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# ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

# CONVERGENCE OF SPECTRAL EXPANSION IN EIGEN AND ASSOCIATED VECTOR-FUNCTIONS OF $2m-$th ORDER DIRAC OPERATOR

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# GENERAL CHARACTERISTICS OF THE WORK

**Rationale and development degree of the topic.**

Spectral theory of differential operators (of Schrodinger, Dirac, etc.) especially has been developed especially in recent decade in connection with the development of quantum mechanics.

Main problems of spectral theory are: study of spectrum basicity of the root functions of the studied operator in this or other space of functions; equiconvergence of spectral expansion of an arbitrary function from this or other class in the system of root functions of the studied operator with the expansion of the same function in trigonometric Fourier series; absolute and uniform convergence of spectral expansion of a function from the class, generally speaking, not coinciding with the domain of definition of this operator; solution of inverse problems of spectral analysis, etc.

Research on the spectral theory of ordinary differential operators originates from the classical works of J. Liouville, Sh. Sturm and later works of V.A. Steklov, Ya.D. Tamarkin, D. Birkhoff, M.L. Rasulov and other authors, where the problems of asymptotics of eigen values and convergence of spectral expansions were studied for various classes of boundary value problems.

For a long time, the main object of the study were spectral properties of self-adjoint differential operators. However, in recent decades, a number of new problems of mathematical physics reducing to the study of spectral property of not self-adjoint differential operators, have arisen. An example of problems of this series can be the Bitszadze-Samarsky problem with nonlocal boundary conditions for a heat conductivity equation.

When studying not self-adjoint problems, it was observed that the system of eigen-functions of a not self-adjoint operator, generally speaking, not only does not form a basis in the class of $L_2$, but it is not complete in $L_2$. Therefore, this system must be supplemented with associated functions. In these problems, eigen and associated functions (root functions), generally speaking, are not orthogonal in

$L_2$, and neither their closedness, nor their minimality imply their basicity in this space. Thus, transition to not self-adjoint problems required new approaches for their study.

The fact of completeness in $L_2$ of specially constructed system of root functions in M.V. Keldysh's works (called by M.V. Keldysh a canonic system) was established for a wide class of boundary value problems. Further, the issue on the completeness was studied for a wide class boundary value problems in the papers of V.B. Lidsky, M.A.Naimark, V.N. Vizitei and A.S. Markus, A.S. Markus, J.E.Allahverdiyev, M.G.Gasymov and M.G.Javadov, A.M.Kroll, A.A. Shkalikov and other authors.

In their works V.P. Mikhailov and G.M. Keselman succeeded in distinguishing a class of strongly regular boundary value problems, providing Riesz basicity of the system of root functions in $L_2$. Block basicity (or basicity with parenthesis) of the system of root functions of the $n$-th order differential operator with regular boundary conditions was established in A.A. Shkalikov's work.

In all the above works (except A.A. Shkalikov's work) the operators whose system of root functions contains a finitely many associated functions, were considered.

In N.I. Ionkin's paper, a boundary value problem (with regular conditions) is studied for a second order operator with zero coefficients. All eigen-values of this problem are double, and a total amount of associated functions is infinitely many. Nevertheless, it turned out that the root functions of this problem (with their special choice) form Riesz basis in $L_2$.

The first a more general result on equiconvergence for ordinary differential operators with the conditions of regularity of boundary conditions and sufficient smoothness of coefficients was first obtained by Ya.D. Tamarkin. Later, M. Stone obtained a similar result for an operator with summable coefficients. In his work A.P. Khromov extended Tamarkin's equiconvergence theorem on integral operators whose kernels generalize the properties of the Green function of a differential operator with regular boundary conditions. We not also the papers where equiconvergence rate is studied for differential operators with regular boundary conditions. We note also the papers V.S.Rikhlov where equiconvergence rate is studied for differential operators with regular boundary conditions.

All these works were based on the resolvent method and the equiconvergence obtained in these works are block-equiconvergences.

Recently, the method developed by V.I. Il'in for studying differential operators is successfully applied. He has noticed that in the availability of infinitely many multiple eigen values, the basicity and equiconvergence properties unlike the completeness properties 1) significantly depend on the choice of root functions; 2) it is not defined only by the specific form of boundary conditions, the values of the coefficients of the differential operator also influence on these properties, and these properties change for any arbitrary small change in the value of the coefficients in the metrics of the classes where these coefficient were given. Thus, in this situation, it is impossible to formulate the basicity and equiconvergence conditions in terms of boundary conditions.

In this connection, V.A. Il'in suggested a new interpretation of root functions that are understood as regular solutions of the appropriate equation with a spectral parameter without respect to the form of boundary conditions. Such an interpretation allows to consider arbitrary boundary conditions (both local and nonlocal) of the system of functions not connected with any boundary conditions, and also some systems obtained by uniting the subsets of root functions of two different boundary value problems.

In his paper V.A. Il'in has considered a system of root functions of an ordinary differential operator and under certain natural conditions he established theorems on equiconvergence and basicity on a compact (and also on componentwise equiconvergence).

Further, the study of these and other issues of spectral theory of ordinary differential operators was developed in the papers of V.A. Il'in and his followers V.V. Tikhonomirov, I.S. Lomov, N.B. Kerimov, V.D. Budayev, I.Io. V.I. Komornik, N.Lazhetich, L.V.

Kritskov, V.M. Kurbanov and others.

Note that unconditional basicity of the system of root functions of the Dirac operator was studied in V.M. Kurbanov's paper by V.A. Il'in method. He has established Bessel property and unconditional basicity criteria in $L_2$ of the system of root functions of the Dirac operator. A theorem on componentwise uniform equiconvergence when the system consists only of eigen functions of the Dirac operator and forms in $L_2$ the Riesz basis, was proved by M. Khovvat. These papers of V.M.Kurbanov and A.I.Ismayilova, L.Z.Buksayeva, G.R.Hajiyeva were devoted to componentwise uniform equiconvergence on a compact, uniform convergence, the Riesz property of the system of root functions of vector-functions of the Dirac operator.

Basicity of the system of root vector-functions of the Dirac operator with specific boundary conditions was studied in the papers of P. Jakov and B. Mityagin, I. Troshin and M. Yamomota, L.L. Oridorog and S. Khassi.

In I. Troshin and M. Yamomota's paper, the Riesz basicity is established for the case when the potential of the Dirac operator belongs to $L_2$ and boundary conditions are separated. P. Jakov and B. Mityagin have studied the case with regular boundary conditions and with a potential from the class $L_2$. They proved the Riesz basicity from subspaces, and in the case of strongly regular boundary conditions, the Riesz basicity. The Dirac operator with a potential from $L_p$, $p \geq 1$, was studied in the papers of A.M. Savchuk and A.A. Shkalikov, A.M. Savchuk and I.V. Sadovnich, and for the case of strongly regular boundary conditions the Riesz basicity was proved, but in the case of regular (but not strongly regular) boundary conditions the Riesz basicity from subspaces was proved. In the papers of A.A. Lunev and M.M. Malamud, the Dirac type $2 \times 2$ system with potentials from the class $L_1$ and with strongly regular boundary conditions was studied and the Riesz basicity was established. For the Dirac $2m \times 2m$ system with Dirichlet boundary conditions and a potential from $L_2$, the Riesz basicity from subspaces was proved by Ya.V. Mukuntuk and D.V. Ruyud in their paper.

Thus, further investigation of differential operators, including Dirac type operators by the V.A. Il'in method is of interest.

**Object and subject of the study.** The main object of the dissertation work is to study basicity and eguiconvergence of expansions in root vector-functions of the $2m$-th order Dirac operator.

**Goals and objectives of the study.** To study Bessel property, unconditional basicity of the system of root vector functions of $2m$-th order Dirac type operators in the space $L_2$ and the problem on componentwise uniform equiconvergence on a compact of orthogonal expansion in eigen vector functions of the given operator with trigonometric expansion.

**Research methods.** In the work we apply the methods of spectral theory of differential operators, functional analysis and theory of harmonic analysis.

**The main hypotheses to be defended.**

- The results of the study of Bessel property of the system of root vector-functions of $2m$-th order Dirac type operator in the space $L_2^{2m}(0,2\pi)$.
- The results of the study of unconditional basicity of the system of root vector-functions of $2m$-th order Dirac type operator in the space $L_2^{2m}(0,2\pi)$.
- The results of the study of equivalent basicity of the system of root vector-functions of $2m$-th order Dirac type operator in the space $L_2^{2m}(0,2\pi)$.
- The results of the study of componentwise uniform equiconvergence on a compact with trigonometric series of expansions of vector-functions from the class $L_2^{2m}(0,2\pi)$ in orthogonal series in eigen vector-functions of $2m$-th order Dirac operator type operator.

- The results of the study of Bessel property and unconditional basicity of the system of root vector-functions of a Dirac type operator in the spaces $L_2^2(0,\pi)$.

**Scientific novelty of the research.**

- To derive shift formula and establish the estimation between different $L_p^{2m}$ norms of the root (eigen and associated) vector-functions of $2m$-th type Dirac operator.
- To establish Bessel property criterion of the system of root vector functions of $2m$-th ordr Dirac type operator in the space $L_2^{2m}(0,2\pi)$.
- To establish unconditional basicity criterion for the system of root vector-functions of $2m$-th order Dirac operator in the space $L_2^{2m}(0,2\pi)$.
- A theorem on equivalent basicity of the system of root vector-functions of $2m$ - th order Dirac type operator in the space $L_2^{2m}(0,2\pi)$ was proved.
- A theorem on componentwise uniform equiconvergence on a compact with a trigonometric series of expansions of vector-functions from the class $L_2^{2m}(0,\pi)$ in orthogonal series in eigen vector-functions of $2m$-th order Dirac type operator was roved.
- The estimations between different $L_p^2$ - norms of the root vector-functions of a Dirac type operator were proved, and criteria on Bessel property and unconditional basicity in $L_2^2(G)$, $G=(a,b)$, $mes\, G<\infty$ of the system of root vector-functions of this operator were established.

**Theoretical and practical value of the research.** The work is of theoretical character. Its results can be used in spectral theory of differential operators (Dirac type operator); when justifying the solution of some mathematical physics problems by the Fourier method

**Approbation of the work.** The main results were reported:

фt the International Conference Azerbaijan-Turkey Ukraine MADEA 7 (Baku, 2015); International Workshop on "Non-harmonic Analysis and Differential Operators" (Baku, 2016); International conference on Mathematical advances and applications Istanbul-Turkey (Baku, 2018); Modern problems of Mathematics and Mechanics. Proceedings of the International conference devoted of the 60th anniversary of the Institute of Mathematics and Mechanics of Azerbaijan National Academy of Sciences (Baku, 2019); at the seminar of the chair of "Mathematical Analysis" (prof. B.A.Aliyev) of Azerbaijan State Pedagogical University.

**Author's personal contribution**. All the conclusions ad results obtained belong to the author personally

**Author's publications.** The main results of the dissertation were published in 10 papers whose list is at the end of the abstract.

**The name of the organization where the dissertation work was performed.** The work was performed at the chair of "Mathematical analysis" of Baku State Pedagogical University.

**Structure and volume of the dissertation (in signs, indicating the volume of each structural unit separately).** Total volume of the work- 214600 signs (title page-320 signs, content 2280 signs, introduction-46000 signs, chapter I-98000 signs, chapter II-66000 signs, introduction-2000). The list of references consists of 102 titles.

## CONTENT OF THE DISSERTATION

We now state the main results of the dissertation.

In the dissertation we study Bessel property and unconditional basicity of the system of root vector-functions of $2m$-th order Dirac type operators in the space $L_2^{2m}(0,2\pi)$, componentwise uniform equiconvergence on a compact of orthogonal expansion in eigen vector-functions of Dirac type one-dimensional operator of $2m$-th order with trigonometric expansion.

The work consists of introduction, two chapters and a list of references. Each chapter is divided into sections.

In **chapter I** we consider a Dirac type $2m$-th order operator:

$$Du = B\frac{du}{dx} + P(x)u, \quad u = (u_1, u_2, ..., u_{2m})^T, \; m \geq 1 \qquad (1)$$

determined on an arbitrary interval $G = (a,b)$, where $B = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$ or $B = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, $I$ is a unit operator in $E^m$, $J = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}$, $P(x) = (p_{ij}(x))_{i,j=1}^{2m}$ is a summable complex valued $2m \times 2m$ matrix function.

The estimations for eigen and associated vector-functions of the operator $D$ are established, Bessel property and basicity of the system of root vector-functions of the given operator are studied, Bessel property and unconditional basicity criteria are established, a theorem on equivalent basicity in $L_2^{2m}(0,2\pi)$ is proved.

Following V.A. Il'in, we will undersand root vector-functions of the operator $D$ without respect to the form of boundary conditions, more exactly, uder the eigen vector-function of the operator $D$ responding to the complex eigen value $\lambda$, we understand any identically non-zero complex-valued vector-function $\overset{0}{u}(x)$, that is absolutely continuous on any closed subinterval $G$ and almost everwhere in $G$ satisfies the equation

$$D\overset{0}{u} = \lambda \overset{0}{u}.$$

Similarly, under the associated vector-function of order $\ell$, $\ell \geq 1$, responding to the same $\lambda$ and the eigen function $\overset{0}{u}(x)$, we understand any complex-valued vector function $\overset{\ell}{u}(x)$, that is absolutely continuous on any closed subinterval $G$ and almost everywhere in $G$ satisfies the equation

$$D\overset{l}{u}=\lambda\overset{l}{u}+\overset{l-1}{u}.$$

***Theorem 1.*** *Let the functions* $p_{ij}(x)$, $i,j=\overline{1,2m}$ *belong to the class* $L_1^{loc}(G)$. *Then for any compact* $K\subset G$ *there exist such constants* $C^r(K,l,m)$, $r=1,2$; $l=0,1,2,...$ *independent of* $\lambda$, *that the following estimations are valid:*

$$\left\|\overset{l-1}{u}\right\|_{L_\infty^{2m}(K)}\le C^1(K,l,m)\left(1+|\operatorname{Im}\lambda|\right)\left\|\overset{l}{u}\right\|_{L_\infty^{2m}(K)}, \tag{2}$$

$$\left\|\overset{l}{u}\right\|_{L_\infty^{2m}(K)}\le C^2(K,l,m)\left(1+|\operatorname{Im}\lambda|\right)^{1/p}\left\|\overset{l}{u}\right\|_{L_p^{2m}(K)},\ 1\le p<\infty, \tag{3}$$

*where* $\overset{-1}{u}\equiv 0$

**Remark 1.** If $G$ is a finite interval, $p_{ij}(x)$ $(i,j=\overline{1,2m})$ belong to the class $L_1(G)$, then the estimations (1) and (2) are valid in the case $K=\overline{G}$ as well.

Note that for root vector-functions of ordinary differential equations such estimations were established and found their application in the papers of V.A. Il'in, N.B. Kerimov, L.V. Kritskov, V.M. Kurbanov, I.S. Lomov, V.V. Tikhomirov. In the case of the Dirac operator, such estimations were established in the papers of V.M. Kurbanov, V.M. Kurbanov and A.I. Ismyilova, A.M. Abdullyeva, L.Z. Buksayeva.

Bessel property and basicity the system of root vector-functions of the operator $D$ with a summable potential are studied in 1.3. Bessel property and unconditional basicity criteria are established and a theorem on equivalent basicity in $L_2^{2m}(0,2\pi)$ is proved.

The system $\{\vartheta_k(x)\}_{k=1}^{\infty}\subset L_2^{2m}(G)$ is called a Bessel system if there exists such a constant $M$ that for any vector-functions $f(x)\in L_2^{2m}(G)$ the following inequality is fulfilled:

$$\sum_{k=1}^{\infty}\left|\left(f,\vartheta_k\right)\right|^2 \le M\,\|f\|_2^2 .$$

The system $\{\vartheta_k(x)\}_{k=1}^{\infty} \subset L_2^{2m}(G)$ is quadrtically close to the system $\{\tau_k(x)\}_{k=1}^{\infty} \subset L_2^{2m}(G)$, if

$$\sum_{k=1}^{\infty}\|\vartheta_k - \tau_k\|_2^2 < \infty .$$

Two sequences of elements in Hilbert space $H$ are called equivalent if there exists a linear bounded and a boundedly invertible in $H$ operator taking one of these sequences to another one.

In subsection 1.3.1 we study Bessel property of the system of root vector-functions of the operator $D$ with a summable potential and establish Bessel property criterion in $L_2^{2m}(G)$, $G=(0,2\pi)$.

Let $\{\psi_k(x)\}_{k=1}^{\infty}$ be an arbitrary system consisting of the root vector-functions of the operator $D$, $\{\lambda_k\}_{k=1}^{\infty}$ be an appropriate system of eigen-values. Furthermore the vector-function $\psi_k(x)$ is included into the system $\{\psi_k(x)\}_{k=1}^{\infty}$ together with appropriate less order associated vector-functions. This means that $D\psi_k = \lambda_k\psi_k + \theta_k\psi_{k-1}$ , where $\theta_k$ equals either 0 (in this case $\psi_k(x)$is an eigen vector-function) or 1 (in this case $\psi_k(x)$ is an associated vector unction, and $\lambda_k = \lambda_{k-1}$), $\theta_1 = 0$.

***Theorem 2. (Bessel property criterion).*** *Let $P(x) \in L_1(G) \otimes C^{2m\times 2m}$, the length of the chains of the root vector-functions be uniformly bounded, and there exist such a constant $C_0$ that*

$$\left|\operatorname{Im}\lambda_k\right| \le C_0,\ k=1,2,\dots . \tag{4}$$

*For the system* $\left\{\psi_k(x)\|\psi_k\|_2^{-1}\right\}_{k=1}^{\infty}$ *to be Bessel in* $L_2^{2m}(G)$, *it is necessary and sufficient that there exists such a constant* $M_1$ *that*

$$\sum_{|\mathrm{Re}\,\lambda_k-\tau|\le 1} 1 \le M_1\,, \tag{5}$$

*where* $\tau$ *is an arbitrary real number.*

Denote by $D^*$ an operator formally adjoint to the operator $D$, i.e.

$$D^* = B\frac{d}{dx} + P^*(x),$$

where $P^*(x)$ is a matrix adjoint to the matrix $P(x)$.

Let the system $\{\psi_k(x)\}_{k=1}^{\infty}$ be minimal in $L_2^{2m}(G)$, and its biorthogonally conjugate system $\{\varphi_k(x)\}_{k=1}^{\infty}$ consist of the root vector-functions of the operator $D^*$ i.e. $D^*\varphi_k = \overline{\lambda}_k\varphi_k + \theta_{k+1}\varphi_{k+1}$.

In 1.3.2 we study the basicity of the system of root vector-functions of the operator $D$. Unconditional basicity criterion is established and a theorem on equivalent basicity in $L_2^{2m}(0,2\pi)$ is proved.

***Theorem 3. (On unconditional basicity).***

*Let* $P(x)\in L_1(G)\otimes C^{2m\times 2m}$, *the lengths of the chains of the root vector-functions be uniformly bounded, one of the systems* $\{\psi_k(x)\}_{k=1}^{\infty}$ *and* $\{\varphi_k(x)\}_{k=1}^{\infty}$ *be complete in* $L_2^{2m}(G)$ *and condition (4) be fulfilled. Then the necessary and sufficient condition of unconditional basicity in* $L_2^{2m}(G)$ *of each of these systems is the existence of the constants* $M_1$ *and* $M_2$, *providing the validity of inequalities (4) and*

$$\|\psi_k\|_2\,\|\varphi_k\|_2 \le M_2\,,\ k=1,2,\ldots \tag{6}$$

**Remark 2.** Note that under the conditions of theorem 3, fulfillment of the inequalities (5) and (6) is a necessary and sufficient condition for the Riesz basicity of each of the systmes $\left\{\psi_k(x)\|\psi_k(x)\|_2^{-1}\right\}_{k=1}^{\infty}$ and $\left\{\varphi_k(x)\|\varphi_k(x)\|_2^{-1}\right\}_{k=1}^{\infty}$ in $L_2^{2m}(G)$.

**Remark 3.** In the part of sufficiency of theorem 3, the condition of uniform boundedness of the length of the chain of root vector-functions should be omitted, because it is contained in the condition (5).

***Theorem 4. (On equivalent basicity).*** *Let $P(x) \in L_1(G) \otimes C^{2m\times 2m}$, conditions (4)-(6) be fulfilled and the system $\left\{\psi_k(x)\|\psi_k\|_2^{-1}\right\}_{k=1}^{\infty}$ be quadratically close to some basis $\{u_k(x)\}_{k=1}^{\infty}$ of the space $L_2^{2m}(G)$.*

*Then the systems $\left\{\psi_k(x)\|\psi_k\|_2^{-1}\right\}_{k=1}^{\infty}$ and $\left\{\varphi_k(x)\|\psi_k\|_2\right\}_{k=1}^{\infty}$ are the bases in $L_2^{2m}(G)$, are equivalent to the basis $\{u_k(x)\}_{k=1}^{\infty}$ and its biorthogonal conjugate system, respectively.*

In 1.4 we study componentwise uniform equiconvergence on a compact with a trigonomterik series of expansions of $2m$-component vector-functions in orthogonal series in eigen vector-functions of the Dirac type operator

$$D\psi = B\frac{d\psi}{dx} + P(x)\psi, \quad x \in G = (0,\pi), \tag{7}$$

where $B = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ or $B = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}$, $I$ is a unit operator in $E^m$, $J = \left(\alpha_{ij}\right)_{i,j=1}^{m}$, $\alpha_{k,\,m-k+1} = 1$, $k = \overline{1,m}$; $\alpha_{ij} = 0$ for $(i,j) \neq (k, m-k+1)$, $k = \overline{1,m}$; $P(x) = diag\ (p_1(x), p_2(x), \ldots, p_{2m}(x))$; $\psi = \left(\psi^1, \psi^2, \ldots, \psi^{2m}\right)^T$; $p_l(x), l = \overline{1,2m}$ are real functions from $L_p(0,\pi)$, $p > 2$. A theorem on componentwise uniform equiconvergence and componentwise localization principle, is proved.

Let $\{\psi_k(x)\}_{k=1}^{\infty}$ be a complete system orthonormed in $L_2^{2m}(G)$ and consisting of eigen vector-functions of the operator $D$, while $\{\lambda_k\}_{k=1}^{\infty}$, $\lambda_k \in R$, be an appropriate system of eigen values.

For arbitrary $f(x) \in L_2^{2m}(G)$ we introduce a partial sum of its spectral expansion in the system $\{\psi_k(x)\}_{k=1}^{\infty}$:

$$\sigma_\nu(x,f) = \left(\sigma_\nu^1(x,f), \sigma_\nu^2(x,f), \dots, \sigma_\nu^{2m}(x,f)\right)^T,$$

where

$$\sigma_\nu^j(x,f) = \sum_{|\lambda_k| \le \nu} (f, \psi_k)\, \psi_k^j(x),$$

$$\psi_k(x) = \left(\psi_k^1(x), \psi_k^2(x), \dots, \psi_k^{2m}(x)\right)^T,$$

$$f(x) = \left(f_1(x), f_2(x), \dots, f_{2m}(x)\right)^T.$$

Along with the partial sum $\sigma_\nu(x,f)$ we define the operator

$$S_\nu(x,f) = \left(S_\nu(x,f_1), S_\nu(x,f_2), \dots, S_\nu(x,f_{2m})\right)^T,$$

where $S_\nu(x,f_j)$, $j=\overline{1,2m}$, is a modified partial sum of trigonometric series of the function $f_j(x)$, i.e.

$$S_\nu(x,f_j) = \frac{1}{\pi}\int_G \frac{\sin\nu(x-y)}{x-y}\, f_j(y)\,dy, \qquad \nu > 0.$$

The main results of this section are in the following two theorems.

***Theorem 5.*** *Let the functions* $p_l(x)$, $l=\overline{1,2m}$, *belong to the class* $L_p(G)$, $p>2$. *Then for arbitrary vector-function* $f(x) \in L_2^{2m}(G)$ *on any compact* $K \subset G$ *the following equality is valid:*

$$\lim_{\nu \to +\infty} \left\| \sigma_\nu^j(\cdot,f) - S_\nu(\cdot,f_j) \right\|_{C(K)} = 0, \quad j=\overline{1,2m}, \tag{8}$$

*i.e. the* $j$ *-th component of the expansion of the vector-function* $f(x) \in L_2^{2m}(G)$ *in orthogonal series in the system* $\{\psi_k(x)\}_{k=1}^{\infty}$

*uniformly equiconverges on any compact* $K \subset G$ *with expansion in trigonometric Fourier series of the corresponding* $j$ *-th component of* $f_j(x)$ *of the vector-function* $f(x)$.

Allowing for localization principle, for trigonometric Fourier series theorem 5 yields a theorem on componentwise localization.

***Theorem 6.*** *Let the conditions of theorem 5 be fulfilled. Then for orthogonal expansion of arbitrary function* $f(x) \in L_2^{2m}(G)$ *in the system* $\{\psi_k(x)\}_{k=1}^{\infty}$ *componentwise localization principle is valid. Convergence or divergence of the* $j$ *-th component of this expansion at the point* $x_0 \in G$ *depends on behavior of this point in small vicinity of this point* $x_0$ *only of corresponding* $j$ *-th component* $f_j(x)$ *of the expanded vector-function* $f(x)$ *(and is independent on behavior of other components).*

In **chapter II** we consider the Dirac type operator

$$D_1 y = By' + P(x)y \ ,$$
$$y(x) = (y_1(x), \ y_2(x))^T ,$$

$B = \begin{pmatrix} 0 & b_1 \\ b_2 & 0 \end{pmatrix}$, $b_2 < 0 < b_1$ , $P(x) = diag\ (p_1(x), \ p_2(x))$ on the interval $G = (a,b)$, $mesG < \infty$. Estimations between different $L_p^2$-norms of the vector-functions of this operator are proved. Bessel property and unconditional basicity criterion in $L_2^2(G)$ of the system of root vector-functions of the operator $D_1$ is established.

In 2.1 we establish estimations between $L_\infty^2$ norms of two neighboring root vectors and between $L_\infty^2$ and $L_p^2$, $1 \le p < \infty$ - norms of one the same root vector of this operator. This section consists of two subsections. In subsection 2.1.1 we establish shift formulas and mean value formulas for the root vector-functions of the operator $D_1$ .

***Lemma 1. (see [30]). (Shift formula).*** *If the functions* $p_1(x)$ *and* $p_2(x)$ *belong to the class* $L_1^{loc}(G)$ *and the points* $x-t,\ x,\ x+t$ *are in the domain* $G$*, then the following formulas are valid:*

$$\overset{l}{u}(x+t)=\left[\cos\frac{\lambda}{\sqrt{|b_1b_2|}}\,t\,I-\sin\frac{\lambda}{\sqrt{|b_1b_2|}}\,t\,\frac{B}{\sqrt{|b_1b_2|}}\right]\overset{l}{u}(x)+$$

$$+B^{-1}\int_x^{x+t}\left(\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-\xi+x)\frac{B}{\sqrt{|b_1b_2|}}-\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-\xi+x)I\right)\times$$

$$\times\left[P(\xi)\overset{l}{u}(\xi)-\overset{l-1}{u}(\xi)\right]d\xi\,, \qquad (9)$$

$$\overset{l}{u}(x-t)=\left[\cos\frac{\lambda}{\sqrt{|b_1b_2|}}\,t\,I+\sin\frac{\lambda}{\sqrt{|b_1b_2|}}\,t\,\frac{B}{\sqrt{|b_1b_2|}}\right]\overset{l}{u}(x)+$$

$$+B^{-1}\int_{x-t}^{x}\left(\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t+\xi-x)\frac{B}{\sqrt{|b_1b_2|}}+\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t+\xi-x)I\right)\times$$

$$\times\left[P(\xi)\overset{l}{u}(\xi)-\overset{l-1}{u}\ \xi\right]d\xi\,, \qquad (10)$$

$$\overset{l}{u}(x+t)+\overset{l}{u}(x-t)=2\overset{l}{u}(x)\cos\frac{\lambda}{\sqrt{|b_1b_2|}}\,t+$$

$$+B^{-1}\int_{x-t}^{x+t}\left(\sin\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)\frac{B}{\sqrt{|b_1b_2|}}-sgn(\xi-x)\cos\frac{\lambda}{\sqrt{|b_1b_2|}}(t-|x-\xi|)I\right)\times$$

$$\times\left[P(\xi)\overset{l}{u}(\xi)-\overset{l-1}{u}(\xi)\right]d\xi\,, \tag{11}$$

*when* $I$ *is a unit operator in* $E^2$*, while* $E^2$ *is a two-dimensional Euclidean space*·

Based on the formulas (9)-(11) in the subsection 2.1.2 we establish estimates for the root vector-functions of the operator $D_1$.

***Theorem 7.*** *Let the functions* $p_1(x)$, $p_2(x)$ *belong to the class* $L_1^{loc}(G)$. *Then for any compact* $K\subset G$ *there exist such constants* $C^i(K,l,b_1,b_2), i=1,2$, $l=0,1,2,...,$ *independent of* $\lambda$ *that the following estimations are valid:*

$$\left\|\overset{l-1}{u}\right\|_{L^2_\infty(K)}\le C^1(K,l,b_1,b_2)\left(1+|Im\,\lambda|\right)\left\|\overset{l}{u}\right\|_{L^2_\infty(K)}, \tag{12}$$

$$\left\|\overset{l}{u}\right\|_{L^2_\infty(K)}\le C^2(K,l,b_1,b_2)\left(1+|Im\,\lambda|\right)^{\frac{1}{p}}\left\|\overset{l}{u}\right\|_{L^2_p(K)},\ \ 1\le p<\infty\cdot \tag{13}$$

**Remark 4.** If $G$ is a finite interval, $p_1(x)$ and $p_2(x)$ belong to the class $L_1(G)$, then the estimations (12) and (13) are valid in the case $K=\overline{G}$.

In 2.2 we establish the Bessel property and unconditional basicity criteria for the system of root vector-functions of the Dirac type operator $D_1$ in $L_2^2(G)$.

Let $\{u_k(x)\}_{k=1}^{\infty}$ be an arbitrary system consisting of eigen and associated functions of the operator $D_1$, $\{\lambda_k\}_{k=1}^{\infty}$ be an appropriate system of eigen values. Furthermore, the function $u_k(x)$ is included into the system $\{u_k(x)\}_{k=1}^{\infty}$ together with all appropriate associated functions of less order. This means that $D_1u_k=\lambda_ku_k+\theta_ku_{k-1}$, where $\theta_k$ either equals 0 (in the case $u_k(x)$ is an eigen vector-function), or 1 (in this case $u_k(x)$ is an associated vector-function, and $\lambda_k = \lambda_{k-1}$), $\theta_1 = 0$.

The main results of this section are in the following two theorems.

***Theorem 8. (Bessel property criterion).*** *Let $G$ be a finite interval, the functions $p_1(x)$ and $p_2(x)$ belong to the class $L_2(G)$, the lengths of the chains of root functions be uniformly bounded, and there exist such a constant $C_0$ that*

$$|\mathrm{Im}\lambda_k| \le C_0,\ k = 1,2,... \qquad (14)$$

*Then for the system of functions $\{\varphi_k(x)\}_{k=1}^{\infty}$, where $\varphi_k(x) = u_k(x)\|u_k\|_{2,2}^{-1}$, to satisfy the Bessel inequality, it is necessary an sufficient the existence of such a constant $K$ that*

$$\sum_{|\mathrm{Re}\,\lambda_k - \nu| \le 1} 1 \le K \qquad (15)$$

*where $\nu$ is an arbitrary real number.*

Denote by $D_1^*$ an operator formally conjugate to the operator $D_1$, i.e. $D_1^* = -B^* \dfrac{d}{dx} + P^*(x)$, where $P^*(x)$ is a matrix conjugate to the matrix $P(x)$.

***Theorem 9. (On unconditional basicity).*** *Let $G$ be a finite interval,, $\{u_k(x)\}_{k=1}^{\infty}$ be an arbitrary complete in $L_2^2(G)$ and minimal system consisting of eigen and associated functions of the operator $D_1$, the system $\{\upsilon_k(x)\}_{k=1}^{\infty}$, biorthogonally conjugated to $\{u_k(x)\}_{k=1}^{\infty}$ in $L_2^2(G)$ and consisting of eigen and associated functions of the operator $D_1^*$ (i.e. $D_1^*\upsilon_k = \overline{\lambda}_k\upsilon_k + \theta_{k+1}\upsilon_{k+1}$), be complete in $L_2^2(G)$, the length of any point of root vectors be uniformly bounded and condition (20) be fulfilled.*

*Then necessary and sufficient condition of unconditional basicity in* $L_2^2(G)$ *of the system* $\{u_k(x)\}_{k=1}^{\infty}$ *is the existence of the constants* $K$ *and* $M_1$*, providing validity of inequalities (15) and*

$$\|u_k\|_{2,2}\|\upsilon_k\|_{2,2} \le M_1 \qquad (16)$$

*for any* $k = 1,2$ .

Note that under the conditions of theorem 9, the fulfillment of inequalities (15) and (16) is a necessary and sufficient condition for the Riesz basicity of the system $\left\{u_k(x)\|u_k\|_{2,2}^{-1}\right\}$ in $L_2^2(G)$.

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## CONSLUSIONS

- A shift formula was derived and the estimations between different $L_p^{2m}$ norms of root (eigen and associated) vector functions of $2m$-th order Dirac type operator, where established.
- Bessel property criterion of the system of root vector-functions of $2m$-th order Dirac type operator was established in the space $L_2^{2m}(0,2\pi)$.
- Unconditional basicity criterion of the system of root vector-functions of $2m$-th order Dirac type operator was established in the space $L_2^{2m}(0,2\pi)$.
- A theorem on componentwise uniform equiconvergence of a compact with trigonometric series of expansions of vector-functions from the class $L_2^{2m}(0,\pi)$ in orthogonal series in eigen vector-functions of $2m$-th order Dirac type operator was proved.
- Estimations between different $L_p^2$-norms of the root vector-functions of a Dirac type operator were proved, and Bessel property and unconditional basicity criteria $L_2^2(G)$, $G=(a,b)$, $mesG<\infty$ of the system of root vector-functions of the given operator were established.

**Main results of the dissertation were published in the following papers:**

1.Kurbanov, V.M., Ismailova, A.I., Hajiyeva, G.R. Basicity of the systems of root vector-functions of one dimensional Dirac operator // Madea-7. Azerbaijan-Turkey-Ukrainian International conference. Mathematical Analysis, Differential Equations and their applications. -Baku: 2015. -pp.96-98.

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