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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

DIRECT AND INVERSE PROBLEMS FOR THE EIGENVALUES OF THE PAUILI OPERATOR WITH RESPECT TO DOMAIN

Specialty: 1211.01- Differential equations

Field of science: Mathematics

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GENERAL CHARACTERISTICS OF THE DISSERTATION

Relevance of the topic and degree of elaboration.

The presented dissertation is devoted to the formulation of the different type problems for the Pauli operator and development their solution methods. It is known that the Pauli operator characterizes the motion of the quantum particle in a magnetic field. The eigenvalues of this operator are equal to the total energy of the particle in the state determined by the corresponding eigenfunction. Therefore, the solution of various types of problems posed for this operator is of both theoretical and certain practical importance. On the other hand, the Pauli operator has a more complex mathematical form, since the motion of the particle is considered in the magnetic field and, unlike the Schrödinger operator, it also takes into account its spin. This makes its research even more difficult.

One of the specific aspects of the dissertation is that the problems discussed here are mostly with the variable domain. Such problems are less studied and require special approaches due to their specificity. The fact that the searched mathematical objects are not functions, as in traditional problems, but domains, means that the application of traditional methods to the study of these problems is not enough. Mainly in the last 40 years, many researches were conducted in this field and serious results were obtained. Several main trends may be noted in the direction of solving the variable domain problems.

The first direction is existence problems in the minimization of the functionals that describe certain characteristics of the considered system. Such problems were investigated in many works [5, 6, 8, 9, 11, 32, 55, 76, 94, 111, 115, 116]. It has been shown that the domain minimization problems are correct when certain geometric conditions are imposed on the domains, the minimizing functionals satisfy some monotonicity conditions and depend on a small number of physical characteristics (indeed eigenvalues) [35, 43].

The second direction is minimization problems with respect to domain. Although such optimization problems have been investigated by many authors, their unified solution method has not yet been developed [109, 110].

The third direction is the study of inverse spectral problems with respect to domain. The problems of this type have been less studied, and their mathematical formulation and the identification of spectral data are serious mathematical problems in themselves.

And finally, the next direction is the development of numerical solution methods of the considered problems. Investigation of the different problems in the analytically obtained optimal domains by the numerical methods development of their effective solutions methods should be considered serious results [15, 22, 41, 71, 78, 82, 85, 86, 104].

Therefore, the topic of the dissertation can be considered relevant .

The object and subject of his research. The object of research of the presented dissertation is direct and inverse problems in variable and fixed domains for the two-dimensional Pauli operator, which is a generalization of the Schrödinger operator. The subject of the research is approaches and solution methods that allow to solve the considered problems with analytical and numerical methods.

The purpose and problems of the study. It is the development of analytical and numerical methods to solve the direct and inverse problems set for the eigenvalues of the two-dimensional Pauli operator and their functions in the variable and fixed domains, simulation of the numerical experiments in various cases, analyzing their results and obtained effects.

Research methods. The methods of functional analysis, spectral theory of operators, optimization theory and approximate calculation theory were used to solve the problems considered in the dissertation.

Main conclusion presented for defense.

- Investigation of the problems of minimization of the function depending on the eigenvalues of the Pauli operator with respect to the domain;

- Study of the problem of maximizing eigenvalues of the Pauli operator;
- Formulation the inverse problem for the Pauli operator and development its solution method;
- Numerical study of the problem considered in the nonlinear case;
- Numerical study of the problem considered in the fixed domain.

Scientific novelty of the research.

- The problem of minimization of the function depending on the eigenvalues of the Pauli operator with respect to the domain has been investigated, theorems about the existence of the optimal domain have been proven;
- The problem of maximizing the eigenvalues of the Pauli operator has been studied and relevant theorems have been proved;
- An inverse spectral problem was set for the Pauli operator and its analytical-numerical solution method was proposed;
- A solution method was proposed for the numerical solution of the problem considered in the nonlinear case, and numerical experiments were performed;
- The problem considered in the fixed domain was studied numerically, numerical experiments were performed, and the resulting effects were analyzed.

Theoretical and practical significance of research. Taking into account the practical importance of the research object of the dissertation, one can state that the methods developed in the dissertation can be used in the spectral theory of the operators, in the field of eigenvalue problems with variable domain, and also has a certain practical importance. The results obtained in the dissertation may be useful in the analytical and numerical solution of a number of direct and inverse problems related to the defining the optimal domains in having a wide practical importance problems.

Approbation and application of the dissertation. The main results of the dissertation were presented at the following scientific seminars and conferences: Seminar of the Department of Differential Equations of Sumgait State University (Head - Prof. F. Feyziyev), Scientific Seminar of the Institute of Applied Mathematics of Baku

State University (Head - Academician F. Aliyev), V Congress of the Turkic World Mathematicians (Bishkek – 2014), Control and Optimization with Industrial Applications (Baku - 2015), The Applied Problems of Mathematics and New Information Technologies III republic scientific conference (Sumgait - 2016), International Scientific Conference on Theoretical and Application Problems of Mathematics (Sumgait – 2017), International Conference on Modern Trends in Physics (Baku – 2017), II Republican Scientific-Practical Conference of Young Researchers (Baku – 2019), 8th International Eurasian Conference on Mathematical Sciences and Applications (Baku – 2019), The Conference "Modern Problems of Mathematics and Mechanics" dedicated to the 60th anniversary of the Institute of Mathematics and Mechanics (Baku-2019), III Republican Scientific-Practical Conference of Young Researchers (Baku - 2020), Republican Scientific Conference "Actual Problems of Mathematics and Mechanics" dedicated to the 99th anniversary of the National Leader Heydar Aliyev (Baku-2022).

Applicant's personal contribution. All the main scientific results obtained in the dissertation are the result of the activity of the applicant personally as a result of application of the ideas of the supervisor in specific direction, formulation of the problems and development of the solution methods.

Publications of the author. The main results of the dissertation were published in 16 scientific works of the author. The list of these works is given at the end of the dissertation.

The name of the institution where the dissertation was performed. The dissertation was performed at the "Differential Equations and Optimization" department of the Sumgait State University.

The total volume of the dissertation with indicating the volume of the structural sections of the dissertation separately. The total volume of the dissertation is 189770 signs (title - 335, table of contents - 2710, introduction - 27912 signs, first chapter - 94000 signs, second chapter - 40000 signs, third chapter - 24830 signs, conclusion-813 signs). The dissertation consists of an introduction, three chapters, a conclusion, 15 pictures, 2 tables and a 116-item bibliography.

BRIEF CONTENTS OF THE DISSERTATION

The relevance of the topic is justified in the introduction of the dissertation, the brief content of the dissertation is explained.

The first chapter of the dissertation, called “Some problems with variable domains for the eigenvalues of the Pauli operator”, consists of three paragraphs.

In this chapter, the minimization problem for the function of the eigenvalues of the Pauli operator, which is a generalization of the Schrödinger operator and is widely used in quantum mechanics, is considered. Existence problems have been investigated for the considered problem and relevant theorems have been proved.

It is known that the eigenvalues of various operators describe certain parameters of various physical processes and natural phenomena. Sometimes, when modeling these processes, it is required to take into account the geometry of the environment in which the processes occur. In many cases, the area where the process takes place is not fixed and is included in the model as a variable parameter. Such problems are called variable domain problems. When it is required to minimize (maximize) some parameter of the process with respect to the domain, such problems are called optimization problems (shape optimization) with respect to the domain. Since the argument of the considered functional, unlike the traditional cases, does not depend on functions, but on the domain, the solution of such problems is connected with serious mathematical difficulties [39, 42]. When studying many natural processes, these or other physical-mechanical parameters are described by the eigenvalues of some operators. For example, the eigenvalues of the Schrödinger operator, which is widely used in quantum physics, characterize the energy levels of the charged particle, and the eigenvalues of the biquadratic operator characterize eigenfrequency during transverse oscillations of the membrane and plates. Sometimes, it is necessary to investigate not only the eigenvalues of these operators (or the appropriate physical-mechanical quantities they characterize), but the functions or functionals that depend on one or more of them. When the domain where the considered process takes place is taken as a variable quantity, these functionals ultimately depend on the area. Thus, we come to domain-dependent eigenvalue problems. These problems have been seriously studied in recent decades due to the wide scope of their application. A large number of works by different authors have been devoted to the existence of solutions to such problems, finding extremal solutions, and developing numerical solution algorithms. However, neither analytical solution of variable domain problems, nor general methods for numerical construction of these solutions have been developed yet.

In this chapter, the minimization problem is considered for the function of the eigenvalues of the Pauli operator, which is a generalization of the Schrödinger operator and is widely used in quantum mechanics. It is known that the Pauli operator describes the motion of a spin particle moving in an external magnetic field. In contrast to the Schrödinger operator, the consideration of both the spin of the particle and its movement in an external magnetic field make the Pauli operator much more formally complicated and the corresponding problems difficult. The eigenvalues of this operator describe the full energy levels in the external magnetic field of the spin particle.

In the first paragraph of the first chapter of the dissertation, we consider the problem of minimization with respect to the domain set for the function of the eigenvalues of the Pauli operator.

Consider the following eigenvalue problem:

$$P\varphi = \lambda\varphi, x \in D \ , \tag{1}$$

$$\varphi = 0, x \in S_D \ . \tag{2}$$

Here P is the Pauli operator defined as follow

$$P = P(a,\nu)\cdot J + \sigma B\ , \tag{3}$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, (a,\nu) = (-i\nabla - a)^2 + V\ ,$$

i is an imaginary unit; V is smooth enough function; $\nabla = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right\}; a = (a_1, a_2) \in R^2$ is a vector potential; B is a magnetic field generated by the vector potential a i.e.,

$$B = \frac{\partial}{\partial x} a_2 - \frac{\partial}{\partial y} a_1 .$$

Considering all these notations two-dimensional Pauli operator can be written as follows

$$P = \begin{pmatrix} (-i\nabla - a)^2 + a_2 \frac{\partial}{\partial x} - a_1 \frac{\partial}{\partial y} + V & 0 \\ 0 & (-i\nabla - a)^2 - a_2 \frac{\partial}{\partial x} + a_1 \frac{\partial}{\partial y} + V \end{pmatrix} =$$

$$= \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}, \tag{4}$$

where

$$b_{11} = -\Delta + (2ia_1 + a_2)\frac{\partial}{\partial x} + (2ia_2 - a_1)\frac{\partial}{\partial y} + a^2 + V\ ,$$

$$b_{22} = -\Delta + (2ia_1 - a_2)\frac{\partial}{\partial x} + (2ia_2 + a_1)\frac{\partial}{\partial y} + a^2 + V\ .$$

By Ω we define the given bounded open subset of and by K we define the set of all quazi-open subsets of Ω. Let K_c is the set of all subsets of Ω having Lebesgue measure c.

In this section we consider the following problem of the function depending on the first two eigenvalues of the Pauli operator in the set K_c

$$\min\left\{F(K), K \in K_c\right\}. \tag{5}$$

Here

$$F(K) = f\left(\lambda_1(K), \lambda_2(K)\right),$$

where f is a continuous non negative function, $\lambda_1(K)$ and $\lambda_2(K)$ are the first and second eigenvalues of the Pauli operator P in the domain K.

Introduce the following subset of R^2

$$E_c = \left\{(x, y) \in R^2 : x = \lambda_1(K), y = \lambda_2(K), K \in K_c\right\}.$$

That is, in fact, the set E_c is a subset of R^2 such that its elements are pairs formed from the first and second eigenvalues of the Pauli operator given by the boundary condition (2). Then problem (5) can be written in this form

$$\min\left\{f(x, y) : (x, y) \in E_c\right\}$$

It follows from this that the fact that the set E_c is closed in R^2 can give a very important information, because in this case the existence of the solution to problem (5) is a direct result.

This section gives auxiliary facts and the following main theorem.

Theorem 1. Let $c \in \left[0, |\Omega|\right]$. The defined above set E_c is closed.

Section 1.2. is devoted to the proof of this theorem. Let B be enough big ball for the given number c. Here is proved that then E_c is a closed set in R^2. This result leads to the existence result of the problem

$$\min\left\{\Phi\left(\lambda_1(A), \lambda_2(A)\right) : A \in A_c(B)\right\}$$

for some classes of functions Φ.

Let for the number $c > 0$ be given and $B \subseteq R^N$ be a union of two not intersected ball satisfying $mesB' = \frac{c}{2}$, $mesB'' = \frac{c}{2}$.

Theorem 2. Under the above conditions the set E is closed in R^2 .

The following lemma used in the proof of the theorem, is proved.

Lemma 1. If E is a set that is convex horizontally and vertically then this set is closed in R^2 .

The proof of the main theorem is given by the following sequence.

First it is assumed that A is horizontally convex. It means that if $A \in A_c(B)$ then the interval connecting $(\lambda_1(A), \lambda_2(A))$ and $(\lambda_2(A), \lambda_2(A))$ belongs to the internal of E .

Then it is supposed that A is vertically convex. It will mean that if $A \in A_c(B)$ then the interval connecting $(\lambda_1(A), \lambda_2(A))$ and $\left(\lambda_1(A), \frac{\lambda_2(B_1)}{\lambda_1(B_1)} \lambda_1(A)\right)$ belongs to the interval of E .

The following lemmas will be used in the process of proof.

Lemma 2. If C_1 and C_2 are quazi-open sets satisfying $cap(C_1 \cap C_2) = 0$ and $u \in H_0^1(C_1 \cup C_2)$ then $u_{|c_1} \in H_0^1(C_1)$ and $u_{|c_2} \in H_0^1(C_2)$.

Lemma 3. Assume that A is a quazi-open set and $\lambda_1(A) < \lambda_2(A)$. In this case internal set of A_1 is connected. Here the cases occur: $A_2 \subseteq A_1$ or the φ_2 is valid $cap(A_1 \cap A_2) = 0$.

Consequence 1. For $A \in A_c(B)$ and positive integer i the mapping $t \mapsto \lambda_i(A^t)$ is both lower and upper right semi continuous.

So we can conclude that the second eigenvalue of the Pauli operator satisfies the condition: if $t_1 < t_2$ and $\lambda_2\left(A^{t_1}\right) < \lambda_2\left(A^{t_2}\right)$, then for $\forall \lambda^* \in \left[\lambda_2\left(A^{t_1}\right), \lambda_2\left(A^{t_2}\right)\right]$ there exists $t^* \in [t_1, t_2]$ such that $\lambda_2(A^{t*}) = \lambda^*$.

Here the following fact is proved for quasi-open sets.

Lemma 4. Let $A \in A_C(B)$. The we have a sequence $\{A_n\}_{n\in N}$ of the Shteiner symmetry of A such that $m\left(A_n \setminus A^{\#}\right) \to 0, n \to \infty$.

Consequence 2. For each $A \in A_C(B)$ there exists a sequence $\{A_n\}_{n\in N}$ such that each weak limit point γ of $\{A_n\}_{n\in N}$ belongs to the internal of $A^{\#}$.

Consequence 3. For the sequence $\{A_n\}_{n\in N}$ is valid

$$\lambda_2\left(A^{\#}\right) \leq \lim_{n\to\infty} \inf \lambda_2(A_n).$$

Proposition 1. Assume that A_0 and A_1 are quazi-open sets and $A_1 \subseteq A_0$. Then there exists such a γ continuous mapping $h:[0,1] \to A\left(R^N\right)$ between A_0 and A_1 that for $t_1 < t_2$ is valid $h(t_1) \supseteq h(t_2)$ and $h(0) = A_0, h(1) = A_1$.

Proposition 2. For each $A \in A_C(B)$ there exists the set $\overline{A} \in A_C(B)$ satisfying $s\left(\overline{A}\right) \in \Re_{\inf}(A)$ and weak connected or is a ball.

In 1.2 the problem of maximizing of the eigenvalues of the Pauli operator is considered.

Let $N \geq 2$ and $\Omega \subseteq R^N$ be a bounded open set. The Sobolev space $H_1(\Omega)$ is compactly included in $L_2(\Omega)$. We call such set as regular sets. In such domains the spectrum of the Pauli operator with Neumann consists only of eigenvalues. These eigenvalues can be numbered as

$$0 = \mu_0(\Omega) \leq \mu_1(\Omega)) \leq \mu_2(\Omega)) \leq \ldots \to \infty$$

considering their multiplicity. Then for each $k \geq 1$

$$\mu_k(\Omega) = \min_{S \in S_k} \max_{u \in S} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega u^2 dx}.$$

Here S_k is the set of all k- dimensional subspaces in $H^1(\Omega)_{/\Re} := \left\{ u \in H^1(\Omega) : \int_\Omega u dx = 0 \right\}$. If Ω is a connected set then $\mu_1(\Omega) > 0$.

Szegö in 1954 proved that among all simply connected two dimensional $\Omega \subseteq \Re^2$ sets with smooth boundary the set maximizing the eigenvalue μ_1 is the ball B. This means that

$$|\Omega| \mu_1(\Omega) \leq |B| \mu_1(B).$$

Weinberger later proved more strong fact. He showed that the following inequality holds for every $N \geq 2$ and regular $\Omega \subseteq \Re^N$ without any topological restrictions

$$|\Omega|^{\frac{2}{N}} \mu_1(\Omega) \leq \mu_1.$$

Here

$$\mu_1^* = |B|^{\frac{2}{N}} \mu_1(B).$$

Polya established that the first part of Weyl's law actually provides the estimation of eigenvalues. That is, in the N dimensional case the relation

$$\forall k \geq 1, \mu_k(\Omega) \leq 4\pi^2 \left(\frac{k}{\omega_N |\Omega|} \right)^{\frac{2}{N}}$$

holds true. Where ω_N is a volume of the unit ball in $\Re^N$. This fact is valid only the some classes of domains. In general case it satisfies only for the case when $k = 1$. Kröger showed that for $k = 2$ the estimation $\mu_2(\Omega) \leq \frac{16\pi}{|\Omega|}$ is valid for the two dimensional case. The quantity $\frac{16\pi}{|\Omega|}$ in fact is twice larger that needed. The problem of finding of the domain that maximize the k-th eigenvalue of the Pauli operator is still a serious mathematical problem. Even for $k = 2$ solution of this problem meets serious difficulties. For $k \geq 2$ we have not meet strong results on the analytical solution of this problem. In some special cases a numerical solution methods have been developed for $k \leq 10$, but in such cases the existence results should be proved.

Here is proved that in the class of domains with the fixed measure the second eigenvalue of the Pauli operator with Neumann boundary condition reaches its maximum in the union of two non-intersected balls.

Also it is proved that the Polya hypothesis is also true for the eigenvalues of the Pauli operator with Neumann boundary condition without any restrictions on the size, geometry and topology of domains $k = 2$.

In any ball B introduce

$$\mu_2^* = 2^{\frac{2}{N}} |B|^{\frac{2}{N}} \mu_1(B).$$

Then we can write

$$\mu_0(B_1 \cup B_2) = 0, \mu_1(B_1 \cup B_2) = 0, \mu_2(B_1 \cup B_2) = \mu_2^*$$

.

Here B_1, B_2 are equal not intersected balls with volume $\frac{1}{2}$.

The main result of this section is formulated as the following theorem.

Theorem 3. Let $\Omega \subseteq \Re^N$ be a regular set. Then

$$|\Omega|^{\frac{2}{N}} \mu_2(\Omega) \leq \mu_2^*.$$

If in the above inequality the equality case is true, then the set Ω almost everywhere coincides with the union of two not intersected spheres.

From here the following result is obtained.

Consequence 2. For $k=2$ Polya hypothesis is true for the eigenvalues of the Pauli operator with Neumann condition, regardless of the size of the space.

Lemma 5. For all $x \in \Re^N$ and $A \neq B$ from $\Re^N$ holds true

$$g_A(x) \cdot \overrightarrow{ab} > g_B(x) \cdot \overrightarrow{ab}.$$

Lemma 6. Let us assume that for the points A and B from $\Re^N$ such that

$$\forall i = 1,\dots, N, \int_{\Omega} g_A(x) \cdot e_i dx = \int_{\Omega} g_B(x) \cdot e_i dx \ .$$

The for all $\nu \in R^N$ is valid

$$\int_{\Omega} g_A(x) \cdot \nu dx = \int_{\Omega} g_B(x) \cdot \nu dx$$

and $A = B$.

Proof. The first statement of the Lemma can be obtained by direct checking. To prove the second statement it is enough to take $\nu = ab$ in Lemma 5.

Proposition 3. There exist two different points A, B that satisfy the orthogonality condition.

Lemma 7. For $(g_i)_i \in G$ satisfying $\forall \varepsilon \in \left(0, \frac{m - \tilde{\mu}_1(\rho)}{2}\right)$ and $\int_{\Re^N} \rho g_i u_1 dx = 0$ there exist a number $C > 0$ for which the relation

$$\forall i = 1,\dots,N, \ \tilde{\mu}_2(\rho) \leq \int_{\Re^N} \rho |\nabla g_i|^2 dx + C_\varepsilon$$

holds true.

In the third chapter of the first chapter of the dissertation called "An inverse problem for the eigenvalues of the Pauli operator" an inverse problem for the eigenvalues of the Pauli operator with respect to the domain is given. A scheme for the analytical solution of this problem is proposed.

Here the problem

$$P\varphi = \lambda \varphi \ , \qquad x \in D \ , \tag{7}$$

$$\varphi = 0, \qquad x \in S_D \ . \tag{8}$$

is considered. Here P is the Pauli operator defined as above. By M we define the set of all bounded convex domains $D \in R^2$ Let

$$K = \{D \in M : S_D \in C^2\}.$$

Introduce the functions

$$S_j(x) = \frac{|\nabla u_j(x)|^2}{\lambda_j}, x \in S_D, j = 1,2,\ldots \tag{9}$$

It is required to find a domain $D \in K$ on the boundary of which the following relations hold

$$S_j(x) = g_j(x), x \in S_D, j = 1,2,\ldots \tag{10}$$

Here $g_j(x), j = 1,2,\ldots$ are given in R^N functions.

In other words, it is required to find a domain D from the set K, such that the functions determined by the spectral data of problem (7)-(8) are equal to the given functions on the boundary of that domain.

Suppose that the function satisfies the following condition

$$t^2 V(xt) = V(x) \tag{11}$$

Theorem 4. Under the condition (11) the coefficients $\alpha_i, i = 1,2,\ldots$ of the support function V of the solution to the inverse spectral problem for defining the domain $D \in K$ formulated above over the basis given on the unit ball B are the solution of the following infinite system of algebraic equations

$$\sum_{k,m=1}^{\infty} A_{k,m}(j)\alpha_k \alpha_m = 2, j = 1,2,\ldots \tag{12}$$

Here

$$A_{k,m}(j) = \lim_{n\to\infty}\left[\int\limits_{S_{G_n^k}} g_j(x)\left[P_{G_n^m}(n(x)) - P_{H_n^m}(n(x))\right]ds - \right.$$

$$-\int_{S_{H_n^k}} g_j(x)\left[P_{G_n^m}(n(x)) - P_{H_n^m}(n(x))\right]ds \Bigg].$$

The second chapter of the dissertation is called “Investigation of the approximate solution in the specified region in the nonlinear case”. In this chapter, the problem is considered for developing an effective numerical solution method for the solution of the initial and boundary value problem set for the considered equation in the fixed domain for the nonlinear case. For this, the Saulyev finite element scheme is used.

It is known that one of the features of the explicit finite difference methods is the restriction of the time step size due to stability requirements. This makes it difficult to apply such methods in many cases. On the other hand, implicit methods are known to be stable, but they require to be solved in a matrix system at each time level. Therefore, one must try to find such explicit schemes that are also stable.

Consider the following nonlinear particular differential equation

$$\frac{\partial u}{\partial \tau} = \frac{\sigma_0^2}{2}\left(1 + \Psi\left(a^2 x^2 \frac{\partial^2 u}{\partial x^2}\right)\right) x^2 \frac{\partial^2 u}{\partial x^2},\ x \in [0,\infty), \tau \in [0,T] \qquad (13)$$

where σ_0 and a are given numbers.

We will consider this equation with the following initial and Dirichlet boundary conditions

$$u(x,0) = \max(x - E, 0), u(0,\tau) = 0, \qquad (14)$$

$$\lim_{x \to \infty} u(x,\tau) = x$$

Section 2.1 presents the problem statement and some facts to be used.

2.2 describes the Saulyev scheme for the nonlinear problem.

The compatibility, monotonicity and stability of the considered scheme are analyzed in 2.3. For the problem considered in this paragraph, the following theorem on the monotonicity of the Saulyev scheme is proved.

Theorem 5. If to take the time step in the numerical scheme (13) as follows

$$k \le \frac{h^2}{\alpha_i^n},\ 0 \le i \le M, 0 \le n \le N, \tag{15}$$

then this scheme is stable.

In 2.4, numerical experiments were performed and the results were presented in the form of graphs. Here, the effectiveness of the applied method has been demonstrated, and the results have been compared with the results obtained by other existing methods.

In the third chapter of the dissertation, we consider the two-dimensional Pauli operator in the fixed domain, and obtain hyperbolic and periodic analytical solutions of the problem set for this operator at different values of the appropriate parameters.

Graphs of these solutions at different values of the parameters are given.

The following problem posed for the Pauli operator is considered

$$\begin{gathered} -\frac{\partial^2}{\partial x^2}\varphi_1 - \frac{\partial^2}{\partial y^2}\varphi_1 + (2ia_1 + a_2)\frac{\partial}{\partial x}\varphi_1 + \\ + (2ia_2 - a_1)\frac{\partial}{\partial y}\varphi_1 + \left(a^2 + V + \frac{\partial}{\partial x}a_2 - \frac{\partial}{\partial y}a_1 \right)\varphi_1 = 0 \end{gathered} \tag{16}$$

$$\begin{gathered} -\frac{\partial^2}{\partial x^2}\varphi_2 - \frac{\partial^2}{\partial y^2}\varphi_2 + (2ia_1 - a_2)\frac{\partial}{\partial x}\varphi_2 + (2ia_2 + a_1)\frac{\partial}{\partial y}\varphi_2 + \\ + \left(a^2 + V - \frac{\partial}{\partial x}a_2 + \frac{\partial}{\partial y}a_1 \right)\varphi_2 = 0 \end{gathered} \tag{17}$$

We can write the obtained the second order partial differential equations in the following form

$$-\frac{\partial^2}{\partial x^2}\varphi_1 - \frac{\partial^2}{\partial y^2}\varphi_1 + A_1\frac{\partial}{\partial x}\varphi_1 +$$

$$+ A_2\frac{\partial}{\partial y}\varphi_1 + \left(a^2 + V + \frac{\partial}{\partial x}a_2 - \frac{\partial}{\partial y}a_1\right)\varphi_1 = 0\,, \qquad (18)$$

$$-\frac{\partial^2}{\partial x^2}\varphi_2 - \frac{\partial^2}{\partial y^2}\varphi_2 + A_3\frac{\partial}{\partial x}\varphi_2 + A_4\frac{\partial}{\partial y}\varphi_2 +$$

$$+\left(a^2 + V - \frac{\partial}{\partial x}a_2 + \frac{\partial}{\partial y}a_1\right)\varphi_2 = 0\,, \qquad (19)$$

$$A_1 = 2ia_1 + a_2,\, A_2 = 2ia_2 - a_1,$$

$$A_3 = 2ia_1 - a_2,\, A_4 = 2ia_2 + a_1. \qquad (20)$$

To solve this problem in 3.2 the following wave transformation is applied

$$\xi = m_1 x + m_2 y$$

Here m_1 and m_2 may be takes constant or variable parameters. Using this transformation the equations under consideration may be written as

$$\left(m_1^2 + m_2^2\right)\varphi_1'' + \left(A_1 m_1 + A_2 m_2\right)\varphi_1' +$$

$$+\left(a^2 + V + \frac{\partial}{\partial x}a_2 - \frac{\partial}{\partial y}a_1\right)\varphi_1 = 0, \qquad (21)$$

$$\left(m_1^2 + m_2^2\right)\varphi_1'' + \left(A_3 m_1 + A_4 m_2\right)\varphi_1' +$$

$$+\left(a^2 + V - \frac{\partial}{\partial x}a_2 + \frac{\partial}{\partial y}a_1\right)\varphi_2 = 0, \qquad (22)$$

where

$$\varphi_i' = \frac{d\varphi_i}{d\xi}, i = 1{,}2 \quad \text{and} \quad a_i(x,y), i = 1{,}2\,. \qquad (23)$$

In 3.3 the tan-expansion $(\varphi/2)$ method is applied.

In 3.3. is shown that the solutions of the considered problem are defined by the expressed

$$\varphi(\xi) = \sum_{k=0}^{m} c_k \left[\tan(\varphi/2)\right]^k + \sum_{k=1}^{m} d_k \left[\tan(\varphi/2)\right]^{-k}, \qquad (24)$$

where $c_k\,(0 \le k \le m), d_k\,(1 \le k \le m)$ are parameters to be found and the functions $c_m, d_m \neq 0$ and $\phi = \phi(\xi)$ are solutions to the following ordinary differential equation

$$\phi'(\xi) = k_1 \sin(\phi(\xi)) + k_2 \cos(\phi(\xi)) + k_3 \qquad (25)$$

In Section 3.4, different cases are considered and the steps to find the solution are given for calculating the specific solutions of the considered equation for each case.

The sets of hyperbolic and trigonometric solutions of the equation considered in 3.5 have been found. Graphs of the obtained solutions are given. The dependence of the obtained solutions on the imposed physical conditions was analyzed. By choosing different values of the parameters, different families of solutions were obtained and their graphs were given.

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CONCLUSION

The presented dissertation is devoted to the formulation of the different type problems for the Pauli operator and development their solution methods.

The following main results were obtained in the dissertation:

- The problem of minimization of the function depending on the eigenvalues of the Pauli operator with respect to the domain was formulated, investigated and the theorems on the existence of the optimal domain were proved [2,3,9,11];
- The problem of maximizing the eigenvalues of the Pauli operator with Neumann condition was studied and relevant theorems have been proved [4,6];
- An inverse spectral problem was formulated for the Pauli operator and its analytical-numerical solution method was proposed [14];
- A solution method was proposed for the numerical solution of the problem considered in the nonlinear case, and numerical experiments were performed [12];
- The problem for the Pauli operator in the fixed domain was considered numerically, numerical experiments were performed, and the obtained effects were analyzed [17].

The main results of the dissertation are published in the

following works:

1. Gasimov, Y.S., Aliyeva, A.R. On a variation of the eigenvalues of Pauli operator with respect to domain // V Congress of the Turkic World Mathematicians Kyrgyztan, “Issyk – Kul Aurora”, - Bishkek: - 5 -7 June, -2014, -p. 70.
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