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# ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

# SOME INVERSE BOUNDARY VALUE PROBLEMS FOR FOURTH ORDER DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

Specialty: 1211.01 – Differential equations

Field of science: Mathematics

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## GENERAL CHARACTERISTICS OF THE WORK

**Rationale and development degree of the topic.** Mathematical modeling methods are widely used for studying processes in the environment that surrounds us. In order to study these processes, one of the effective ways is to model it in the form of differential equations. Boundary value problems for partial differential equations occupies an important place in theory of differential equations. The problems of finding unknown coefficients of the equation alongside with its solution is called an inverse problem in theory of partial differential equations. If along with the solution of the problem, the right hand side of the equation is also sought these inverse problems are called linear and if along with the solution of the equation even if one of the coefficients is sought, such inverse problems are called nonlinear problems.

Inverse problems is one of the actively developing fields of modern mathematics. Inverse problems can be encountered in many fields of human activities as geophysics, biology, ecology, medicine, etc. Inverse problems for partial differential equations were studied in the papers of Tikhonov A.N., Budak B.M., Anikonov Y.Y., Kostin D.B., Denisov A.M., Lavrentyev M.M., Iskenderov A.D., Akhundov A.Y., Ayda-zadeh K.R., Iskenderov A.D., Ivanchov N.I., Ivanov V.K., Amirov A.X., Kabanikhin S.I., Kaminin V.L.[1], Namazov G.K., Khudaverdiyev K.I., Kojanov A.I., Guliyev M.A., Malyshev I.G., Isgandarov N.Sh., Mehraliyev Y.T., Prilepko A.I., Romanov V.G., Cannon J.R.[2], Karimov N.B., Ismayilov A.I., Ismayilov M.I. and of other authors.

Recently, nonlocal problems for partial differential equations

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[1] Камынин, В.Л. Об обратной задаче определения правой части в параболическом уравнении с условием интегрального переопределения // Математические заметки, -2005. т.77, № 4, -с. 522–534

[2] Cannon, J.R. The solution of the heat equation subject to the specification of energy // Quart. Appl. Math. –1963. v.5, №21. -p. 155-160

are widely studied. These problems are important from both of theoretical and practical point of view. The integral condition problems are most studied problems among nonlocal problems. Such conditions are encountered in plasma physics, heat transfer, in mathematical modeling of hydration process of capillary-simple enviroment, in demography, in some problems of mathematical biology.

The dissertation work was devoted to the existence and uniqueness of the classic solution of nonlocal, nonlinear boundary value problems reduced to self-adjoint, not self-adjoint boundary value problems for finding time-dependent unknown coefficient, the right-hand side in a bounded domain for a linearized Benny-Luke equation contained in the class of partial differential equations of fourth order. As such problems are encountered in mechanics, physics, the topic of the work is urgent.

**Objective of the research.** The objective of the research is to study classical solution of nonlocal, nonlinear boundary value problems for finding time-dependent unknown coefficient and the right-hand side in a rectangle for a linearized Benny-Luke equation, contained in the class of fourth order partial differential equations.

**Research methods.** For studying the problems under consideration, the appropriate auxuliary problem is considered and equivalent relation between these problems is shown. Using the method of separation of variables and compressed mapping principles the existence and unigueness of the auxiliary problem is proved. And then by means of the equivalence relation, the existence and uniqueness of the problem under consideration is shown.

**Main theses to be defended.**

1. Studying nonlinear, nonlocal problem reduced to a self-adjoint boundary condition for fourth order linear partial Benny-Luke equation;

2. Studying the existence and uniqueness of the classic solution of an inverse boundary value problem with a not self-adjoint boundary condition for fourth order linear partial Benny-Luke equation.

**Scientific novelty of the study.** In the dissertation work the following main results were obtained:

- Theorems on the existence and uniqueness of the solution of self-adjoint boundary condition, nonlocal, nonlinear inverse boundary value problems with for a linearized Benny-Luke equation contained in the class of fourth order partial differential equations;
- Theorems on the existence and uniqueness of the classic solution of nonlocal, nonlinear inverse boundary value problems that can be reduced to a Benny-Luke equation contained in the class of fourth order partial differential equations are proved;
- Theorems on the existence and uniqueness of the solution of self-adjoint boundary condition, nonlinear inverse boundary value problems for Benny-Luke equation contained in the class of fourth order partial differential equations are proved;
- Theorems on the existence and uniqueness of the classic solution of nonlocal, nonlinear inverse boundary value problems that can be reduced to not self-adjoint boundary condition for a linearized Benny-Luke equation contained in the class of fourth order partial differential equations are proved.

**Theoretical and practical importance of the study.**

The dissertation work is of theoretical character. The results obtained can be determined by the application of the study of inverse problems generated by various problems.

**Approbation and application.** The results of the dissertation work were reported in the seminars of "Mathematical Analysis" department of Ganja State University, in the conference "Actual problems of applied mathematics, informatics and mechanics" held in RF Voronej city in 2019 and 2020, in the scientific seminar "Operators, system of functions and mathematical physics" held in Baku and organized by Khazar University in 2019, in the Republican Scientific Conference "Function theory, functional analysis and their applications"devoted to 110-th jubilee, of acad. Ibrahim Ibish oglu Ibrahimov.

**Applicants personal contribution.** All the results of the dissertation work belong to the author.

**Authors publications.** Main results of the dissertation work were published in the works [1]-[11] and given at the end of the thesis.

**The name of the organization where the dissertation work was executed.** The dissertation work was executed in Ganja State University of Azerbaijan.

**Total volume of the dissertation work indicating separately the volume of structural units in signs**.

The total volume of the dissertation is -232828 signs (title page 404 signs, content -1674 signs, introduction -34750 signs, chapter I -76000 signs, chapter II -80000 signs, chapter III -40000 signs), a list of references with 93 names. The total volume of the work is 127 pages.

## THE CONTENT OF THE WORK

The dissertation work consists of introduction and 3 chapters. The rationale of the topic, brief review of works related to the topic were justified in introduction. And the obtained results of the work were reflected.

Chapter I of the work consists of 2 sections. Here, the existence and unigueness of the classic solutions of nonlocal boundary condition inverse boundary value problems for a linearized Benny-Luke equation contained in the class of fourth order partial differential equations, that can be reduced to self-adjoint boundary condition is studied.

In section 1 of chapter I in the rectangle $D_T = \{(x,t): 0 \le x \le 1, 0 \le t \le T\}$ for the linearized Benny-Luke equation

$$u_{tt}(x,t) - u_{xx}(x,t) + \alpha u_{xxxx}(x,t) - \beta u_{xxtt}(x,t) =$$

$$= a(t)u(x,t) + b(t)g(x,t) + f(x,t) \qquad (1)$$

we consider an inverse boundary value problem within nonlocal[3]

$$u(x,0)=\int_0^T p(t)u(x,t)dt+\varphi(x)\ ,$$

$$u_t(x,0)+\delta u(x,T)=\psi(x)\quad (0\le x\le 1) \tag{2}$$

boundary

$$u_x(0,t)=0, u_x(1,t)=0, u_{xxxx}(0,t)=0 \quad (0\le t\le T) \tag{3}$$

integral

$$\int_0^1 u(x,t)dx=0 \quad (0\le t\le T), \tag{4}$$

and additional

$$u(0,t)=h_1(t) \quad (0\le t\le T), \tag{5}$$

$$u(1,t)=h_2(t) \quad (0\le t\le T) \tag{6}$$

conditions, here $\alpha>0$, $\beta>0$, $\delta\ge 0$ are the given numbers, $f(x,t)$, $g(x,t)$, $p(t)$, $\varphi(x)$, $\psi(x)$, $h_i(t)(i=1,2)$ are the given functions, while $u(x,t), a(t), b(t)$ are desired functions.

**Definition 1.1.1** If the functions $u(x,t,)\in \tilde{C}^{4,2}(D_T)$, $a(t)\in C[0,T]$, $b(t)\in C[0,T]$ on the rectangle $D_T$ satisfy the boundary condition (2) of equation (1) in the interval $[0,1]$ the conditions (3)- (6) on the interval $[0,T]$ in the usual sense, then the triple $\{u(x,t), a(t), b(t)\}$ is called a classic solution of the inverse boundary value problem (1)-(6), here

$$\tilde{C}^{4,2}(D_T)=\{u(x,t): u(x,t)\in C^2(D_T), u_{ttx}(x,t),$$
$$u_{ttxx}(x,t), u_{xxx}(x,t), u_{xxxx}(x,t)\in C(D_T)\}.$$

Along with problem (1)-( 6) we consider the following auxiliary problem. It is required to find such a triple

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[3] Mehraliyev, Y.T., Valiyeva, B.K., Ramazanova, A.T. An inverse boundary value problem for a linearized Bonny-Luc equation with nonlocal boundary conditions // Cogent Mathematics & Statistics, -2019. №6, -p.1-19

$u(x,t)\in \tilde{C}^{4,2}(D_T)$, $a(t)\in C[0,T]$, $b(t)\in C[0,T]$ consisting of the functions $\{u(x,t),a(t),b(t)\}$ that this triple along with conditions (1) -(3) satisfies the conditions

$$u_{xxx}(1,t)=0 \ (0\le t\le T) \tag{7}$$

$$h_1''(t)-u_{xx}(0,t)+\alpha u_{xxxx}(0,t)-\beta u_{xxtt}(0,t)=$$
$$=a(t)h_1(t)+b(t)g(0,t)+f(0,t) \ (0\le t\le T) \tag{8}$$

$$h_2''(t)-u_{xx}(1,t)+\alpha u_{xxxx}(1,t)-\beta u_{xxtt}(1,t)=$$
$$=a(t)h_2(t)+b(t)g(1,t)+f(1,t) \ (0\le t\le T) \tag{9}$$

The following theorem is proved.

**Theorem 1.1.1.** Suppose that the coherence conditions

$$\varphi(x),\ \ \psi(x)\in C[0,1],\ \ p(t)\in C[0,T],\quad h_i(t)\in C^2[0,T]\quad (i=1,2),$$

$$f(x,t),\ \ g(x,t)\in C(D_T),\ h(t)\equiv h_1(t)g(1,t)-\ -h_2(t)g(0,t)\ne 0,$$

$$\int_0^1 f(x,t)dx=0,\quad \int_0^1 g(x,t)dx=0(0\le t\le T)\quad \text{and}$$

$$\int_0^1 \varphi(x)dx=0,\quad \int_0^1 \psi(x)dx=0,$$

$$\varphi(0)=h_1(0)-\int_0^T p(t)h_1(t)dt,\quad \psi(0)=h_1'(0)+\delta h_1(T),$$

$$\varphi(1)=h_2(0)-\int_0^T p(t)h_2(t)dt,\quad \psi(1)=h_2'(0)+\delta h_2(T)\ .$$

are satisfied. Then the following statements are valid:

A. Every classic solution of inverse boundary value problem (1)-( 6) is the solution of inverse boundary value problem (1)-( 3) (7)-(9) .

B. Every solution of problem (1)-( 3) (7)-(9) satisfying the condition

$$\left(\|p(t)\|_{C[0,T]}+2T\|a(t)\|_{C[0,T]}\right)T<1$$

is the classic solution of inverse boundary value problem (1)-( 6).

To study the solution of problem (1)-(3),(7)-(9) we introduce the following spaces.

$B_{2,T}^{5(1)}$ denotes the functions representable in the form of

$$u(x,t)=\sum_{k=0}^{\infty}u_k(t)\cos\lambda_k x \ \ (\lambda_k=k\pi)$$

in the rectangle $D_T$, here each of the functions $u_k(t)\,(k=0,1,2,...)$ is continuous in the interval $[0,T]$ and the condition

$$I(u)\equiv\|u_0(t)\|_{C[0,T]}+\left\{\sum_{k=1}^{\infty}(\lambda_k^5\|u_k(t)\|_{C[0,T]})^2\right\}^{\frac{1}{2}}<+\infty.$$

is satisfied. In this space we determine the norm in the form of $\|u(x,t)\|_{B_{2,T}^{5(1)}}=I(u)$.

By $E_T^{5(1)}$ we denote the topological product $B_{2,T}^{5(1)}\times C[0,T]\times C[0,T]$. In this space the norm of the element $z=\{u,a,b\}$ is determined by the formula

$$\|z\|_{E_T^{5(1)}}=\|u(x,t)\|_{B_{2,T}^{5(1)}}+\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]}.$$

It is known that $B_{2,T}^{5(1)}$ and $E_T^{5(1)}$ are Banach spaces[4].

It is assumed that the data of the problem (1)-(3), (7)-(9) satisfy the following conditions.

1.1. $\alpha>0$, $\beta>0$, $0\le\delta<\sqrt{\dfrac{\alpha}{1+\beta}}\,\pi$, $p(t)\in C[0,T]$.

1.2. $\varphi(x)\in C^4[0,1], \varphi^{(5)}(x)\in L_2(0,1),\ \varphi'(0)=\varphi'(1)=\varphi'''(0)=\varphi'''(1)=0.$

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[4] Худавердиев К.И., Велиев А.А. Исследование одномерной смешанной задачи для одного класса псевдогиперболических уравнений третьего порядка с нелинейной операторной правой частью, Баку: Чашыоглы, 2010, 168 с.

1.3. $\psi(x)\in C^3[0,1], \psi^{(4)}(x)\in L_2(0,1),\ \psi'(0)=\psi'(1)=\psi'''(0)=\psi'''(1)=0.$

1.4. $f(x,t),\ f_x(x,t)\in C(D_T),\ f_{xx}(x,t)\in L_2(D_T),$

$f_x(0,t)=f_x(1,t)=0\ (0\le t\le T).$

1.5. $g(x,t), g_x(x,t)\in C(D_T), g_{xx}(x,t)\in L_2(D_T),$

$g_x(0,t)=g_x(1,t)=0\ (0\le t\le T).$

1.6. $h_i(t)\in C^2[0,T] (i=1,2),$

$h(t)\equiv h_1(t)g(1,t)-h_2(t)g(0,t)\ne 0\ \ (0\le t\le T).$

The following theorem is proved.

**Theorem 1.1.2.** Let the conditions 1.1- 1.6 be satisfied. Furthermore, it is assumed that the inequality

$$(B(T)(A(T)+2)+C(T)+D(T))(A(T)+2)<1$$

is valid. Then the problem (1)-(3), (7)-(9) has a unigue solution in the sphere

$$K=K_R(\|z\|_{E_T^{5(1)}}\le R\le A(T)+2)$$

taken from the space $E_T^{5(1)}$. Here, the expressions of $A(T)$, $B(T),\ C(T)$ and $D(T)$ are determined in section 1.2.

By means of theorem 1.1.1 from theorem 1.1.2 we obtain the validaty of the following theorem.

**Theorem 1.1.3.** Assume that all the conditions of theorem 1.1.2, the coherence conditions

$$\int_0^1 f(x,t)dx=0,\ \int_0^1 g(x,t)dx=0\ \ (0\le t\le T)\ \text{və}\ \int_0^1 \varphi(x)dx=0,\ \ \int_0^1 \psi(x)dx=0\ ,$$

$$\varphi(0)=h_1(0)-\int_0^T p(t)h_1(t)dt,\quad \psi(0)=h_1'(0)+\delta h_1(T),$$

$$\varphi(1)=h_2(0)-\int_0^T p(t)h_2(t)dt,\quad \psi(1)=h_2'(0)+\delta h_2(T)\ .$$

are satisfied. If the inequality $(\|p(t)\|_{C[0,T]}+2T(A(T)+2))T<1$ is satisfied, then the inverse boundary value problem (1)-( 6) has a unigue classic solution, in the sphere $K = K_R(\|z\|_{E_T^{5(1)}} \le R = A(T)+2)$ taken from the space $E_T^{5(1)}$.

In section 2 of chapter I it is assumed that we are given the rectangle $D_T = \{(x,t): 0 \le x \le 1, 0 \le t \le T\}$. In this rectangle we consider the equation

$$u_{tt}(x,t) - u_{xx}(x,t) + \alpha u_{xxxx}(x,t) - \beta u_{xxtt}(x,t) =$$
$$= a(t)u(x,t) + b(t)g(x,t) + f(x,t) \tag{10}$$

and assume that the function $u(x,t)$ being the solution of this equation satisfies the non-local

$$u(x,0) = \int_0^T p(t)u(x,t) + \varphi(x),\ u_t(x,0) = \psi(x)(0 \le x \le 1) \tag{11}$$

periodic

$$u(0,t) = u(1,t), u_x(0,t) = u_x(1,t),\ u_{xx}(0,t) = u_{xx}(1,t) \quad (0 \le t \le T) \tag{12}$$

nonlocal integral

$$\int_0^1 u(x,t)dx = 0 \ \ (0 \le t \le T) \tag{13}$$

and additional

$$u(x_i,t) = h_i(t)\ (i = 1,2; x_1 \ne x_2,\ 0 \le t \le T) \tag{14}$$

conditions, here $\alpha > 0,\ \beta > 0,\ x_i \in (0,1)\ (i = 1,2)$ are the given numbers, $f(x,t),\ g(x,t),\ p(t),\ \varphi(x),\ \psi(x),\ h_i(t)\ (i = 1,2)$ are the given functions, $u(x,t), a(t), b(t)$ are the desired functions.

**Definition 1.2.1.** Under the classic solution of inverse boundary value problem (10)-(14) we understand such a triple

$$u(x,t) \in \tilde{C}^{4,2}(D_T),\ a(t) \in C[0,T],\ b(t) \in C[0,T]$$

consisting of the functions $\{u(x,t), a(t), b(t)\}$ that these functions satisfy equation (10) in the rectangle $D_T$, the conditions (11) in the interval $[0,1]$ the conditions (12), (13) and (14) in the interval $[0,T]$ in the usual sense.

For studying the classic solution of inverse boundary value problem (10) - (14) we consider the following auxiliary inverse boundary value problem.

It is required to find such a triple

$$u(x,t) \in \tilde{C}^{4,2}(D_T),\ a(t) \in C[0,T],\ b(t) \in C[0,T]$$

consisting of the functions $\{u(x,t), a(t), b(t)\}$ that this triple along with conditions (10) – (12) satisfy the conditions

$$u_{xxx}(0,t) = u_{xxx}(1,t) \qquad (0 \le t \le T) \tag{15}$$

$$h_i''(t) - u_{xx}(x_i,t) + \alpha u_{xxxx}(x_i,t) - \beta u_{xxtt}(x_i,t) =$$
$$= a(t)h_i(t) + b(t)g(x_i,t) + f(x_i,t)(i = 1,2;\ 0 \le t \le T) \tag{16}$$

in the usual sense.

**Theorem 1.2.1.** It is assumed that the functions $\varphi(x)$, $\psi(x)$ are continuous in the interval $[0,1]$ the function $p(t)$ in the interval $[0,T]$ the functions $f(x,t)$, $g(x,t)$ in the rectangle $D_T$,

$$\int_0^1 f(x,t)dx = 0,\ \int_0^1 g(x,t)dx = 0(0 \le t \le T),\ h_i(t) \in C^2[0,T],\ i = 1,2.$$

Assume that the coherence conditions

$$\int_0^1 \varphi(x)dx = 0,\ \int_0^1 \psi(x)dx = 0,$$

$$\varphi(x_i) = h_i(0) - \int_0^T p(t)h_i(t)dt,\ \psi(x_i) = h_i'(0)\ \ (i = 1,2)$$

are satisfied. Then the following statements are valid.

1. Each classic solution of the inverse boundary value problem (10) – (14) is the solution of the inverse boundary value problem (10) – (12), (15), (16) as well.
2. The solution of the interval boundary value problem (10)–(12), (15), (16) satisfying the condition

$$(\|p(t)\|_{C[0,T]} + 2T\|a(t)\|_{C[0,T]})T < 1$$

is the solution of the inverse boundary value problem (10)–(14).

Imposing the following conditions on the data of the auxiliary inverse problem, we prove a theorem on the existence and unigueness of the solution.

1.7. $\alpha > 0$, $\beta > 0$, $p(t) \in C[0,T]$.

1.8. $\varphi(x) \in C^4[0,1], \varphi^{(5)}(x) \in L_2(0,1)$,

$\varphi(0) = \varphi(1), \varphi'(0) = \varphi'(1), \varphi''(0) = \varphi''(1), \varphi'''(0) = \varphi'''(1), \varphi^{(4)}(0) = \varphi^{(4)}(1)$.

1.9. $\psi(x) \in C^3[0,1], \psi^{(4)}(x) \in L_2(0,1)$,

$\psi(0) = \psi(1), \psi'(0) = \psi'(1), \psi''(0) = \psi''(1), \psi'''(0) = \psi'''(1)$.

1.10. $f(x,t), f_x(x,t) \in C(D_T), f_{xx}(x,t) \in L_2(D_T)$,

$f(0,t) = f(1,t), \ f_x(0,t) = f_x(1,t) \ (0 \le t \le T)$.

1.11. $g(x,t), g_x(x,t) \in C(D_T), g_{xx}(x,t) \in L_2(D_T)$,

$g(0,t) = g(1,t), g_x(0,t) = g_x(1,t) (0 \le t \le T)$.

1.12. $h_i(t) \in C^2[0,T] (i = 1,2)$,

$h(t) \equiv h_1(t) g(1,t) - h_2(t) g(0,t) \ne 0 \ (0 \le t \le T)$.

**Theorem 1.2.2**. Assume that the conditions 1.7-1.12 and the inequality

$$(B(T)(A(T)+2) + C(T) + D(T))(A(T)+2) < 1$$

is satisfied. Then the inverse problem (10) – (12), (15), (16) has a unigue solution in the sphere $K = K_R(\|z\|_{E_T^{5(2)}}) \le A(T)+2)$ taken from the space $E_T^{5(2)}$. Here the space $E_T^{5(2)}$ the expressions of $A(T)$, $B(T)$, $C(T)$ and $D(T)$ were determined in the section 1.2.

Using the equivalence theorem and theorem 1.2.1, we prove the following theorem.

**Theorem 1.2.3.** Assume that all the conditions of theorem 1.2.1 are satisfied. Furthermore, assume that the coherence conditions

$$\int_0^1 f(x,t)dx = 0, \ \int_0^1 g(x,t)dx = 0 (0 \le t \le T)$$

and

$$\int_0^1 \varphi(x)dx = 0, \quad \int_0^1 \psi(x)dx = 0,$$

$$\varphi(x_i) = h_i(0) - \int_0^T p(t)h_i(t)dt, \ \psi(x_i) = h_i'(0) \ \ (i = 1,2)$$

are satisfied. If the inequality $(\|p(t)\|_{C[0,T]} + 2T(A(T)+2))T < 1$ is satisfied, then the inverse boundary value problem (10)-(14) has a unigue solution in the sphere $K = K_R(\|z\|_{E_T^{5(2)}}) \le A(T)+2)$ taken from the space $E_T^{5(2)}$.

Unlike chapter I, in chapter II we study the existence and unigueness of the classic solution of nonlocal inverse boundary value problem for a Benny-Luke equation with not self-adjoint boundary condition.

In section 1 of chapter II it is assumed that,

$$D_T = \{(x,t):\ 0 \le x \le 1, \ \ 0 \le t \le T\}.$$

$f(x,t)$, $g(x,t)$, $p(t)$, $\varphi(x)$, $\psi(x)$, $h_i(t)(i = 1,2)$ are the given functions such that $x \in [0,1],\ t \in [0,T]$.

We consider the following boundary value problem[5]: it is reguired to find such a triple $u(x,t), a(t), b(t)$ consisting of the function $\{u(x,t),\ a(t), b(t)\}$ that this triple satisfies the linearized Benny-Luke equation[6]

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[5] Мегралиев, Я,Т., Велиева, Б.К. Обратная краевая задача для линеаризованное уравнения Бенни-Люка с нелокальными условиями // Вестник Удмуртского Университета. Математика, механика, компьютерные науки, -2019. т.29. вып.2. –с.166-182с

[6] Benney D.J., Luke J.C. On the interactions of permanent waves of finite amplitude // Journal of Mathematical Physics. 1964. v.43. p. 309–313. https://doi.org/10.1002/sapm1964431309

$$u_{tt}(x,t)-u_{xx}(x,t)+\alpha u_{xxxx}(x,t)-\beta u_{xxtt}(x,t)=$$
$$=a(t)u(x,t)+b(t)g(x,t)+f(x,t),\quad (x,t)\in D_T \qquad (17)$$

nonlocal conditions

$$u(x,0)=\int_0^T p(t)u(x,t)dt+\varphi(x)\,,\ u_t(x,0)+\delta u_t(x,T)=\psi(x)\ \ (0\le x\le 1) \qquad (18)$$

with respect to time, not self-adjoint boundary conditions

$$u(0,t)=u(1,t)\ ,u_x(0,t)=0,\ u_{xx}(0,t)=u_{xx}(1,t)\ ,u_{xxx}(0,t)=0\,(0\le t\le T) \qquad (19)$$

and additional conditions

$$\int_0^1 u(x,t)dx=h_1(t)\quad (0\le t\le T), \qquad (20)$$

$$u_x\left(\frac{1}{2},t\right)=h_2(t)\quad (0\le t\le T) \qquad (21)$$

here $\alpha>0$, $\beta>0$, $\delta\ge 0$ are the fixed real numbers.

**Definition 2.1.1.** If the triple $u(x,t)\in\tilde{C}^{4,2}(D_T)$ $a(t)\in C[0,T]$, $b(t)\in C[0,T]$ consisting of the functions $\{u(x,t),\ a(t),b(t)\}$ satisfies the equation (17), the conditions (18)-(21) in the usual sense, this triple is called the classic solution of the problem (17)-(21).

Now, along with inverse boundary value problem (17)-(21) we consider the following auxiliary problem: it is reguired to find such a triple $u(x,t)\in\tilde{C}^{5,2}(D_T)$, $a(t)\in C[0,T]$, $b(t)\in C[0,T]$ consisting of the functions $\{u(x,t),a(t),b(t)\}$ that this triple along with conditions (17)-(19) satisfy the conditions

$$h_1''(t)-u_x(1,t)+\alpha u_{xxx}(1,t)-\beta u_{ttx}(1,t)=$$
$$=a(t)h_1(t)+b(t)\int_0^1 g(x,t)dx+\int_0^1 f(x,t)dx\quad (0\le t\le T) \qquad (22)$$

$$h_2''(t)-u_{xxx}\left(\frac{1}{2},t\right)+\alpha u_{xxxxx}\left(\frac{1}{2},t\right)+\beta u_{xxxtt}\left(\frac{1}{2},t\right)=$$
$$=a(t)h_2(t)+b(t)g_x\left(\frac{1}{2},t\right)+f_x\left(\frac{1}{2},t\right)\ \ (0\le t\le T) \qquad (23)$$

here

$$\tilde{C}^{5,2}(D_T)=\left\{u(x,t):u(x,t)\in\tilde{C}^{4,2}(D_T),u_{ttxxx}(x,t),u_{xxxxx}(x,t)\in C(D_T)\right\}$$

Between the stated inverse boundary value problem (17)-(21) and the auxiliary inverse boundary value problem (17)-(19), (22), (23) we prove the following theorem.

**Theorem 2.1.1.** Assume that the following coherence conditions are satisfied $\varphi(x),\ \psi(x)\in C^1[0,1],\ p(t)\in C[0,T],$ $h_i(t)\in C^2[0,T](i=1,2),\ g(x,t),\ g_x(x,t)\in C(D_T),\ h(t)\equiv h_1(t)g_x\left(\frac{1}{2},t\right)$

$$-h_2(t)\int_0^1 g(x,t)dx\neq 0\quad (0\le t\le T),\qquad f(x,t),\ f_x(x,t)\in C(D_T)\ \text{and}$$

$$\int_0^1\varphi(x)dx=h_1(0)-\int_0^T p(t)h_1(t)dt,\ \int_0^1\psi(x)dx=h_1'(0)+\delta\, h_1'(T),$$

$$\varphi'\left(\frac{1}{2}\right)=h_2(0)-\int_0^T p(t)h_2(t)dt,\ \psi'\left(\frac{1}{2}\right)=h_2'(0)+\delta\, h_2'(T)\qquad (24)$$

then the following equations are valid.

**A.** The classic solution $\{u(x,t),a(t),b(t)\}$ of the inverse boundary value problem (17)-(21) satisfying the condition $u(x,t)\in\tilde{C}^{5,2}(D_T)$ is the solution of the auxiliary problem (17)-(19), (22), (23) as well;

**B.** The solution of the auxiliary inverse boundary value problem (17)-(19), (22), (23) satisfying the condition $\left(\|p(t)\|_{C[0,T]}+\frac{(2\delta+1)T}{1+\delta}\|a(t)\|_{C[0,T]}\right)T<1$ is the classic solution of the inverse boundary value problem (17)-(21) as well.

In section 2 of chapter II it is assumed that the data of the auxiliary problem satisfy the following conditions:

2.1. $\alpha>0,\ \beta>0,\ 0\le\delta<1,\ p(t)\in C[0,T].$

2.2. $\varphi(x)\in C^5[0,1],\varphi^{(6)}(x)\in L_2(0,1),\ \varphi'(0)=\varphi'''(0)=\varphi^{(5)}(0)=0,$

$\varphi(0)=\varphi(1), \varphi''(0)=\varphi''(1), \varphi^{(4)}(0)=\varphi^{(4)}(1)$.

2.3. $\psi(x)\in C^4[0,1], \psi^{(5)}(x)\in L_2(0,1),\ \psi'(0)=\psi'''(0)=0$,

$\psi(0)=\psi(1), \psi''(0)=\psi''(1)$.

2.4. $f(x,t),\ f_x(x,t),\ f_{xx}(x,t)\in C(D_T),\ f_{xxx}(x,t)\in L_2(D_T)$,

$f_x(0,t)=0,\ \ f(0,t)=f(1,t),\ \ f_{xx}(0,t)=f_{xx}(1,t)\ \ (0\le t\le T)$.

2.5. $g(x,t), g_x(x,t), g_{xx}(x,t)\in C(D_T), g_{xxx}(x,t)\in L_2(D_T)$,

$g_x(0,t)=0,\ \ g(0,t)=g(1,t),\ \ g_{xx}(0,t)=g_{xx}(1,t)\ \ (0\le t\le T)$.

2.6. $h_i(t)\in C^2[0,T] (i=1,2)$,

$$h(t)\equiv h_1(t)g_x\left(\frac{1}{2},t\right)-h_2(t)\int_0^1 g(x,t)dx\neq 0\ (0\le t\le T).$$

The following theorem is proved.

**Theorem 2.2.1**. If the conditions 2.1 – 2.6 and the inequality

$$(B(T)(A(T)+2)+C(T)+D(T))(A(T)+2)<1$$

are satisfied, the problem (17)-(19), (22), (23) has a unigue solution in the sphere $K=K_R(\|z\|_{E_T^6}\le R\le A(T)+2)$ taken from the space $E_T^6$. Here, the space $E_T^6$ the expressions of $A(T)$, $B(T)$, $C(T)$ and $D(T)$ - were determined in section 2.2.

**Theorem 2.2.2**. Let all the conditions of theorem 2.1.2 and coherence conditions (24) be satisfied. If the condition

$$\left(\|p(t)\|_{C[0,T]}+\frac{(2\delta+1)T(A(T)+2)}{1+\delta}\right)T<1$$

is satisfied, the problem (17)-(21) has a unigue solution in the sphere $K=K_R$ taken from the space $E_T^6$.

In chapter III consisting of two sections, unlike chapters I and II, we study the existence and unigueness of the classic solution of a nonlocal inverse boundary value problem for a Benny-Luke equation reduced to the self-adjoint boundary condition.

In section 1 of chapter III it is assumed that we are given a rectangle $D_T = \{(x,t): 0 \le x \le 1,\ 0 \le t \le T\}$ and consider the following inverse boundary value problem. It is reguired to find such triple $\{u(x,t), a(t), b(t)\}$ that this triple satisfies the equation

$$u_{tt}(x,t) - u_{xx}(x,t) + \alpha u_{xxxx}(x,t) - \beta u_{xxtt}(x,t) =$$
$$= a(t)u(x,t) + b(t)g(x,t) + f(x,t) \quad (x,t) \in D_T \tag{25}$$

nonlocal conditions

$$u(x,0) = \varphi(x) + \int_0^T p_1(t)u(x,t)dt,$$
$$u_t(x,0) = \psi(x) + \int_0^T p_2(t)u(x,t)dt \quad (0 \le x \le 1) \tag{26}$$

with respect to time, the boundary conditions

$$u(1,t) = 0,\ u_x(0,t) = u_x(1,t), \quad u_{xx}(1,t) = 0 \quad (0 \le t \le T) \tag{27}$$

the integral conditions

$$\int_0^1 u(x,t)dx = 0 \quad (0 \le t \le T) \tag{28}$$

and additional conditions

$$u(0,t) = h_1(t) \qquad (0 \le t \le T), \tag{29}$$
$$u\left(\frac{1}{2}, t\right) = h_2(t) \qquad (0 \le t \le T) \tag{30}$$

here $\alpha > 0,\ \beta > 0$ are the given numbers $g(x,t),\ f(x,t),\ \varphi(x), \psi(x),\ p_1(t),\ p_2(t),\ h_1(t), h_2(t)$ are the given functions.

**Definition 3.1.1.** The triple $\{u(x,t), a(t), b(t)\}$ satisfying the equation (25) in the domain $D_T$ nonlocal conditions (26) on the interval $[0,1]$ the conditions (27)-(30) on the interval $[0,T]$ in the usual sense, is said to be a classic solution of the inverse boundary value problem (25)-(30).

Along with inverse boundary value problem (25)–(30) we consider the following auxiliary inverse boundary value problem. It is reguired to find a triple

$$u(x,t)\in \tilde{C}^{4,2}(D_T),\ a(t)\in C[0,T],\ b(t)\in C[0,T]$$

consisting of the functions $\{u(x,t),a(t),b(t)\}$ that along with conditions (25) – (27) this triple satisfies the conditions

$$u_{xxx}(0,t)=u_{xxx}(1,t)\quad (0\le t\le T), \tag{31}$$

$$h_1''(t)-u_{xx}(0,t)+\alpha u_{xxxx}(0,t)-\beta u_{xxtt}(0,t)=$$
$$=a(t)h_1(t)+b(t)g(0,t)+f(0,t)\ (0\le t=T), \tag{32}$$

$$h_2''(t)-u_{xx}\left(\frac{1}{2},t\right)+\alpha u_{xxxx}\left(\frac{1}{2},t\right)-\beta u_{xxtt}\left(\frac{1}{2},t\right)=$$
$$=a(t)h_2(t)+b(t)g\left(\frac{1}{2},t\right)+f\left(\frac{1}{2},t\right)\quad (0\le t=T) \tag{33}$$

The following eguivalence theorem is valid.

**Theorem 3.1.1**. Assume that $\varphi(x),\ \psi(x)\in C[0,1]$,
$p_1(t),\ p_2(t)\in C[0,T],\ f(x,t),\ g(x,t)\in C(D_T)$,
$h_i(t)\in C^2[0,T]\,(i=1,2)$,

$$h(t)\equiv h_1(t)\,g(\frac{1}{2},t)-h_2(t)g(0,t)\ne 0\ (0\le t\le T),$$

$$\int_0^1 f(x,t)dx=0,\ \int_0^1 g(x,t)dx=0(0\le t\le T)\quad \text{və}$$

$$\int_0^1\varphi(x)dx=0,\quad \int_0^1\psi(x)dx=0, \tag{34}$$

$$\varphi(0)=h_1(0)-\int_0^T p_1(t)h_1(t)dt,\ \psi(0)=h_1'(0)-\int_0^T p_2(t)h_1(t)dt,$$

$$\varphi\left(\frac{1}{2}\right)=h_2(0)-\int_0^T p_2(t)h_2(t)dt,\ \psi\left(\frac{1}{2}\right)=h_2'(0)-\int_0^T p_2(t)h_2(t)dt \tag{35}$$

Then the following statements are valid.

- Each classic solution $\{u(x,t), a(t), b(t)\}$ of the inverse boundary value problem (25) –(30) is the solution of the auxiliary inverse boundary value problem (25)-(27), (31)- (33) as well.
- The solution $\{u(x,t), a(t), b(t)\}$ of the auxiliary inverse boundary value problem (25)-(27), satisfying the condition

$$\left( \|p_1(t)\|_{C[0,T]} + T\|p_2(t)\|_{C[0,T]} + \frac{1}{2}\|a(t)\|_{C[0,T]} \right) < 1$$

(31)-(33) is the classic solution of the inverse boundary value problem (25) – (30) as well.

In section 2 of chapter III we prove the existence and unigueness of the solution of the auxiliary problem. Then, using the eguivalence theorem, we show the existence and unigueness of the classic solution of the main problem. It is assumed that the data of the auxiliary inverse boundary value problems satisfy the following conditions

3.1. $\varphi(x) \in C^4[0,1], \varphi^{(5)}(x) \in L_2(0,1)$,

$\varphi(1) = 0, \varphi'(0) = \varphi'(1),\ \varphi''(1) = 0, \varphi'''(0) = \varphi'''(1),\ \varphi^{(4)}(1) = 0.$

3.2. $\psi(x) \in C^3[0,1], \psi^{(4)}(x) \in L_2(0,1)$,

$\psi(1) = 0,\ \psi'(0) = \psi'(1),\ \psi''(1) = 0, \psi'''(0) = \psi'''(1)$

3. 4. $f(x,t),\ f_x(x,t) \in C(D_T),\ f_{xx}(x,t) \in L_2(D_T)$,

$f(1,t) = 0,\ \ f_x(0,t) = f_x(1,t)\ \ (0 \le t \le T).$

3. 5. $g(x,t), g_x(x,t) \in C(D_T), g_{xx}(x,t) \in L_2(D_T)$,

$g(1,t) = 0,\ \ g_x(0,t) = g_x(1,t)\ \ (0 \le t \le T).$

3.6.. $\alpha > 0$, $\beta > 0, p_i(t) \in C[0,T],\ h_i(t) \in C^2[0,T] (i = 1,2)$,

$$h(t) \equiv h_1(t) g\left(\frac{1}{2}, t\right) - h_2(t) g(0,t) \ne 0\ (0 \le t \le T).$$

The following eguivalence theorem is proved.

**Theorem 3.2.1.** Let conditions 3.1-3.6 be satisfied. If the inequality

$$(A(T)+2)(B(T)A(T)+2)+C(T)+D(T))<1$$

is satisfied the inverse boundary value problem (25)-(27), (31)- (33) has a unigue solution in the sphere $K=K_R(\|z\|_{E_T^{5(3)}}\leq A(T)+2)$ taken from the space $E_T^{5(3)}$. Here the space $E_T^{5(3)}$ the expressions of $A(T)$, $B(T)$, $C(T)$ and $D(T)$ were determined in section 2.3.

**Theorem 3.2.2.** Let all the conditions of theorem 3.2.1. the $\int_0^1 f(x,t)dx=0,$ $\int_0^1 g(x,t)dx=0(0\leq t\leq T)$ and coherence conditions (34), (35) be satisfied. İf the inequality

$$\left(\|p_1(t)\|_{C[0,T]}+T\|p_2(t)\|_{C[0,T]}+\frac{1}{2}(A(T)+2)\right)T<1$$

is satisfied, then the problem (25) –(30) has a unigue solution in the sphere $K=K_R(\|z\|_{E_T^{5(3)}}\leq R)$ taken from the space $E_T^{5(3)}$.

# CONCLUSIONS

The dissertation work was devoted to the study of some inverse boundary value problems for some nonlocal boundary condition fourth order differential equations and the following results were obtained:

- - Theorems on the existence and uniqueness of the solution of a self-adjoint boundary condition, nonlocal, nonlinear inverse boundary value problems for a linearized a fourth order linearized partial Benny-Luke equation were proved;
- Theorems on the existence and uniqueness of the classic solution of nonlocal, nonlinear inverse boundary value problems reduced to self-adjoint boundary condition for a fourth order linearized partial Benny-Luke equation were proved;
- Theorems on the existence and uniqueness of the solution of a not self-adjoint boundary condition, nonlocal, nonlinear inverse boundary value problems for fourth order linearized partial Benny-Luke equation were proved;
- Theorems on the existence and uniqueness of the classic solution of nonlocal, nonlinear inverse boundary value problems reduced to not self-adjoint boundary condition for a linearized, fourth order partial Benny-Luke equation were proved.

**The main results of the dissertation work were published in the following works:**


1. Мегралиев, Я.Т., Велиева, Б.К. Об одной нелокальной обратной краевой задаче для уравнения Бенни-Люка // -Воронеж: Актуальные проблемы прикладной математики, информатики и механики. –17-19 декабря, -2018. -с. 99-104.
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