$

if $f \in H_p^\omega(G)$. *Here*

$$R(\nu) = \inf_{m\ge 2}\left\{\omega_1(P_2,m^{-1})\ln\nu + \|P_2\|_1 \ln m\right\}$$

Corollary 0.0.7. *Let* $r = 1$ *and the conditions of theorem 0.0.7 be fulfilled. Then for any function* $f(x) \in H_p^\omega(G)$ *the following estimation is valid*

$$\Delta_\nu(K,f) = O\left(\omega\left(\nu^{-1}\right)\ln\nu\right),\ \ \nu \to +\infty, \quad (0.0.19)$$

and if in addition we require $\omega_1(P_2,\delta) = O(\ln^{-\gamma}\delta^{-1})$, $\delta \to +0$, $\gamma > 0$, *then as* $\nu \to +\infty$ *the following estimation is fulfilled*

$$\Delta_\nu(K,f) = O\left(\omega\left(\nu^{-1}\right)\ln^{\frac{1}{1+\gamma}}\nu\right)B(\gamma), \quad (0.0.20)$$

where the symbol «O» depends on the function $f(x)$, $B(\gamma)=2^{\gamma}\gamma^{-\frac{\gamma}{1+\gamma}}+2\gamma^{-\frac{1}{1+\gamma}}$. *In particular for* $\gamma=1$, $p=1$, $f(x)\in W_1^1(G)$ *the estimation* $\Delta_\nu(K,f)=O\left(\nu^{-1}\ln^{\frac{1}{2}}\nu\right)$, $\nu\to+\infty$ *is valid.*

Note that uniform equiconvergence on a compact was studied in an exhaustive manner in the papers of V.A.Il'in, V.M.Kurbanov, R.A.Safarov, A.T.Garayev for Schrodinger operator with a summable potential. Influence of degree of summability of the coefficients of a differential operator on uniform equiconvergence rate was studied in the papers V.S.Rykhlov, V.M.Kurbanov, R.A.Safarov, A.T.Garayeva. In his papers of V.M.Kurbanov established estimations of uniform equiconvergence in the terms of integral module of continuity of the expanded function. Dependence of uniform equiconvergence rate on a modulus of continuity of the potential of one dimensional Schrodinger operator was studied in the papers V.M.Kurbanov and A.T.Garayeva.

In the last section of the dissertation we consider a formal integral operator

$$Lu=u^{(3)}+P_1(x)u^{(2)}+P_2(x)u^{(1)}+P_3(x)u \quad ,x\in G=(0,1) \ ,$$

with complex-valued coefficients $P_\ell(x)\in W_1^{3-\ell}(G)$, $\ell=\overline{1,3}$, $W_1^0(G)\equiv L_1(G)$. In this section, we impose some conditions on the system $\{u_k(x)\}_{k=1}^{\infty}$ and on the numbers μ_k and prove the analog of theorem 0.0.2. for biorthogonal expansions of a function from the class $W_2^1(G)$ in eigen and associated functions of this operator.

In conclusion the author expresses her deep gratitude to her supervisor prof. V.M.Kurbanov for the problem statement, constant attention and useful advices.

CONCLUSIONS

- Absolute and uniform convergence on the segment $\overline{G}=[0,1]$of spectral expansion of a function from the Sobolev class $W_{\rho}^{1}(G),\ p\geq 1,\ G=(0,1)$, in eigen functions of a third order ordinary differential operator with summable coefficients was studied and uniform convergence rate on this segment was estimated.
- Absolute and uniform convergence of orthogonal expansion of the function $f(x)\in W_{1}^{1}(G),\ G=(0,1)$, in eigen functions of a third order ordinary differential operator with summable coefficients is proved and the residue of the given expansion in the metric was $C(\overline{G})$ estimated.
- Theorems on uniform equiconvergence on a compact with trigonometric series of expansions in root functions of a third order differential operator with summable coefficients for a function from the class $L_p(G),\ p\geq 1$ were proved. Uniform equiconvergence rate for a function from the class $H_p^{\omega}(G),\ B_{\rho,\theta}^{\alpha}(G)$ was estimated.
- Theorem on absolute and uniform convergence of biorthogonal expansions of a function from the class $W_2^1(G),\ G=(0,1)$, in the system of root functions of a third order ordinary differential operator were proved and uniform convergence rate on $\overline{G}=[0,1]$. was set up.

Authors publications

1. Akhundova, E.B. Representation for eigen functions of third order differential operator // Proc. of IMM of NAS of Azerbaijan, - Baku: - 2012. v.XXXVII (XLV), - p. 25-32.

2. Ахундова, Э.Б. О скорости сходимости разложения функции из класса $W_p^1(G),\ p>1$ по собственным функциям дифференциального оператора третьего порядка // Riyaziyyat və informatikanın aktual problemləri Heydər Əliyevin anadan olmasının 90 illik yubileyinə həsr olunmuş Beynəlxalq konfransın tezisləri, - Bakı: 29 – 31 May, - 2013. -s. 133-134.

3. Ахундова, Э.Б. Сходимости спектрального разложения функции из класса $W_1^1(G)$ по собственным функциям обыкновенного дифференциального оператора третьего порядка // - Баку: Известия Педагогического Университета, - 2014. № 3, - с. 17-22.

4. Ахундова, Э.Б.Скорость сходимости спектрального разложения по собственным функциям дифференциального оператора третьего порядка // “Riyaziyyat və mexanikanın aktual problemləri” Riyaziyyat və Mexanika İnstitutunun 55 illiyinə həsr olunmuş Beynəlxalq konfransın materialları, - Bakı: 15 – 16 may, - 2014, -s. 67-68.

5. Kurbanov, V.M., Akhundova, E.B. Absolute convergence of spectral expansion in eigenfunctions of third order ordinary differential operator // - Baku: Proc. of the İMM of NAS of Azerbaijan, - 2014. v.40, special issue , - p.264-274.

6. Kurbanov, V.M., Akhundova, E.B. Convergence of spectral expansion of an absolutely continuous function eigenvalues of third order differential operator. // Madea -7. Azerbaijan – Turkey - Ukrainian International conference. Mathematical Analysis , Differential Equations and their applications. - Baku: September -8-13, - 2015, - p.93-94.

7. Ахундова, Э.Б. Абсолютная и равномерная сходимость биортогонального разложения функции из класса $W_2^1(G)$ по

корневым функциям дифференциального оператора третьего порядка // - Баку: Известия Педагогического Университета, - 2015, № 2, - с. 18-23.

8. Курбанов, В.М., Ахундова, Э.Б. О скорости равносходимости спектрального разложения с тригонометрическим рядом для дифференциального оператора третьего порядка // "Funksional analiz və onun tətbiqləri" adlı Əməkdar elm xadimi, professor Əmir Şamil oğlu Həbibzadənin anadan olmasının 100-cü ildönümünə həsr olunmuş respublika elmi konfransının materialları. - Bakı: - 2016. - s.157-158.

9. Kurbanov, V.M., Akhundova, E.B. Absolute and uniform convergence of spectral expansion of the function from the class $W_p^1(G)$, $p>1$, in eigen functions of third order differential operator//-Serbia (Beograd): Publications De'L Institut Mathematique, - 2017. v.101, № 115, - p.169-182.

10. Garayeva, A.T., Akhundova, E. On dependence of uniform equiconvergence rate on modulus of continuity of coefficient of third order differential operator //- Baku: Transactions of Pedaqogical University, - 2017. v.65. №2, - p.61-71.

The defense will be held on 23 april 2021 year at 14^{00} at the meeting of the Dissertation council ED 1.04 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan.

Address: AZ 1141, Baku city, B.Vahabzade str, 9.

Dissertation is accessible at the Institute of Mathematics and Mechanics of the NASA library.

Electronic versions of dissertation and its abstract are available on the official website of the Institute of Mathematics and Mechanics of the NASA.

Abstract was sent to the required addresses on 09 march 2021.

Signed for print: 09.03.2021
Paper format: 60x84 1/16
Volume: 80000
Number of hard copies: 20