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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

INVERSE PROBLEM FOR DIRAC OPERATOR WITH SPECTRAL PARAMETER IN BOUNDARY CONDITION

Speciality: 1211.01 – Differential equations

Field of science: Mathematics

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GENERAL CHARACTERISTICS OF THE WORK

Relevance of the topic and degree of processing. Starting from results to discover causes has interest researchers for centuries. In science, identifying the reasons of what happened through observation is called an inverse problem. Inverse problems are one of the most important fields in science (including nature study) and techniques. They provide information about the parameters we can't see directly.

Spectral theory of differential operators is a rapidly growing branch of modern mathematics. The main problems of this theory include spectrum studies, expansion of the given function in the eigenfunctions of differential operator, direct and inverse problems of spectral analysis, etc. Recovery of differential operators based on some spectral data (spectra, sequence of normalizing numbers, spectral function, Weyl function, scattering data, etc) is called an inverse problem of spectral analysis. The theory of inverse spectral problems has wide applications in mechanics, physics, geophysics, electronics, meteorology and various branches of natural science and techniques. With the advance of quantum mechanics, recent years saw increased interest in inverse spectral problems for various differential operators. That's why the various versions of inverse problems became one of the actively developing fields of modern mathematics.

Dirac equation is one of the most extensively studied differential equations. It is a relativistic wave equation introduced by the English physicist Paul Dirac in 1928. The canonical form of one-dimensional stationary Dirac system was given by M.G. Gasymov and B.M. Levitan in 1966.

Inverse problems with various propositions for the canonical Dirac system were considered in the scientific works of M.G. Gasymov, B.M. Levitan, H.M. Huseynov, M. Klaus, Kh.R. Mamedov, R.Kh. Amirov, V.A. Yurko, D.B. Hinton, A.K. Jordan, J.K. Shaw, M. Kiss, B.A. Watson, S. Albeverio, R. Hryniv, Y. Mykytyuk, M.M. Malamud, I.M. Nabiev, B. Keskin, A.S. Ozkan and other researchers. Two-spectra inverse problem for Dirac operator with separated boundary conditions has been completely solved by M.G. Gasymov and T.T. Jabiyev.

Direct and inverse spectral problems for the systems of differential equations with analytic coefficients, summable potential, discontinuous coefficients, separable boundary conditions with spectral parameter, discontinuity condition and singularity inside the considered interval have been considered (in interval, in semi-axis, in whole axis) byM.G. Gasymov, H.M. Huseynov, F. Gesztesy, N.B. Kerimov, M. Horvath, Kh.R. Mamedov, V.M. Kurbanov, K. Li, M. Zhang, R.Kh. Amirov, Y. Mykytyuk, Z.S. Aliyev, A.A. Shkalikov, E.S. Panakhov, C-Fu Yang, N.P. Bondarenko and their students.To solve the recovery problems, the authors above used two or more spectra, spectral function, spectrum and normalizing numbers, Weyl matrix function and scattering data. Recovery problems with nonseparated, periodic, antiperiodic, quasiperiodic and generalized periodic boundary conditions have been considered by T.V. Misyura, I.M. Nabiev, Y.L. Korotyaev and A.S. Makin.

Note that no author except for T.Sh. Abdullayev and I.M. Nabiev has ever considered an inverse problem for Dirac operator with nonseparated boundary condition involving a spectral parameter. That’s why it is so important to develop further this field.

In this dissertation, the inverse problem for Dirac operator with nonseparated boundary condition involving a linear function of spectral parameter is fully solved. The most difficult and the most important thing in this kind of inverse problems is to find the appropriate necessary and sufficient conditions. As spectral data, we use two spectra, some sign sequence and one complex number. We provide full characteristics of these spectral data.

The foregoing confirms the relevance of the topic considered in this work.

The object and the subject of research. The object of this research is a Dirac operator with a nonseparated boundary condition involving a linear function of spectral parameter, and the subjects are the solution of inverse problem, recovery algorithms, necessary and sufficient conditions.

The goal and the objectives of research. The main goal of this research is to solve the inverse spectral problem for Dirac operator with one of nonseparated boundary conditions involving a linear function of spectral parameter. The main objectives are to explore the spectral properties of the considered boundary value problem, to state the uniqueness theorems for different inverse problems, to compile the recovery algorithms for the considered Dirac system, and to obtain the necessary and sufficient conditions for the solution of the considered inverse spectral problem.

Research methods. We use the methods of mathematical analysis, differential equations, mathematical physics, theories of real and complex functions, and functional analysis.

Main points to be defended in this dissertation. The main points to be defended in this dissertation (in case one of nonseparated boundary conditions involves a linear function of spectral parameter) are:

- to explore the main properties of the spectral data for Dirac operator;
- to obtain an asymptotic formula for the eigenvalues of boundary value problem;
- to show the rules of multiplicity and intermittency of the eigenvalues of boundary value problem;
- to prove uniqueness theorems by putting inverse problems in different forms for the boundary problem;
- to compile recovery algorithms for Dirac operator;
- to find necessary and sufficient conditions for the solution of inverse spectral problem.

Scientific novelty of research. The following new scientific results are obtained in this work (in case one of nonseparated boundary conditions involves a linear function of spectral parameter):

- main properties of spectral data for Dirac system are explored;

- asymptotics of the eigenvalues of boundary value problem is found;
- necessary and sufficient conditions for the multiplicity of eigenvalues (and zeros of characteristic function) are found;
- rules of intermittency of eigenvalues are shown;
- representation for the characteristic function of boundary value problem in the form of infinite product is obtained;
- uniqueness theorems for the solutions of differently stated inverse problems are proved;
- recovery algorithms are built for boundary value problems;
- characteristics of spectral data are given.

Theoretical and practical significance of research. Results obtained in this work are theoretical, and they can be successfully used in the problems of spectral theory of operators, as well as in physics and techniques. Also, these results can be useful in the integration of some nonlinear evolution equations of mathematical physics, while solving various problems of quantum mechanics and theory of oscillations.

Approbation and applications. The main results of the dissertation work were presented at the republican scientific conference of doctoral students and young researchers on "Information, science, technology and university perspectives" held at Lankaran State University (Lankaran, Azerbaijan, 2020), "6th International IFS and Contemporary Mathematics Conference" (Mersin, Turkey, 2019), 2nd International Scientific and Practical Internet Conference "Integration of Education, Science and Business in Modern Environment: Winter Debates" (Dnipro, Ukraine, 2021), "4th International E-Conference on Mathematical Advances and Applications" (Yıldız Technical University, Istanbul, Turkey, 2021), "1st International Symposium on Recent Advances in Fundamental and Applied Sciences" **(**Atatürk University, Erzurum, Turkey, 2021), XI International Youth Scientific-Practical Conference "Mathematical modeling of processes and system" (Bashkir State University, Ufa Respublic of Bashkortostan, Russia, 2021), XXXVII International Conference Problems of Decision Making Under

Uncertainties (Sheki-Lankaran, Republic of Azerbaijan, Kyiv, Ukraine, 2022), XII International Youth Scientific-Practical Conference “Mathematical modeling of processes and system” (Bashkir State University, Ufa Respublic of Bashkortostan, Russia, 2022), “International Conference on Spectral Theory of Operators and Related Problems” (Ufa, Republic of Bashkortostan, Russia, 2023).

The results obtained in this work can be used in various problems of mathematical physics, relativistic quantum theory and spectral theory of differential operators.

Author’s personal contribution. All the results obtained in this work belong to the author.

Author’s publications. Author’s publications include 5 articles in scientific journals recommended by the Presidential Higher Certifying Commission of Azerbaijan (one of them being cited by the Web of Science: Science Citation Index –Expanded (SCIE), the another one – by the Web of Science: Emerging Sources Citation Index (ESCI)), and 9 conference theses (8 in international and 1 in national conferences), making a total of 14 publications. The full list of author’s publications is available at the end of the abstract.

The name of the organization where the dissertation work was executed. The work was executed at the department of “Mathematics and Computer Science” of Lankaran State University, Lankaran, Azerbaijan.

Structure and volume of the dissertation work (in signs, indicating the volume of each structural unit separately). Total volume of the dissertation work – 215.485 characters, characters (title page – 333 characters, contents – 1057 characters, introduction – 33.340 characters, chapter I – 110.000 characters, chapter II – 32.000 characters, chapter III – 38.000 characters, and conclusion – 755 characters). The reference list consists of 102 entries.

THE CONTENT OF THE DISSERTATION

This dissertation consists of introduction, three chapters, conclusion and a list of references. The first chapter has six paragraphs, and the second and third chapters have three paragraphs each. Introduction emphasizes the relevance and informs about the background of the work. The object, the subject, the purpose, the objectives of the work, the methods used, main points, scientific novelty, theoretical and practical importance, approbation, applications and the obtained results are also highlighted in Introduction. Then, brief information about the results obtained in the dissertation was given.

The canonical form of one-dimensional stationary Dirac system has the following form:

$$BY'(x)+Q(x)Y(x)=\lambda Y(x), \tag{1}$$

where $B=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $Q(x)=\begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}$, $Y(x)=\begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$, λ is a spectral parameter, $p(x), q(x)\in W_2^1[0,\pi]$ – real functions, and $W_2^1[0,\pi]$ is a Sobolev space of absolutely continuous functions whose derivatives are square summable in the interval $[0,\pi]$ (i.e. belong to $L_2[0,\pi]$). Consider in $[0,\pi]$ the boundary value problem generated by the equation (1) and the general nonseparated boundary conditions

$$A_0Y(0)+A_1Y(\pi)=0, \tag{2}$$

where

$$A_0=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},\ A_1=\begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix},$$

and a_{ik} $\left(i=1,2;\ k=\overline{1,4}\right)$ are arbitrary complex numbers.

Definition 1. If the problem (1), (2) has a non-trivial solution $Y_0(x)$ for $\lambda=\lambda_0$, then the number λ_0 is called an eigenvalue of this problem, and the vector function $Y_0(x)$ is called the vector eigenfunction of this problem corresponding to the eigenvalue λ_0. The number of linearly independent solutions of the problem (1), (2) corresponding to the given eigenvalue λ_0 is called the multiplicity of the eigenvalue λ_0.

If

$$A_0=\begin{pmatrix}\alpha\lambda+\beta & 1\\ -\overline{\omega} & 0\end{pmatrix},\ A_1=\begin{pmatrix}\omega & 0\\ \gamma & 1\end{pmatrix},$$

then the boundary conditions (2) become

$$\begin{aligned}&y_2(0)+(\alpha\lambda+\beta)y_1(0)+\omega y_1(\pi)=0,\\ &y_2(\pi)+\gamma y_1(\pi)-\overline{\omega}y_1(0)=0,\end{aligned}\tag{3}$$

where α,β,γ are real numbers, and ω is a complex number. We consider the case $\alpha\omega\neq 0$, i.e. the case where the boundary conditions are nonseparated and one of them includes a linear function of spectral parameter. Denote the boundary value problem (1), (3) by $D(\omega,\alpha,\beta,\gamma)$.

Let $C(x,\lambda)=\begin{pmatrix}c_1(x,\lambda)\\ c_2(x,\lambda)\end{pmatrix}$ and $S(x,\lambda)=\begin{pmatrix}s_1(x,\lambda)\\ s_2(x,\lambda)\end{pmatrix}$ be the solutions of the equation (1) satisfying the initial conditions

$$C(0,\lambda)=\begin{pmatrix}1\\ 0\end{pmatrix},\ S(0,\lambda)=\begin{pmatrix}0\\ 1\end{pmatrix}.$$

The Wronskian of these solutions is identically equal 1, i.e.

$$c_1(x,\lambda)s_2(x,\lambda)-c_2(x,\lambda)s_1(x,\lambda)\equiv 1.$$

The general solution of the equation (1) has the form

$$Y(x,\lambda)=M_1C(x,\lambda)+M_2S(x,\lambda),$$

where M_1 and M_2 are arbitrary constants. Consider this solution in the boundary conditions (3), then the characteristic function of the boundary value problem $D(\omega,\alpha,\beta,\gamma)$ has the following form:

$$d(\lambda)=\det(A_0+A_1e(\pi,\lambda)),\tag{4}$$

where $e(\pi,\lambda)=\begin{pmatrix}c_1(\pi,\lambda) & s_1(\pi,\lambda)\\ c_2(\pi,\lambda) & s_2(\pi,\lambda)\end{pmatrix}$. The zeros of this function are the eigenvalues of problem $D(\omega,\alpha,\beta,\gamma)$. If we consider the expressions of A_0 and A_1 in relation (4), then the characteristic function becomes

$$\begin{aligned}d(\lambda)=2\operatorname{Re}\omega-c_2(\pi,\lambda)-\gamma c_1(\pi,\lambda)+\\ +|\omega|^2s_1(\pi,\lambda)+(\alpha\lambda+\beta)[s_2(\pi,\lambda)+\gamma s_1(\pi,\lambda)].\end{aligned}\tag{5}$$

In chapter 1 of this dissertation, the representations for the components of fundamental solutions of the Dirac system are obtained and the spectral properties of the boundary value problem $D(\omega,\alpha,\beta,\gamma)$ are established.

In Paragraph 1 of Chapter 1, the represenations for the components $c_1(x,\lambda)$, $c_2(x,\lambda)$, $s_1(x,\lambda)$, $s_2(x,\lambda)$ of the fundamental solutions of the Dirac system are obtained.

Theorem 1. If $p(x),\ q(x)\in W_2^1[0,x]$, then the following representations are true:

$$c_1(x,\lambda)=\cos\lambda x+\frac{b+q(x)+d(x)}{2\lambda}\sin\lambda x-\frac{a-p(x)}{2\lambda}\cos\lambda x+\frac{1}{\lambda}\int_0^x R_1(x,t)e^{2i\lambda t}dt,$$

$$c_2(x,\lambda)=\sin\lambda x-\frac{b+d(x)-q(x)}{2\lambda}\cos\lambda x-\frac{a+p(x)}{2\lambda}\sin\lambda x+\frac{1}{\lambda}\int_0^x R_2(x,t)e^{2i\lambda t}dt,$$

$$s_1(x,\lambda)=-\sin\lambda x+\frac{d(x)+q(x)-b}{2\lambda}\cos\lambda x-\frac{a+p(x)}{2\lambda}\sin\lambda x+\frac{1}{\lambda}\int_0^x R_3(x,t)e^{2i\lambda t}dt,$$

$$s_2(x,\lambda)=\cos\lambda x-\frac{b+q(x)-d(x)}{2\lambda}\sin\lambda x+\frac{a-p(x)}{2\lambda}\cos\lambda x+\frac{1}{\lambda}\int_0^x R_4(x,t)e^{2i\lambda t}dt,$$

where $a=p(0),\ b=q(0),\ d(x)=\int_0^x\left[p^2(t)+q^2(t)\right]dt$, and $R_q(x,t)\ \left(q=\overline{1,4}\right)$ is a square summable function with respect to t for every $x\in[0,\pi]$.

For the special case of $x=\pi$, this theorem has a corollary below:

Corollary. The following representations are true for the functions $c_1(\pi,\lambda)$, $c_2(\pi,\lambda)$, $s_1(\pi,\lambda)$ and $s_2(\pi,\lambda)$:

$$c_1(\pi,\lambda)=\cos\pi\lambda+B_1\frac{\sin\pi\lambda}{\lambda}-C_1\frac{\cos\pi\lambda}{\lambda}+\frac{\psi_1(\lambda)}{\lambda},$$

$$c_2(\pi,\lambda)=\sin\pi\lambda+B_2\frac{\cos\pi\lambda}{\lambda}+C_2\frac{\sin\pi\lambda}{\lambda}+\frac{\psi_2(\lambda)}{\lambda},$$

$$s_1(\pi,\lambda)=-\sin\pi\lambda+B_3\frac{\cos\pi\lambda}{\lambda}+C_2\frac{\sin\pi\lambda}{\lambda}+\frac{\psi_3(\lambda)}{\lambda},$$

$$s_2(\pi,\lambda)=\cos\pi\lambda+B_4\frac{\sin\pi\lambda}{\lambda}+C_1\frac{\cos\pi\lambda}{\lambda}+\frac{\psi_4(\lambda)}{\lambda},$$

where

$$B_1 = A - Q_1,\ B_2 = -A + Q_2,\ B_3 = A + Q_2,\ B_4 = A + Q_1,$$

$$C_1 = \frac{p(0) - p(\pi)}{2},\ C_2 = -\frac{p(0) + p(\pi)}{2},$$

$$A = \frac{1}{2}\int_0^{\pi}\left[p^2(x) + q^2(x)\right]dx,\quad Q_1 = -\frac{q(0) + q(\pi)}{2},\quad Q_2 = \frac{q(\pi) - q(0)}{2},$$

$$\psi_p(\lambda) = \int_{-\pi}^{\pi}\tilde{\psi}_p(t)e^{i\lambda t}dt,\ \tilde{\psi}_p(t) \in L_2[-\pi, \pi],\ p = \overline{1,4}.$$

These representations play an important role in the solution of various direct and inverse spectral problems for Dirac operator.

In paragraph 2 of chapter 1, the simple spectral properties of the considered boundary value problem are established. The conditions are found which provide that the eigenvalues are non-zero real numbers. Also, the theorem on the associated functions of the eigenvalues is proved.

Theorem 2. When $\alpha < 0$ the eigenvalues of the problem $D(\omega, \alpha, \beta, \gamma)$ are real.

Note that throughout this work we assume $\alpha < 0$, i.e. we consider the case where the eigenvalues are real.

Definition 2. If the vector functions

$$Y_1(x) = \begin{pmatrix} y_{1,1}(x) \\ y_{2,1}(x) \end{pmatrix},\ Y_2(x) = \begin{pmatrix} y_{1,2}(x) \\ y_{2,2}(x) \end{pmatrix},\ .\ .\ .\ .\ ,\ Y_r(x) = \begin{pmatrix} y_{1,r}(x) \\ y_{2,r}(x) \end{pmatrix}$$

have an absolutely continuous derivative, satisfy the differential equation

$$B\ Y_j'(x) + Q(x)\ Y_j(x) - Y_{j-1}(x) = \lambda_0 Y_j(x)$$

and the boundary conditions

$$(\alpha\lambda + \beta)y_{1,j}(0) + y_{2,j}(0) + \omega y_{1,j}(\pi) + \alpha y_{1,j-1}(0) = 0,$$

$$-\overline{\omega}y_{1,j}(0) + \gamma y_{1,j}(\pi) + y_{2,j}(\pi) = 0,$$

$$j = 1,2,3,\ldots,r\ ,$$

then these functions are called the associated vector functions of the vector eigenfunction $Y_0(x)=\begin{pmatrix} y_{1,0}(x) \\ y_{2,0}(x) \end{pmatrix}$.

Theorem 3. If $\alpha<0$, then the vector eigenfunctions of the boundary value problem $D(\omega,\alpha,\beta,\gamma)$ have no associated vector functios.

Theorem 4. If

$$2\operatorname{Re}\left[\omega\overline{y_1(0)}y_1(\pi)\right]-\gamma|y_1(\pi)|^2+\beta|y_1(0)|^2+\int_0^\pi p(x)\left[|y_1(x)|^2+|y_2(x)|^2\right]dx\neq 0,$$

then the eigenvalues of the boundary value problem $D(\omega,\alpha,\beta,\gamma)$ are different from zero.

In paragraph 3 of chapter 1, the asymptotic formula for the sequence of eigenvalues of $D(\omega,\alpha,\beta,\gamma)$ is obtained.

By means of the representations stated in the corollary of Theorem 1, we obtain the equality

$$d(\lambda)=2\operatorname{Re}\omega+\alpha\lambda(\cos\pi\lambda-\gamma\sin\pi\lambda)+(\alpha\gamma B_3+\alpha C_1+\beta-\gamma)\cos\pi\lambda+$$
$$+\left(\alpha B_4+\alpha\gamma C_2-1-|\omega|^2-\beta\gamma\right)\sin\pi\lambda+l(\lambda)$$

for the characteristic function (5), where $l(\lambda)=\int_{-\pi}^{\pi}\tilde{l}(t)e^{i\lambda t}dt$, $\tilde{l}(t)\in L_2[-\pi,\pi]$. Using Rouche's theorem, we show that the roots γ_k $(k=\pm 0,\pm 1,\pm 2,...)$ of the equation $d(\lambda)=0$ have the asymptotics

$$\gamma_k=k+a+\varepsilon_k$$

as $k\to\pm\infty$. By means of Taylor expansions of the functions $\sin x$ and $\cos x$, we find the asymptotics of ε_k.

Theorem 5. The following asymptotic formula is true for the eigenvalues γ_k $(k=\pm 0,\pm 1,\pm 2,...)$ (as $|k|\to\infty$) of the boundary value problem $D(\omega,\alpha,\beta,\gamma)$:

$$\gamma_k=k+a+\frac{(-1)^{k+1}\tilde{A}-\tilde{B}}{\pi k}+\frac{\xi_k}{k},$$

where

$$\tilde{A} = -\frac{2\operatorname{Re}\omega}{\alpha b},$$

$$\tilde{B} = -A - \frac{q(\pi)(\gamma^2 - 1) - b^2 q(0) - 2\gamma p(\pi)}{2b^2} + \frac{b^2 + |\omega|^2}{\alpha b^2},$$

$$a = \frac{1}{\pi}\operatorname{arcctg}\gamma, \quad b = \sqrt{1+\gamma^2}, \quad \{\xi_k\} \in l_2 .$$

In paragraph 4 of chapter 1, the necessary and sufficient conditions for the eigenvalues (also for the zeros of the characteristic function of the boundary value problem) to have a multiplicity of 2 are found. Theorem about multiplicity of the eigenvalues of the problem $D(\omega,\alpha,\beta,\gamma)$ is also proved here.

Theorem 6. For the number λ_0 to be a multiple eigenvalue of the boundary value problem $D(\omega,\alpha,\beta,\gamma)$ (also a multiple zero of the characteristic function $d(\lambda)$), it is necessary and sufficient that ω is a non-zero real number and

$$\alpha\lambda_0 + \beta + \omega c_1(\pi,\lambda_0) = 0,$$
$$s_2(\pi,\lambda_0) + \gamma s_1(\pi,\lambda_0) = 0 .$$

Theorem 7. The multiplicity of the zero of the characteristic function $d(\lambda)$ cannot be greater than 2.

The conditions obtained above play a very important role in investigation of spectral structure, in determining the order of mutual arrangement of eigenvalues of operators, in recovery problems and in deriving sufficient conditions.

In paragraph 5 of chapter 1, the formula for the recovery of the characteristic function $d(\lambda)$ of $D(\omega,\alpha,\beta,\gamma)$ using spectrum is obtained in the form of infinite product.

Theorem 8. If ω is given, then the characteristic function $d(\lambda)$ of the boundary value problem $D(\omega,\alpha,\beta,\gamma)$ can be uniquely determined through the spectrum $\{\gamma_k\}$ $(k = \pm 0, \pm 1, \pm 2, \ldots)$ by the following formula:

$$d(\lambda) = -\pi\alpha b(\gamma_{-0} - \lambda)(\gamma_{+0} - \lambda)\prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\gamma_k - \lambda}{k}, \tag{6}$$

where

$$\alpha b = \frac{2\operatorname{Re}\omega}{\pi \lim\limits_{k\to\infty} k(\gamma_{2k} - \gamma_{2k+1} + 1)}.$$

The representation of characteristic function in the form of infinite product plays an important role in the solution of inverse problem. When solving the inverse problem, the first thing to do is to recover the characteristic function by means of spectral data. Then, using characteristic function and spectral data, the coefficients of the boundary value problem are found.

In paragraph 6 of chapter 1, when $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$, the rules for mutual arrangement of eigenvalues of two pairs of boundary value problems $D(\omega,\alpha_1,\beta,\gamma)$, $D(\omega,\alpha_2,\beta,\gamma)$ and $D(\omega,\alpha,\beta_1,\gamma)$, $D(\omega,\alpha,\beta_2,\gamma)$ is investigated.

Theorem 9. If $\operatorname{Im}\omega \neq 0$, then the eigenvalues $\gamma_k^{(1)}$ and $\gamma_k^{(2)}$ $(\gamma_k^{(j)} \neq 0)$ $(k = \pm 0, \pm 1, \pm 2, \ldots)$ of the boundary value problems $D(\omega,\alpha_1,\beta,\gamma)$ and $D(\omega,\alpha_2,\beta,\gamma)$ $(\alpha_1 < \alpha_2 < 0)$ are intermittency, i.e.

$$0 < \gamma_{+0}^{(1)} < \gamma_{+0}^{(2)} < \gamma_1^{(1)} < \gamma_1^{(2)} < \gamma_2^{(1)} < \gamma_2^{(2)} < \ldots,$$
$$0 > \gamma_{-0}^{(1)} > \gamma_{-0}^{(2)} > \gamma_{-1}^{(1)} > \gamma_{-1}^{(2)} > \gamma_{-2}^{(1)} > \gamma_{-2}^{(2)} > \ldots,$$

if $\operatorname{Im}\omega = 0$, then the following inequalities hold:

$$0 < \gamma_{+0}^{(1)} \le \gamma_{+0}^{(2)} \le \gamma_1^{(1)} \le \gamma_1^{(2)} \le \gamma_2^{(1)} \le \gamma_2^{(2)} \le \ldots,$$
$$0 > \gamma_{-0}^{(1)} \ge \gamma_{-0}^{(2)} \ge \gamma_{-1}^{(1)} \ge \gamma_{-1}^{(2)} \ge \gamma_{-2}^{(1)} \ge \gamma_{-2}^{(2)} \ge \ldots,$$

so that the a multiplicites eigenvalue of the problem $D(\omega,\alpha_1,\beta,\gamma)$ is a simple eigenvalue of the problem $D(\omega,\alpha_2,\beta,\gamma)$.

Theorem 10. If $\operatorname{Im}\omega \neq 0$, then the eigenvalues $\mu_k^{(1)}$, $\mu_k^{(2)}$ $(k = \pm 0, \pm 1, \pm 2, \ldots)$ of the boundary value problems $D(\omega,\alpha,\beta_1,\gamma)$ and $D(\omega,\alpha,\beta_2,\gamma)$ are intermittency, i.e.

$$\ldots < \mu_k^{(1)} < \mu_k^{(2)} < \mu_{k+1}^{(1)} < \mu_{k+1}^{(2)} < \mu_{k+2}^{(1)} < \mu_{k+2}^{(2)} < \ldots,$$

if $\operatorname{Im}\omega = 0$, then the inequalities

$$\ldots \le \mu_k^{(1)} \le \mu_k^{(2)} \le \mu_{k+1}^{(1)} \le \mu_{k+1}^{(2)} \le \mu_{k+2}^{(1)} \le \mu_{k+2}^{(2)} \le \ldots$$

hold, and a multiplicites eigenvalue of one of these problems is a simple eigenvalue of another one.

In chapter 2 of this dissertation, we define the spectral data for the boundary value problems $D(\omega,\alpha_j,\beta,\gamma)$ and $D(\omega,\alpha,\beta_j,\gamma)$ $(j=1,2)$ and we prove the uniqueness theorems for the solutions of inverse problems. Then, based on the spectral data and uniqueness theorems, we construct the recovery algorithms for Dirac system. As spectral data, we consider here the sequences of eigenvalues, the sign sequence $\{\delta_n\}$ and the number ω.

In paragraph 1 of chapter 2, we consider an inverse problem for the boundary value problem $D(\omega,\alpha_j,\beta,\gamma)$.

Consider the conditions

$$\begin{gathered} y_2(0)+(\alpha_j\lambda+\beta)y_1(0)+\omega y_1(\pi)=0, \\ y_2(\pi)+\gamma y_1(\pi)-\overline{\omega}y_1(0)=0, \\ j=1,2, \end{gathered} \tag{7}$$

which correspond to the boundary conditions (3). Denote the problem (1), (7) by $D(\omega,\alpha_j,\beta,\gamma)$, and the sequence of eigenvalues of this problem by $\left\{\gamma_k^{(j)}\right\}$ $(j=1,2;\ k=\pm 0,\pm 1,\pm 2,\ldots)$. By virtue of characteristic function (5), the eigenvalues of the problem $D(\omega,\alpha_j,\beta,\gamma)$ are the zeros of the characteristic function

$$d_j(\lambda)=2\operatorname{Re}\omega+U_+(\lambda)+(\alpha_j\lambda+\beta)\sigma(\lambda),$$

where

$$U_+(\lambda)=|\omega|^2 s_1(\pi,\lambda)-c_2(\pi,\lambda)-\gamma c_1(\pi,\lambda),$$
$$\sigma(\lambda)=s_2(\pi,\lambda)+\gamma s_1(\pi,\lambda).$$

Also, very important role is played here by the eigenvalues of the boundary value problems generated by the equation (1) and the boundary conditions

$$y_1(0)=y_1(\pi)=0, \tag{8}$$
$$y_1(0)=y_2(\pi)+\gamma y_1(\pi)=0. \tag{9}$$

Denote the sequence of eigenvalues of the problem (1), (8) by $\{\lambda_n\}$, $(n = 0, \pm 1, \pm 2, \ldots)$, and the one of the problem (1), (9) by $\{\upsilon_n\}$ $(n = 0, \pm 1, \pm 2, \ldots)$. The asymptotic formula

$$\upsilon_n = n + \frac{1}{\pi} \operatorname{arcctg} \gamma + m_n \tag{10}$$

is true for the eigenvalues $\{\upsilon_n\}$ of the problem (1), (9), where $\sum_{n=-\infty}^{\infty} m_n^2 < \infty$.

We state the inverse spectral problem for the boundary value problem $D(\omega, \alpha_j, \beta, \gamma)$ as follows:

Inverse problem 1. Given the spectra $\{\gamma_k^{(1)}\}$, $\{\gamma_k^{(2)}\}$ $(k = \pm 0, \pm 1, \pm 2, \ldots)$, the sign sequence

$$\delta_n = \operatorname{sign}\left(1 - |\omega s_1(\pi, \upsilon_n)|\right) \quad (n = 0, \pm 1, \pm 2, \ldots)$$

and the number ω, recover the matrix coefficient function $Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}$ of the equation (1) and the coefficients $\alpha_1, \alpha_2, \beta, \gamma$ of the boundary conditions (7).

As the boundary conditions are nonseparated, the spectra $\{\gamma_k^{(1)}\}$, $\{\gamma_k^{(2)}\}$ are not enough to uniquely solve this problem. Therefore, additional spectral data must be introduced.

Theorem 11. The spectra $\{\gamma_k^{(1)}\}$, $\{\gamma_k^{(2)}\}$ $(k = \pm 0, \pm 1, \pm 2, \ldots)$, the sign sequence

$$\delta_n = \operatorname{sign}\left(1 - |\omega s_1(\pi, \upsilon_n)|\right) \quad (n = 0, \pm 1, \pm 2, \ldots)$$

and the number ω uniquely recovered the boundary value problems $D(\omega, \alpha_1, \beta, \gamma)$ and $D(\omega, \alpha_2, \beta, \gamma)$.

In paragraph 2 of chapter 2, we consider an inverse problem for the boundary value problem $D(\omega, \alpha, \beta_j, \gamma)$ $(j = 1, 2)$ and we prove a uniqueness theorem. Note that for simplicity we assume here $p(\pi) = q(\pi) = q(0) = 0$.

Consider the conditions (3)

$$y_2(0)+(\alpha\lambda+\beta_j)y_1(0)+\omega y_1(\pi)=0,$$
$$y_2(\pi)+\gamma y_1(\pi)-\overline{\omega}y_1(0)=0, \quad (11)$$
$$j=1,2,$$

which correspond to the boundary conditions (3). Denote the problem (1), (11) by $D(\omega,\alpha,\beta_j,\gamma)$ and the eigenvalues of this problem by $\{\mu_k^{(j)}\}$ $(j=1,2;\ k=\pm0,\pm1,\pm2,...)$.

Inverse problem 2. Given the spectra $\{\mu_k^{(1)}\}$, $\{\mu_k^{(2)}\}$ $(k=\pm0,\pm1,\pm2,...)$, the sign sequence

$$\delta_n=\text{sign}\,(1-|\omega s_1(\pi,\upsilon_n)|),\ (n=0,\pm1,\pm2,...)$$

and the number ω, recover the matrix coefficient function $Q(x)=\begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x)\end{pmatrix}$ of the equation (1) and the coefficients α, β_1, β_2, γ of the boundary conditions (11).

The following uniqueness theorem, important for the unique solution of this inverse problem, is true.

Theorem 12. The spectra $\{\mu_k^{(1)}\}$, $\{\mu_k^{(2)}\}$ $(k=\pm0,\pm1,\pm2,...)$, the sign sequence

$$\delta_n=\text{sign}\,(1-|\omega s_1(\pi,\upsilon_n)|)\ (n=0,\pm1,\pm2,...)$$

and the number ω, uniquely recovered the boundary value problems $D(\omega,\alpha,\beta_1,\gamma)$, $D(\omega,\alpha,\beta_2,\gamma)$.

In paragraph 3 of chapter 2, the recovery algorithms are constructed for the boundary value problems $D(\omega,\alpha_j,\beta,\gamma)$ and $D(\omega,\alpha,\beta_j,\gamma)$ $(j=1,2)$.

Algorithm. Let the spectra $\{\gamma_k^{(1)}\}$, $\{\gamma_k^{(2)}\}$ of the boundary value problems $D(\omega,\alpha_1,\beta,\gamma)$ and $D(\omega,\alpha_2,\beta,\gamma)$, the sign sequence $\{\delta_n\}$ and the number ω be given.

Step 1. Recovery of characteristic functions $d_j(\lambda)$ in the form of infinite product (6) using the sequences $\{\gamma_k^{(j)}\}$.

Step 2. Determination of parameters $\gamma,\alpha_j,(j=1,2),\beta,$ of the boundary conditions (7) using the formulas

$$\gamma = \lim_{k\to\infty} ctg\pi\left(\gamma_k^{(j)} - k\right),$$

$$\alpha_j = \frac{1}{2}\lim_{k\to\infty}\frac{d_j(2k)}{k},$$

$$\beta = -2\operatorname{Re}\omega + \gamma + \frac{1}{\alpha_1 - \alpha_2}\lim_{k\to\infty}\left[\alpha_1 d_2(2k) - \alpha_2 d_1(2k)\right].$$

Step 3. Recovery of function $\sigma(\lambda)$ from the formula

$$\sigma(\lambda) = \frac{d_1(\lambda) - d_2(\lambda)}{(\alpha_1 - \alpha_2)\lambda}$$

and finding the zeros υ_n of this function.

Step 4. Recovery of function $U_+(\lambda)$ from the equality

$$U_+(\lambda) = \frac{\alpha_2 d_1(\lambda) - \alpha_1 d_2(\lambda)}{\alpha_2 - \alpha_1} - \beta\sigma(\lambda) - 2\operatorname{Re}\omega.$$

Step 5. Calculation of the values of function

$$U_-(\lambda) = -|\omega|^2 s_1(\pi,\lambda) - c_2(\pi,\lambda) - \gamma c_1(\pi,\lambda) \tag{12}$$

at the points $\{\upsilon_n\}$ using the relation

$$U_-(\upsilon_n) = (-1)^{n+1}\delta_n\sqrt{U_+^2(\upsilon_n) - 4|\omega|^2}.$$

Step 6. Recovery of function

$$\eta(\lambda) = U_+(\lambda) - U_-(\lambda) + 2|\omega|^2 \sin\lambda\pi \tag{13}$$

using the interpolation formula

$$\eta(\lambda) = \sigma(\lambda)\sum_{n=-\infty}^{\infty}\frac{\eta(\upsilon_n)}{(\lambda - \upsilon_n)\sigma'(\eta_n)}, \tag{14}$$

where

$$\eta(\upsilon_n) = U_+(\upsilon_n) + (-1)^n\delta_n\sqrt{U_+^2(\upsilon_n) - 4|\omega|^2} + 2|\omega|^2\sin\upsilon_n\pi.$$

Step 7. Using the interpolation formula (14) and the function $U_+(\lambda)$ construct the function $U_-(\lambda)$ given by (12) from the equality (13).

Step 8. Recovery of characteristic function of the boundary value problem (1), (8) using the functions $U_\pm(\lambda)$ and the formula

$$s_1(\pi,\lambda)=\frac{1}{2|\omega|^2}[U_+(\lambda)-U_-(\lambda)],$$

and finding its zeros λ_n.

Step 9. Using the functions $\sigma(\lambda)$, $s_1(\pi,\lambda)$ and the parameter γ, find the characteristic function $s_2(\pi,\lambda)$ of the problem generated by the equation (1) and the boundary conditions $y_1(0)=y_2(\pi)=0$ from the formula

$$s_2(\pi,\lambda)=\sigma(\lambda)-\gamma s_1(\pi,\lambda).$$

Step 10. Recovery of coefficient $Q(x)$ of Dirac equation (1) by means of a well-known algorithm using the sequences $\{\lambda_n\}$ and $\{\upsilon_n\}$.

Also, the algorithm for the recovery of boundary value problem $D(\omega,\alpha,\beta_j,\gamma)$ is constructed in a similar way.

In Chapter 3 of this dissertation, the necessary and sufficient conditions for the recovery of boundary value problem $D(\omega,\alpha,\beta,\gamma)$ are obtained. Derivation of these conditions is the most difficult and the most important part of our work.

In Paragraph 1 of Chapter 3, we prove the lemmas to be used in obtaining the characteristics of spectral data of the problem $D(\omega,\alpha_j,\beta,\gamma)$.

Lemma 1. The asymptotic formula

$$\upsilon_n=n+a+\frac{A}{\pi n}+\frac{\eta_n}{n}$$

is true for the zeros υ_n $(n=0,\pm1,\pm2,...)$ of the entire function

$$\sigma(\lambda)=\cos\pi\lambda-\gamma\sin\pi\lambda+\frac{A}{\lambda}[\sin\pi\lambda+\gamma\cos\pi\lambda]+\frac{\psi(\lambda)}{\lambda},$$

where

$$A=\frac{1}{2}\int_0^\pi\left[p^2(x)+q^2(x)\right]dx,$$

$$\psi(\lambda)=\int_{-\pi}^{\pi}\tilde{\psi}(t)e^{i\lambda t}dt\,,\ \tilde{\psi}(t)\in L_2[-\pi,\pi],$$

$$a=\frac{1}{\pi}\operatorname{arcctg}\gamma,\quad \{\eta_n\}\in l_2\,.$$

Lemma 2. The representation

$$d(\lambda)=2\operatorname{Re}\omega+\alpha\lambda(\cos\pi\lambda-\gamma\sin\pi\lambda)+$$

$$+\left(\alpha a\gamma+\alpha A+\frac{\alpha\left(q(\pi)(\gamma^2-1)-b^2q(0)-2\gamma p(\pi)\right)}{2b^2}-\frac{b^2+|\omega|^2}{b^2}-\alpha\gamma\tilde{C}\right)\sin\pi\lambda+$$

$$+\left(\alpha\tilde{C}-\alpha a+\alpha\gamma A+\frac{\alpha\gamma\left(q(\pi)(\gamma^2-1)-b^2q(0)-2\gamma p(\pi)\right)}{2b^2}-\frac{\gamma\left(b^2+|\omega|^2\right)}{b^2}\right)\cos\pi\lambda-$$

$$-\alpha bf(\lambda-a)$$

is true for the entire function

$$d(\lambda)=-\pi\alpha b(\gamma_{-0}-\lambda)(\gamma_{+0}-\lambda)\prod_{\substack{k=-\infty\\k\neq0}}^{\infty}\frac{\gamma_k-\lambda}{k},$$

where $\tilde{C}$ is a real number and

$$f(\lambda)=\int_{-\pi}^{\pi}\tilde{f}(t)e^{i\lambda t}dt,\quad \tilde{f}(t)\in L_2[-\pi,\pi].$$

In paragraph 2 of chapter 3, the necessary conditions are found for the recovery of boundary value problems $D(\omega,\alpha_1,\beta,\gamma)$ and $D(\omega,\alpha_2,\beta,\gamma)$ $(\alpha_1\neq\alpha_2)$.

Theorem 13. The spectral data of the boundary value problems $D(\omega,\alpha_1,\beta,\gamma)$ and $D(\omega,\alpha_2,\beta,\gamma)$ $\left(|\omega|^2<\gamma+1,\ \alpha_1<\alpha_2<0\right)$ satisfy the following conditions:

1) $$\gamma_k^{(j)}=k+a+\frac{(-1)^{k+1}\tilde{A}_j-\tilde{B}_j}{\pi k}+\frac{\xi_k^{(j)}}{k},\qquad(15)$$

where

$$a = \frac{1}{\pi}\operatorname{arcctg}\gamma, \ \ \tilde{A}_j = -\frac{2\operatorname{Re}\omega}{\alpha_j b},$$

$$\tilde{B}_j = -A - \frac{q(\pi)(\gamma^2 - 1) - b^2 q(0) - 2\gamma p(\pi)}{2b^2} + \frac{b^2 + |\omega|^2}{\alpha_j b^2}, \ b = \sqrt{1+\gamma^2}, \ \{\xi_k^j\} \in l_2;$$

2) for $\operatorname{Im}\omega \neq 0$, the eigenvalues $\gamma_k^{(1)}$ and $\gamma_k^{(2)}$ are intermittency, i.e.

$$0 < \gamma_{+0}^{(1)} < \gamma_{+0}^{(2)} < \gamma_1^{(1)} < \gamma_1^{(2)} < \gamma_2^{(1)} < \gamma_2^{(2)} < \ldots,$$
$$0 > \gamma_{-0}^{(1)} > \gamma_{-0}^{(2)} > \gamma_{-1}^{(1)} > \gamma_{-1}^{(2)} > \gamma_{-2}^{(1)} > \gamma_{-2}^{(2)} > \ldots,$$

while for $\operatorname{Im}\omega = 0$ the inequalities

$$0 \le \gamma_{+0}^{(1)} \le \gamma_{+0}^{(2)} \le \gamma_1^{(1)} \le \gamma_1^{(2)} \le \gamma_2^{(1)} \le \gamma_2^{(2)} \le \ldots,$$
$$0 \ge \gamma_{-0}^{(1)} \ge \gamma_{-0}^{(2)} \ge \gamma_{-1}^{(1)} \ge \gamma_{-1}^{(2)} \ge \gamma_{-2}^{(1)} \ge \gamma_{-2}^{(2)} \ge \ldots$$

hold, so that the a multiplicites eigenvalue of the problem $D(\omega, \alpha_1, \beta, \gamma)$ is a simple eigenvalue of the problem $D(\omega, \alpha_2, \beta, \gamma)$;

3)

$$\lim_{k\to\infty}\left\{\frac{1}{\alpha_j b}\left[2\operatorname{Re}\omega - d_j\left(2k + a + \frac{1}{2}\right)\right] - 2k - \frac{1}{2}\right\} = a + \frac{\beta}{\alpha_j} + \frac{\gamma|\omega|^2}{\alpha_j b^2}, \qquad (16)$$

where $d_j(\lambda)$ is a characteristic function of the problem $D(\omega, \alpha_j, \beta, \gamma)$;

4) the following relations hold for $b_n = d_j(\upsilon_n) - 2\operatorname{Re}\omega$:

a) $b_n = (-1)^{n+1}\dfrac{b^2 + |\omega|^2}{b} + \dfrac{\chi_n}{n}, \ \{\chi_n\} \in l_2$;

b) $|b_n| \ge 2|\omega|$, where the numbers $\{\upsilon_n\}$ $(n = 0, \pm 1, \pm 2, \ldots)$ are the zeros of the function $d_1(\lambda) - d_2(\lambda)$, i.e. the eigenvalues of the problem (1), $y_1(0) = y_2(\pi) + \gamma\, y_1(\pi) = 0$;

5) if $|b_n| = 2|\omega|$, then $\delta_n = 0$, otherwise $\delta_n = \pm 1$ and there is a positive integer N such that $\delta_n = 1$ for $|n| > N$.

In paragraph 3 of chapter 3, the main theorem for inverse problem is obtained.

Theorem 14. The sufficient conditions for the sequences $\{\gamma_k^{(1)}\}$, $\{\gamma_k^{(2)}\}$ $(k=\pm 0,\pm 1,\pm 2,...)$, $\{\delta_n\}$ $(n=0,\pm 1,\pm 2,...)$ and the number ω to be the spectral data of the boundary value problems $D(\omega,\alpha_1,\beta,\gamma)$ and $D(\omega,\alpha_2,\beta,\gamma)$ $\left(|\omega|^2<\gamma^2+1,\ \alpha_1<\alpha_2<0\right)$ are:

1) the asymptotic formula (15) holds, where $\tilde{A}_j=-\dfrac{2\operatorname{Re}\omega}{\alpha_j b}$, $b=\dfrac{1}{\sin\pi a}$, α_j, a, $\tilde{B}_j$ are real numbers,

$$0<a<1,\quad \tilde{A}_1<\tilde{A}_2,\quad \{\xi_k^j\}\in l_2;$$

2) the numbers $\gamma_k^{(1)}$ and $\gamma_k^{(2)}$ satisfy the inequalities in the condition 2) of Theorem 13;

3) the equality (16) holds, where

$$d_j(\lambda)=-\alpha_j b\pi\left(\gamma_{-0}^{(j)}-\lambda\right)\left(\gamma_{+0}^{(j)}-\lambda\right)\prod_{\substack{k=-\infty\\ k\neq 0}}^{\infty}\frac{\gamma_k^{(j)}-\lambda}{k};$$

4) the following relations are true for $b_n=d_j(\upsilon_n)-2\operatorname{Re}\omega$:

a) $b_n=(-1)^{n+1}\dfrac{b^2+|\omega|^2}{b}+\dfrac{\chi_n}{n}$, $\{\chi_n\}\in l_2$;

b) $|b_n|\geq 2|\omega|$, where the numbers $\{\upsilon_n\}$ $(n=0,\pm 1,\pm 2,...)$ are the zeros of the function $d_1(\lambda)-d_2(\lambda)$;

5) if $|b_n|=2|\omega|$, then $\delta_n=0$, otherwise $\delta_n=\pm 1$ and there is a positive integer N such that $\delta_n=1$ for $|n|>N$.

Remark. It can be seen from Theorem 13 that the conditions 1), 3)-5) are also necessary conditions for the spectral data. But the condition 2) is a necessary condition when the inequality

$$2\operatorname{Re}\left[\omega\overline{y_1(0)}y_1(\pi)\right]-\gamma|y_1(\pi)|^2+\beta|y_1(0)|^2+\int_0^{\pi}p(x)\left[|y_1(x)|^2+|y_2(x)|^2\right]dx\neq 0$$

holds. Note that the function $p(x)$ recovered in the proof of this theorem may not satisfy this inequality

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CONCLUSION

This dissertation deals with the inverse spectral problem for Dirac operator in case one of nonseparated boundary conditions includes a linear function of spectral parameter.

The following main results have been obtained in this work:

1. The spectral properties of Dirac operator are studied in case one of separated boundary conditions includes a linear function of spectral parameter.
2. The asymptotics of the eigenvalues of the boundary value problem is found.
3. The multiplicity and the intermittency of the eigenvalues of considered operators are studied.
4. Inverse problems for Dirac system are stated and uniqueness theorems for the solutions of these problems are proved.
5. The algorithms for the solutions of inverse problems are constructed.
6. Sufficient conditions for the solution of inverse spectral problem are obtained (under some additional conditions these conditions are also necessary).

The main results of the dissertation are published in the following works:

1. Ferzullazadeh, A., Nabiev, I. On the spectrum of the Dirac operator with a spectral parameter in the boundary condition // 6th International IFS and Contemporary Mathematics Conference, – Mersin, Turkey: – 07-10 June, 2019, – p. 69.
2. Ferzullazadeh, A. Representation for solving the Dirac equation // – Lankaran: Scientific news of Lankaran State University, Mathematics and natural sciences series, – 2019. № 2, – p. 52-58. (Azerbaijani)
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researchers on the theme "Information, science, technology and university perspectives", – Lankaran, Azerbaijan: – 18 December, 2020, – p. 36-37. (Azerbaijani)
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5. Ferzullazadeh, A.G. On the multiplicities of the eigenvalues of the Dirac operator // – Baku: Journal of Baku Engineering University, Mathematics and computer science, – 2020, 4 (2), – p. 111-119. (Azerbaijani)
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10. Ferzullazadeh, A.G. Solution algorithm of the inverse spectral problem for Dirac operator with a spectral parameter in the boundary

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13. Ferzullazadeh, A.G. The multiplicity of zeros of the characteristic function of the boundary value problem for the Dirac operator // XXXVII International Conference on Problems of Decision Making Under Uncertainties, – Sheki-Lankaran, Azerbaijan, Kyiv, Ukraine: – 23-25 November, 2022, – p. 44-45.

14. Ferzullazadeh, A.G., Nabiev, I.M. A sufficient condition for the solution of the inverse problem for a Dirac operator with a spectral parameter in the boundary condition // Spectral Theory of Operators and Related Issues, – Ufa, Russia: – 26-27 October, 2023, – p. 36-37.

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