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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

CONVERGENCE OF SPECTRAL EXPANSIONS FOR ORDINARY DIFFERENTIAL OPERATORS OF FOURTH ORDER WITH A SPECTRAL PARAMETER IN BOUNDARY CONDITIONS

Specialty: 1202.01 – Analysis and functional analysis

Field of science: Mathematics

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GENERAL CHARACTERISTICS OF THE WORK

Rationale of the topic and development degree.

One of the most important branches of modern mathematics is the spectral theory of ordinary differential operators with a spectral parameter in boundary conditions. Boundary value problems that include a spectral parameter both in the equation and in the boundary conditions arise when constructing a mathematical model of many problems in mechanics, physics and other areas of natural science. The presented thesis is devoted to the study of the uniform convergence of spectral expansions in a system of root functions of ordinary differential operators of fourth order whose boundary conditions contain a spectral parameter.

The uniform convergence of expansions in Fourier series in the system of root functions of Sturm-Liouville problems with a spectral parameter in boundary conditions was studied in the works of N.Yu. Kapustin and E.I. Moiseev, N.Yu. Kapustin, D.A. Gulyaev, N.B. Kerimov and E.A. Maris, N.B. Kerimov, S. Goktas and E.A. Maris, S. Goktas and E.A. Maris.

The spectral properties of ordinary fourth-order differential operators with a spectral parameter in boundary conditions were studied by V. Stenberg, H. Berner, H.J. An, S.V. Meleshko and Yu.V. Pokorny, E.M.E. Zaid and S.F.M. Ibrahim, J. Ben Amara and A.A. Vladimirov, N.B. Kerimov and Z.S. Aliyev[1], Z.S. Aliev[2,3], Z.S.Aliev and S.B.Gulieva, Z.S.Aliev, N.B.Kerimov and V.A.Mehrabov, S. Gao, S. Li and R. Man, S. Gao and M. Ran, Z.S.

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[1] Kerimov N.B., Aliev Z.S. On the basis property of the system of eigenfunctions of a spectral problem with spectral parameter in the boundary condition // Differential Equations, – 2007. v. 43, no. 7, – p. 905-915.

[2]Aliyev, Z.S. Basis properties of a fourth order differential operator with spectral parameter in the boundary condition // Cent. Eur. J. Math., - 2010. v. 8, no 2, - p. 378-388

[3] Aliev, Z.S. Basis properties in L_p of systems of root functions of a spectral problem with spectral parameter in a boundary condition // Differential Equations, – 2011. v. 47, no. 6, – p. 766–777

Aliev and F.M. Namazov, V.A. Mehrabov, Z.S. Aliyev and G.T. Mamedova, J. Qin, Q. Li, Z. Zheng, J. Cai. Basis properties of root functions of problems of this type studied in the works of N.B. Kerimov and Z.S. Aliyev, Z.S. Aliyev, Z.S. Aliyev and S.B. Guliyeva, Z.S. Aliyev and F.M. Namazov, Z.S. Aliyev, N.B. Kerimov and V.A. Mehrabov, Z.S. Aliyev and G.T. Mammadova, where the sufficient conditions for the basisity of subsystems of the system of root functions in the space L_p, $1 < p < \infty$, are established.

The uniform convergence of expansions in Fourier series in a system of root functions of ordinary differential operators of fourth order was studied by V.M. Kurbanov and Y.I. Huseynova[4], V.M. Kurbanov[5] only in the case when the boundary conditions do not depend on the spectral parameter.

Thus, the study of expansions in Fourier series in the system of root functions of spectral problems for fourth-order ordinary differential equations with a spectral parameter in boundary conditions is important and relevant from both practical and theoretical points of view.

Object and subject of the study.

The object of the research is the ordinary differential operators of fourth order with spectral parameter in the boundary conditions, and the subject is the uniform convergence of spectral expansions in the the system of root functions.

Goal and tasks of the study.

The main goal and objective of the dissertation work is to study the structure of root subspaces and oscillatory properties of eigenfunctions, obtain refined asymptotic formulas for eigenvalues

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[4] Kurbanov, V.M. Conditions for the absolute and uniform convergence of the biorthogonal series corresponding to a differential operator // Doklady Mathematics, - 2008, v. 78, no. 2, - p. 748-750

[5] Kurbanov, V.M., Huseynova, Y.I. On convergence of spectral expansion of absolutely continuous vector-function in eigenvector-functions of fourth order differential operator // Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics–Baku: – 2014. v. 34, no. 1, – p. 83–90.

and eigenfunctions, study the basic properties in the L_p, $1 < p < \infty$, p space of subsystems of the system of root functions and the uniform convergence of spectral expansions in these subsystems.

İnvestigation methods.

The dissertation work uses methods of mathematical analysis, differential equations, functional analysis, complex analysis, theory of operators in a space with an indefinite metric, spectral theories of linear differential operators.

Basic statements to be defended.

The main propositions in the defense of the thesis are as follows: for the ordinary differential operators of fourth order with spectral parameter in the third of the boundary conditions

– the study of the structure of root subspaces;

– giving a general characteristic of the location of the eigenvalues on the real axis;

– determination of the number of zeros in the interval of all eigenfunctions;

– obtaining refined asymptotic formulas for eigenvalues and eigenfunctions;

– determining the conditions for the system of root functions after removing one function to form a basis in the space $L_p, 1 < p < \infty$;

– finding conditions for the uniform convergence of spectral expansions in a system of root functions after removing one function;

for the ordinary differential operators of fourth order with spectral parameter in the fourth of the boundary conditions

– obtaining refined asymptotic formulas for eigenvalues and eigenfunctions;

– finding conditions for the uniform convergence of spectral expansions in a system of root functions after removing one function.

Scientific novelty of the study.

The following main results were obtained in the dissertation work: for the ordinary differential operators of fourth order with spectral parameter in the third of the boundary conditions

– the structure of rooted subspaces is studied;;

– the general characteristic of the location of the eigenvalues on the real axis is given;

– the number of zeros contained in the interval of all eigenfunctions is defined;

– refined asymptotic formulas for eigenvalues and eigenfunctions were obtained;

– conditions have been determined under which the system of root functions, after removing one function, forms a basis in the space $L_p, 1 < p < \infty$;

– conditions were found for the uniform convergence of spectral expansions in the system of root functions after removing one function;

for the ordinary differential operators of fourth order with spectral parameter in the fourth of the boundary conditions;

– refined asymptotic formulas for eigenvalues and eigenfunctions were obtained;

– conditions were found for the uniform convergence of spectral expansions in the system of root functions after removing one function.

Theoretical and practical value of the study.

The results obtained in the dissertation are mainly theoretical in nature. They can be used in the study of various issues in the spectral theory of differential operators, in the study of various processes in mechanics and physics.

Approbation and application.

The results obtained in the dissertation by the author were reported at seminars held at the Sumgait State University at the Department of Mathematical analysis and theory of functions (headed by Prof. N.T. Kurbanov), at the Baku State University at the Department of Mathematical Analysis (headed by Prof. R.A. Aliyev), at the Khazar University at the Department of Mathematics (headed by Prof. N.B. Kerimov), at the Departments of Differential Equations (headed by Prof. A.B. Aliyev) and Functional analysis (headed by Prof. H.I. Aslanov) of the Institute of Mathematics and

Mechanics of the Ministry of Science and Education of the Republic of Azerbaijan, at the International seminar "Spectral theory and its applications" dedicated to the 80th anniversary of Academician Mirabbas Gasimov (Baku, 2019), at the International Voronezh Winter Mathematical School "Modern Methods of Function Theory and Related Problems" (Voronezh, Russia, 2021), at the international scientific conference "Herzen’s Readings - 2021" on "Some Current Problems of Modern Mathematics and Mathematical Education" (Russia, St. Petersburg, 2021) and at the Republican scientific conference on "Actual problems of mathematics and mechanics" dedicated to the 99th anniversary of the birth of Heydar Aliyev, the national leader of the Azerbaijani people were reported .

The author's personal contribution. All research results obtained belong to the author.

Author’s publications.

The author's publications in scientific journals recommended by the Higher Attestation Commission under the President of the Republic of Azerbaijan are 5 (including 2 WOS, 2 SCOPUS), conference materials are 4 (3 international conferences, 1 republican, 2 of which were held abroad).

The institution where the dissertation work was performed.

The dissertation work was performed at the "Mathematical analysis and theory of functions" department of Sumgayit State University.

Structure and volume of the dissertation (in signs indicating the volume of each structural subdivision separately).

Total volume of the dissertation work– 206011 signs (title page – 394 signs, contents – 2658 signs, introduction – 62959 signs, chapter I – 78000 signs, chapter II – 60000 signs, conclusion – 2000 signs). The list of references consists of 79 names.

THE MAIN CONTENT OF THE WORK

The dissertation work consists of introduction, 2 chapters, conclusion and a list of references.

The first chapter examines the eigenvalue problem for fourth-order ordinary differential equations, in which the spectral parameter is included in the third of the boundary conditions. A general characteristic of the location of eigenvalues on the real axis (complex plane) is given, the structure of root subspaces and the oscillatory properties of eigenfunctions and basis properties in the space $L_p,\ 1 < p < \infty,$ are studied. Moreover, the uniform convergence of expansions in Fourier series in eigenfunctions of the problem under consideration is studied.

In 1.1, the setting of the issue is explained.

Consider the following boundary value problem

$$\ell(y)(x) \equiv y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ 0 < x < l, \tag{1}$$

$$y'(0)\cos\alpha - y''(0)\sin\alpha = 0, \tag{2}$$

$$y(0)\cos\beta + Ty(0)\sin\beta = 0, \tag{3}$$

$$(a\lambda + b)y'(l) + (c\lambda + d)y''(l) = 0, \tag{4}$$

$$y(l)\cos\delta - Ty(l)\sin\delta = 0, \tag{5}$$

where $\lambda \in C$ is a spectral parameter, $Ty \equiv y''' - qy',$ $q(x)$ is a positive absolutely continuous function on $\alpha, \beta, \delta, a, b, c, d$ are real constants such that $0 \le \alpha, \beta \le \pi/2,$ $\pi/2 \le \delta < \pi$ (except the case $\beta = \delta = \pi/2$), $\sigma = bc - ad > 0.$

Note that problem (1)-(5) arises when describing small bending vibrations of a homogeneous elastic cantilever beam with alongitudinal force acting in the cross-sections and with a load attached to the free end by a weightless rod and held in equilibrium by an elastic spring.

The eigenvalue problem (1)-(5) in the case of $\alpha = \beta = 0$ was studied in the work of Z.S. Aliyev[2]. In that work, it was shown that the eigenvalues of this problem are real and simple and form an infinitely increasing sequence. In addition, the oscillation properties of eigenfunctions were studied, asymptotic formulas for the eigenvalues and eigenfunctions were obtained, and the basis property of the eigenfunction system of this problem without one arbitrarily chosen function was established in the space $L_p(0,l),\ 1<p<\infty$.

In 1.2 gives an operator interpretation of problem (1)-(5) and some auxiliary information.

Consider the following boundary condition

$$y'(l)\cos\gamma + y''(l)\sin\gamma = 0, \qquad (6)$$

where $\gamma \in [0,\ \pi/2]$.

Problem (1)- (3), (6), (5) for $\delta \in [0,\pi)$ it was investigated in the works of D.O. Banks and G.J. Kurowski[6], N.B. Kerimov and Z.S. Aliyev[7]. In these works it is shown that the eigenvalues of problem (1)-(3), (6), (5) with $\alpha,\beta,\gamma \in [0,\pi/2]$ and $\delta \in [0,\pi)$ are real and simple and form an infinitely increasing sequence $\{\lambda_k(\alpha,\beta,\gamma,\delta)\}_{k=1}^{\infty}$ such that $\lambda_k(\alpha,\beta,\gamma,\delta)>0$ for $k \ge 2$, and there is an angle $\delta_0(\alpha,\beta,\gamma) \in [\pi/2,\pi)$ such that $\lambda_1(\alpha,\beta,\gamma,\delta)>0$ if $\delta \in [0,\delta_0(\alpha,\beta,\gamma))$, $\lambda_1(\alpha,\beta,\gamma,\delta)=0$ if $\delta=\delta_0(\alpha,\beta,\gamma))$, and $\lambda_1(\alpha,\beta,\gamma,\delta)<0$ if $\delta \in (\delta_0(\alpha,\beta,\gamma),\pi)$. Moreover, the eigenfunction $y_k(\alpha,\beta,\gamma,\delta)$ corresponding to the eigenvalue $\lambda_k(\alpha,\beta,\gamma,\delta)$ has exactly $k-1$ simple zeros for $k \ge 2$; for $k=1$ it does not have zeros if $\delta \in [0,\delta_0(\alpha,\beta,\gamma)]$ and has an arbitrary number of zeros in the interval $(0,\ \pi)$ if $\delta \in (\delta_0(\alpha,\beta,\gamma),\pi)$.

To study the spectral properties of problem (1)-(5), we will study the properties of the solution of the initial-boundary value problem (1)-(3), (5).

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[6] Banks, D.O., Kurowski, G.J. A Prüfer transformation for the equation of a vibrating beam subject to axial forces // J. Differential Equations, – 1977. v. 24 no. 1, – p. 57-74.

[7] Kerimov, N.B., Aliyev, Z.S. On oscillation properties of the eigenfunctions of a fourth order differential operator // – Baku: Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., – 2005, v. 25, no. 4, – p. 63–76.

Lemma 1. *For each* $\lambda \in C$, *there exists a unique (up to a constant factor) nontrivial solution* $y(x,\lambda)$ *of problem (1)-(3), (5).*

Corollary 1. For each fixed $x \in [0, l]$ the function $y(x,\lambda)$ is an entire function of parameter λ.

Denote $\mu_k = \lambda_k(\alpha, \beta, 0, \delta)$ and $\nu_k = \lambda_k(\alpha, \beta, \pi/2, \delta)$. Note than the following relation holds:

$$\nu_1 < \mu_1 < \nu_2 < \mu_2 < \ldots < \nu_k < \mu_k < \ldots \ . \tag{7}$$

Let $m(\lambda) = ay'(l) + cy''(l)$.

Lemma 2. *If* λ *is an eigenvalue of problem (1)-(5), then* $m(\lambda) \neq 0$.

In subsection 1.3, the oscillatory properties of solutions of problem (1)-(3), (5) are studied. Consider the following equation

$$y(x,\lambda) = 0, \ x \in [0, l], \ \lambda \in R. \tag{8}$$

Obviously, each solution of this equation is a function of the parameter λ.

Lemma 3. *Each root* $x(\lambda) \in (0, l)$ *of equation (8) is a simple continuously differentiable function of the parameter* λ.

Corollary 2. *As the parameter* λ *varies,* $\lambda > 0$ *(respectively,* $\lambda \leq 0$*), the function* $y(x,\lambda)$ *may lose a zero or gain a new one if it is inside the interval* $(0,l)$ *or outside it beyond the boundary point* $x = l$ *(respectively,* $x = 0$*).*

Let $s(\lambda)$, $\lambda \in R$, be the number of zeros of the function $y(x,\lambda)$ contained in the interval $(0, l)$. According to the above arguments about the oscillatory properties of problem (1)-(3), (6), (5), Lemmas 1, 3 and Corollary 2, the following oscillatory theorem holds for the function $y(x,\lambda)$ for $\lambda > 0$.

Teorem 1. *If* $\lambda \in (\mu_{k-1}, \nu_k)$ *for* $k \geq 3$, *then* $k-2 \leq s(\lambda) \leq k-1$; *if* $\lambda \in [\nu_k, \mu_k]$ *for* $k \geq 3$, *then* $s(\lambda) = k-1$. *Moreover, if* $\delta \in [0, \delta_0(\alpha, \beta, \pi/2))$, *then* $s(\lambda) = 0$ *for* $\lambda \in [0, \mu_1]$, $0 \leq s(\lambda) \leq 1$ *for* $\lambda \in (\mu_1, \nu_2)$ *and* $s(\lambda) = 1$ *for* $\lambda \in [\nu_2, \mu_2]$; *if* $\delta \in [\delta_0(\alpha, \beta, \pi/2), \ \delta_0(\alpha, \beta, 0))$, *then* $0 \leq s(\lambda) \leq 1$ *for* $\lambda \in [0, \nu_2]$

and $s(\lambda)=1$ *for* $\lambda\in[\nu_2,\mu_2]$; *if* $\delta\in[\delta_0(\alpha,\beta,0),\pi)$, *then* $s(\lambda)=1$ *for* $\lambda\in[0,\mu_2]$.

Let $\varepsilon>0$ be a sufficiently small given number, let μ be a real eigenvalue of Eq. (1) with the boundary conditions $y(0)=y'(0)$ $=y''(0)=0$, (5) if $\beta=0$ and with the boundary conditions (2), $y(0)=Ty(0)=0$, and (5) if $\beta\in(0,\pi/2]$. The oscillation index of the eigenvalue μ is the difference between the numbers of zeros of the function $y(x,\lambda)$ on the interval $(0,l)$ for $\lambda\in(\mu-\varepsilon,\mu)$ and for $\lambda\in(\mu,\mu+\varepsilon)$. It can be seen from this definition that the number of zeros of $y(x,\lambda)$ on the interval $(0,l)$ is equal to the sum of the oscillation indices of all eigenvalues on the interval $(\lambda,0)$ of problem (1), $y(0)=y'(0)=y''(0)=0$, (5) if $\beta=0$ and problem (1), (2), $y(0)=Ty(0)=0$, and (5) if $\beta\in(0,\pi/2]$.

Let $i(\varsigma_k)$, $k\in\mathrm{N}$, be the oscillation index of the eigenvalue ς_k of problem (1), $y(0)=y'(0)=y''(0)=0$, (5) if $\beta=0$ and problem (1), (2), $y(0)=Ty(0)=0$, and (5) if $\beta\in(0,\pi/2]$. Then the preceding implies that the number of zeros of the function $y(x,\lambda)$ on the interval $(0,l)$ for $\lambda<0$ is determined by the formula

$$s(\lambda)=\sum_{\varsigma_k\in(\lambda,0)} i(\varsigma_k) \tag{9}$$

In Section 1.4, the multiplicities of the eigenvalues of problem (1)-(5) are determined, a general characteristic of their location on the real axis is given, and the oscillatory properties of the corresponding eigenfunctions are investigated.

In the case of $a\neq 0$ $(c\neq 0)$, we determine the natural number k_a (k_c) by the following inequality:

$$\nu_{k_a-1}\leq -b/a<\nu_{k_a}\quad (\mu_{k_c-1}<-d/c\leq\mu_{k_c}).$$

Remark 1. If $ac\neq 0$, then $k_a\leq k_c+1$ for $ac>0$ and $k_a\geq k_c$ for $ac<0$.

Theorem 2. *The eigenvalues of problem (1)-(5) are real and* *simple and form an unboundedly increasing sequence* $\{\lambda_k\}_{k=1}^{\infty}$ *such that* $\lambda_k > 0$ *for* $k \geq 3 + \operatorname{sgn}|c|$.

In this subchapter, the oscillatory properties of the eigenfunctions of problem (1)-(5) are completely studied. For simplicity, we will only interpret these properties here for the case $c = 0$.

Teorem 3. *Let* $c = 0$. *Then the eigenfunction* $y_k(x)$ *corresponding to the eigenvalue* λ_k, $k \in \mathrm{N}$, *of problem (1)-(5) (where* $k \geq 1$ *if* $\delta \leq \delta_0(\alpha, \beta, \pi/2)$ *and* $k_a \geq 2$; $k \geq 2$ *if* $\delta \leq \delta_0(\alpha, \beta, \pi/2)$ *and* $k_a = 1$, *or* $\delta_0(\alpha, \beta, \pi/2) < \delta \leq \delta_0(\alpha, \beta, 0)$, *or* $\delta > \delta_0(\alpha, \beta, 0)$ *and* $k_a \geq 3$; $k \geq 3$ *if* $\delta > \delta_0(\alpha, \beta, 0)$ *and* $k_a \leq 2$) *has exactly* $k-1$ *simple zeros for* $k < k_a$ *and has either* $k-2$ *or* $k-1$ *zeros for* $k \geq k_a$; *the function* $y_1(x)$ *for* $\delta < \delta_0(\alpha, \beta, 0)$ *either has no zeros or has* $s(\lambda_1) = \sum_{\varsigma_k \in (\lambda_1, 0)} i(\varsigma_k)$ *simple zeros; for* $\delta \geq \delta_0(\alpha, \beta, 0)$ *it has* $s(\lambda_1) = \sum_{\varsigma_k \in (\lambda_1, 0)} i(\varsigma_k)$ *simple zeros; the function* $y_2(x)$ *for* $\delta > \delta_0(\alpha, \beta, 0)$ *and* $k_a \leq 2$ *either has* $s(\lambda_2) = \sum_{\varsigma_k \in (\lambda_2, 0)} i(\varsigma_k)$ *simple zeros, or has no zeros, or has one simple zero in the interval* $(0, l)$.

Consider the following boundary condition

$$ay'(l) + cy''(l) = 0. \tag{10}$$

In 1.5, asymptotic formulas are obtained for the eigenvalues and eigenfunctions of problem (1)-(5) and problem (1)-(3), (10), (5) for $q \equiv 0$. For simplicity, we note from these formulas those that correspond to the case $\alpha = 0$, $c = 0$.

Theorem 4. *Let* $q \equiv 0$, $\alpha = 0$ *and* $c = 0$.*Then the following asymptotic formulas for the eigenvalues and eigenfunctions of problem (1)-(3), (10), (5) hold:*

$$\sqrt[4]{\tau_k} = \left(k - (1 + 3\operatorname{sgn}\beta)/4\right)\pi/l + O\left(1/k^2\right), \tag{11}$$

$$\vartheta_k(x) = \sqrt{(1+\operatorname{sgn}\beta)/l}\ \{(1-\operatorname{sgn}\beta)\sin\sqrt[4]{\tau_k}\,x - (-1)^{\operatorname{sgn}\beta}\cos\sqrt[4]{\tau_k}\,x +$$

$$+(1-\operatorname{sgn}\beta)\, e^{-\sqrt[4]{\tau_k}\, x} + O\left(1/k^2\right)\Big\}, \tag{12}$$

where relations (12) holds uniformly for $x \in [0,\ l]$.

Remark 2. For each $k \in \mathrm{N}$, let $\Psi_k(x)$ denote the normalized eigenfunction corresponding to eigenvalue τ_k of problem (1)-(3), (10), (5) with $q(x) \equiv 0$, i.e. $\Psi_k(x) = \vartheta_k / \| \vartheta_k \|_2$.

We define the number q_0 and the function $q_0(x),\ x \in [0,\ l]$, accordingly as follows:

$$q_0 = \int_0^l q(x)dx \quad \text{and} \quad q_0(x) = \int_0^x q(t)dt.$$

Theorem 5. *Let* $\alpha = 0,\ c = 0$ *in Eq. (1). Then the following asymptotic formulas for the eigenvalues and eigenfunctions of problem (1)-(5) hold:*

$$\sqrt[4]{\lambda_k} = \left(k - (5 + 3\operatorname{sgn}\beta)/4\right)\pi/l + q_0/4k\pi + O\left(1/k^2\right), \tag{13}$$

$$y_k(x) = \sqrt{(1+\operatorname{sgn}\beta)/l}\ \Big\{(1-\operatorname{sgn}\beta)\sin\sqrt[4]{\lambda_k}\, x - (-1)^{\operatorname{sgn}\beta}\cos\sqrt[4]{\lambda_k}\, x +$$

$$(1-\operatorname{sgn}\beta)\ e^{-\sqrt[4]{\lambda_k}\, x} +$$

$$(-1)^{\operatorname{sgn}\beta}\,(((1-\operatorname{sgn}\beta)\, q_0 - q_0(x))/4\sigma_k)\ \sin\sqrt[4]{\lambda_k}\, x +$$

$$+((q_0 + (1-\operatorname{sgn}\beta)\, q_0(x))/4\sigma_k)\cos\sqrt[4]{\lambda_k}\, x +$$

$$+\ (1-\operatorname{sgn}\beta)\ ((q_0 - q_0(x))/4\sigma_k)\, e^{-\sqrt[4]{\lambda_k}\, x} + O\left(1/k^2\right)\Big\}, \tag{14}$$

where relations (12) holds uniformly for $x \in [0,\ l]$.

In subsection 1.6 we study the basiы properties in the space $L_p, 1 < p < \infty,$ of subsystems of the system of eigenfunctions of problem (1)–(5).

Theorem 6. *Let* r *be an arbitrary fixed natural number. Then the system* $\{y_k\}_{k=1,\, k\neq r}^{\infty}$ *of eigenfunctions of problem (1)-(5) is a basis in the space* $L_p(0,\ l),\ 1 < p < \infty,$ *which is an unconditional basis for* $p = 2$. *In addition, the system* $\{u_k\}_{k=1,\, k\neq r}^{\infty}$ *adjoint to the system* $\{y_k\}_{k=1,\, k\neq r}^{\infty}$ *is determined by the formula*

$$u_k(x) = \delta_k^{-1}\{y_k(x) - m_k m_r^{-1} y_r(x)\},\ \ k \in \mathrm{N},\ k \neq r, \qquad (15)$$

where $\delta_k = \| y_k \|_2^2 + \sigma^{-1} m_k^2 > 0.$

In 1.7 we study the uniform convergence of Fourier series expansions in eigenfunctions of problem (1)-(5).

Let r be an arbitrary fixed natural number. Then by Theorem 6, Fourier series expansions

$$f(x) = \sum_{k=1, k\neq r}^{\infty} (f, u_k) y_k(x) \qquad (16)$$

of any function $f(x) \in C[0,\ l]$ in the system $\{y_k\}_{k=1,\, k\neq r}^{\infty}$ of eigenfunctions of problem (1)-(5) converges in the space $L_p(0,\ l)$, $1 < p < \infty$, and in $L_2(0,\ 1)$ this series converges unconditionally.

Theorem 7. *Let* r *be an arbitrary fixed natural number, and let* $f(x)$ *be a function that is continuous on the interval* $[0,\ l]$ *and has a uniformly converging Fourier series in the function system* $\{\Psi_k(x)\}_{k=1}^{\infty}$ *on the interval* $[0, l]$. *Then the series (16) converges uniformly on the interval* $[0, l]$.

The second chapter considers a problem describing bending vibrations of a homogeneous rod, in the cross section of which a longitudinal force acts, and a load is attached to the right end. This problem is an eigenvalue problem for ordinary differential equations of fourth order in which the fourth of the boundary conditions contains a spectral parameter. Here the conditions are established for the uniform convergence of spectral expansions in the system of eigenfunctions of this problem.

In 2.1, the formulation and physical meaning of the problem is explained.

Applying the variable separation method to the boundary value problem for the partial diferential equations of fourth order corresponding to the mechanics problem considered here, we obtain the following spectral problem:

$$y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ x \in (0,\ 1), \qquad (17)$$

$$y(0) = y'(0) = y''(1) = 0, \qquad (18)$$

$$(a\lambda + b)y(1) - Ty(1) = 0. \tag{19}$$

Here, the function $q(x)$ is an absolutely continuous function on the interval [0, 1], a,b are real constants such that $a > 0,\ b < 0.$

The spectral properties of problem (17)-(19) (when both the equation and the boundary conditions are more general), including the oscillatory and basis properties of eigenfunctions, were studied in detail in the work of Z.S. Aliyev[3]. The main goal of this chapter is to use these properties to study the uniform convergence of Fourier series expansions of a continuous function on [0, l] in the system of root functions of problem (17)-(19).

Section 2.2 provides some supporting information.

Consider the following boundary condition:

$$(a\lambda + b)y(1) - (c\lambda + d)Ty(1) = 0, \tag{20}$$

wher c is a real constant such that

$$\theta = bc - ad \neq 0. \tag{21}$$

Note that condition (19) is obtained from condition (20) for $c = 0,$ $d = 1$ and $\theta < 0.$ Note also that the spectral properties of problem (17), (18), (20) (in the case when condition (18) has a more general form), including the basis properties in the space $L_p(0,\ 1),\ 1 < p < \infty,$ of the system of root functions of this problem, for $\theta > 0$ were studied in detail in the work of N.B. Kerimov and Z.S. Aliev[1], and for $\theta < 0$ in the work of Z.S. Aliev[3]. In the case of $\theta > 0$ the eigenvalues of problem (17), (18), (20) are real and simple, and forms an infinitely increasing equence $\{\lambda_k\}_{k=1}^{\infty}$. In the case of $\theta < 0$ the eigenvalues of this problem forms an unboundedly nondecreasing sequence $\{\lambda_k\}_{k=1}^{\infty}$ such that (a) all eigenvalues are real and simple; (b) all eigenvalues are real and, with the exception of one double eigenvalue, are simple; (c) all eigenvalues are real and, with the exception of one triple eigenvalue, are simple; (d) all eigenvalues are simple, and with the exception of one pair of conjugate non-real eigenvalues, are real (in this case we will assume that $\lambda_1 \in C \setminus R$ and $\lambda_2 = \overline{\lambda_1}$).

It is well known that problem (17), (18), (20) can be reduced to the eigenvalue problem for a linear operator A in the Hilbert space $H = L_2(0, 1) \oplus C$ with scalar product

$$(\hat{y}, \hat{\vartheta})_H = (\{y, m\}, \{\vartheta, s\}) = \int_0^1 y(x)\,\overline{\vartheta(x)}\,dx + |\sigma|^{-1}\, m\bar{s},$$

where the operator

$$A\hat{y} = A\{y, m\} = \{(Ty)', d\,Ty(1) - by(1)\}$$

is defined on the domain

$$D(A) = \{\hat{y} = \{y, m\} \in H : y \in W_2^4(0, 1), (Ty)' \in L_2(0, 1),$$
$$y(0) = y'(0) = y''(1) = 0,\ m = ay(1) - cTy(1)\}.$$

Here problem (1)-(5) acquires the form

$$A\hat{y} = \lambda\hat{y}, \hat{y} \in D(A).$$

Note that in the case $\theta > 0$ the operator A is a self-adjoint lower semibounded operator with discrete spectrum on the space H. The system $\{\hat{y}_k\}_{k=1}^{\infty}$ $\hat{y}_k = \{y_k, m_k\}$, $m_k = ay_k(1) - cTy_k(1)\}$, of eigen-vectors of this operator is an orthogonal basis in H.

In the case $\theta < 0$ the operator A is a non-self-adjoint closed operator in H with a compact resolvent. In this case we introduce the operator $J : H \to H$ by $J\{y, m\} = \{y, -m\}$. This operator generate Pontryagin space $\Pi_1 = L_2(0, 1) \oplus C$ with the inner product

$$(\hat{y}, \hat{\vartheta})_{\Pi_1} = (\{y, m\}, \{\vartheta, s\})_{\Pi_1} = (J\hat{y}, \hat{y})_H = \int_0^1 y(x)\,\overline{\vartheta(x)}\,dx + \theta^{-1} m\bar{s}.$$

Note that the opertaor A is $J-$self-adjoint in Π_1. Moreover, the operator A^*, adjoint to the operator A in H, is determined as follows: $A^* = JAJ$; the system $\{\hat{y}_k\}_{k=1}^{\infty}$ of root vectors of the operator A forms a basis in H.

Let the system $\{\hat{\vartheta}_k^*\}_{k=1}^{\infty}$, $\hat{\vartheta}_k^* = \{\vartheta_k^*, s_k^*\}$, be a system of root vectors of the operator A^*. Then, each element $\hat{\vartheta}_k$, $k \in \mathrm{N}$, of the system $\{\hat{\vartheta}_k\}_{k=1}^{\infty}$, $\hat{\vartheta} = \{\vartheta_k, s_k\}$, which is adjoint to the system $\{\hat{y}_k\}_{k=1}^{\infty}$

, is defined by the formula $\hat{\vartheta}_k = \delta_k^{-1}\hat{\vartheta}_k^*$, where $\delta_k \neq 0$, $k \in \mathrm{N}$, are some numbers.

Note that the main results obtained in the work of N.B. Kerimov and Z.S. Aliyev[1], as well as in the work of Z.S. Aliyev[3] consists of the following: if r arbitrary fixed natural number and $\theta > 0$ or $\theta < 0$ and $s_r \neq 0$, then the system $\{y_k\}_{k=1,k\neq r}^{\infty}$ of root functions of problem (17), (18), (20) is a basis in the space $L_p(0, 1)$, $1 < p < \infty$, and for $p = 2$ this basis is an unconditional basis; if $\theta < 0$ and $s_r = 0$, then the system $\{y_k\}_{k=1,k\neq r}^{\infty}$ is incomplete and nonminimal in $L_p(0, 1)$, $1 < p < \infty$. Moreover, each element u_k, $k \in \mathrm{N}$, of the system $\{u_k\}_{k=1,k\neq r}^{\infty}$, adjoint to the system $\{y_k\}_{k=1,k\neq r}^{\infty}$ is defined by the formula

$$u_k(x) = \delta_k^{-1}\{\vartheta_k(x) - s_k s_r^{-1}\vartheta_r(x)\}.$$

Consider the following boundary condition:

$$y(1) = 0. \tag{22}$$

Along with problem (17)-(19) we consider problem (17), (18) and (22). This problem was studied in the work of D.O. Banks and G.J. Kurowski[6], where it was shown that its eigenvalues are real and simple, and form an infinitely increasing sequence $\{\mu_k\}_{k=1}^{\infty}$ such that $\mu_k > 0$ for any $k \in \mathrm{N}$.

It is known from the works of N.B. Kerimov and Z.S.Aliyev[1], and Z.S. Aliyev[3] that the following asymptotic formulas are true for the eigenpairs $(\mu_k, \vartheta_k(x))$ and $(\lambda_k, y_k(x))$ of problem (17), (18), (22) with $q \equiv 0$ and problem (17)-(19) respectively:

$$\sqrt[4]{\mu_k} = (k + 1/4)\pi + O(1/k), \tag{23}$$

$$\upsilon_k(x) = \sin(k + 1/4)\pi x - \cos(k + 1/4)\pi x + e^{-(k+1/4)\pi x} + O(1/k), \tag{24}$$

$$\sqrt[4]{\lambda_k} = (k - 3/4)\pi + O(1/k), \tag{25}$$

$$y_k(x) = \sin(k - 3/4)\pi x - \cos(k - 3/4)\pi x + e^{-(k-3/4)\pi x} + O(1/k), \tag{26}$$

where relations (24) and (26) hold uniformly for $x \in [0, 1]$.

Remark 3. Note that using asymptotic formulas (23)-(26) it is not possible to determine the conditions for uniform convergence of Fourier series expansions of the continuous function in the system that is obtained from the system of root functions of problem (17)-(19) after removing one function. Therefore, it is necessary to refine the asymptotic formulas (23)-(26) for the eigenvalues and eigenfuctions of problem (17), (18), (22) for $q \equiv 0$ and problem (17)-(19) to order $O(1/k^2)$.

In 2.3, refined asymptotic formulas for the eigenvalues and eigenfunctions of problem (17), (18), (22) (in the case $q \equiv 0$) and (17)-(19) are obtained.

Theorem 8. *For the eigenvalues and eigenfunctions of problem (17), (18), (22) with* $q \equiv 0$ *the following asymptotic formulas hold:*

$$\sqrt[4]{\mu_k} = (k + 1/4)\pi + O\left(1/e^{k\pi}\right), \tag{27}$$

$$\upsilon_k(x) = \sin\left(k + 1/4\right)\pi x - \cos\left(k + 1/4\right)\pi x + e^{-\left(k+\frac{1}{4}\right)\pi x} + O\left(1/e^{k\pi}\right), \tag{28}$$

where relation (28) holds uniformly for $x \in [0, 1]$.

Theorem 9. *For the eigenvalues and eigenfunctions of problem (17)-(19) the following asymptotic formulas hold:*

$$\sqrt[4]{\lambda_k} = (k - 3/4)\pi + (q_0 - 2/a)/4k\pi + O\left(1/k^2\right), \tag{29}$$

$$\begin{aligned} y_k(x) = {} & \sin\left(k - 3/4\right)\pi x - \cos\left(k - 3/4\right)\pi x + e^{-\left(k-\frac{3}{4}\right)\pi x} + \\ & (((q_0 - 2/a)\, x - - q_0(x)/4k\pi)\sin\left(k - 3/4\right)\pi x + \\ & + ((q_0 - 2/a)x - q_0(x)/4k\pi) \times \cos\left(k - 3/4\right)\pi x \\ & - ((q_0 - 2/a)\, x + q_0(x)/4k\pi)\, e^{-\left(k-\frac{3}{4}\right)\pi x} + O\left(1/k^2\right), \end{aligned} \tag{30}$$

where relation (30) holds uniformly for $x \in [0, 1]$.

In section 2.4, the uniform convergence of the expansions in the Fourier series of the continuous function on the subsystems of the system of root functions of the spectral problem (17)-(19) is studied.

Let $\Phi_k(x) = \upsilon_k(x) / \| \upsilon_k \|_2$, $x \in [0, 1]$, and let r be the arbitrary fixed natural number such that $s_r \neq 0$. Then for any continuous function $f(x)$ on the interval [0, 1], the Fourier series

$$\sum_{k=1, k\neq r}^{\infty} (f, u_k) y_k(x) \tag{31}$$

of this function in the system $\{y_k(x)\}_{k=1, k\neq r}^{\infty}$ of root functions of problem (17)-(19) converges in the space $L_p(0, 1), 1 < p < \infty,$ and for $p = 2$ this series converges unconditionally.

One of the main results of this chapter is the following theorem.

Theorem 10. *Let* r *be the arbitrary fixed natural number such that* $s_r \neq 0,$ $f(x) \in C[0, 1]$ *and the Fourier series of this function in the system* $\{\Phi_k(x)\}_{k=1}^{\infty}$ *uniformly converges on the interval* $[0, 1].$ *If* $(f, \vartheta_r) \neq 0,$ *then the series (31) uniformly converges on the interval* $[0, \chi]$ *for each* $\chi \in (0, 1),$ *if* $(f, \vartheta_r) = 0,$ *then this series uniformly converges on the interval* $[0, 1].$

In 2.5, we investigate the uniform convergence of spectral expansions in subsystems of the system of root functions of problem (17), (18), (20) for $c \neq 0.$

Consider the following boundary condition

$$ay(1) - cTy(1) = 0. \tag{32}$$

As it can be seen, problem (17), (18), (22) is obtained from problem (17), (18), (32) for $c = 0.$

Note that the problem (17), (18), (32) was studied in the work of Z.S. Aliyev[3], where it was shown that its eigenvalues are real and simple, and form an infinitely increasing sequence $\{\nu_k\}_{k=1}^{\infty}$ such that $\nu_k > 0$ for any $k \geq 2.$

As in the previous subsections, for the uniform convergence of the Fourier series expansions of any continuous function in the system of root functions of problem (17), (18), (20) for $c \neq 0,$ here we refine the known asymptotic formulas for the eigenvalues and eigenfunctions of problem (17), (18), (32) for $q \equiv 0$ and problem (17), (18), (20) up to order $O(1/k^2)$.

The following is one of the main results of this chapter.

Teorem 11. *Let* r *be the arbitrary fixed natural number such that* $s_r \neq 0$ *for* $\theta < 0$, $f(x)$ *be the continuous function on the interval* $[0, 1]$ *and the Fourier series expansion of this function in the system of* $\{\Phi_k(x)\}_{k=1}^{\infty}$ *converges uniformly on the interval* $[0, 1]$. *If* $(f, \vartheta_r) = 0$ *or* $(f, \vartheta_r) \neq 0$ *and* $d = 0$, *then the Fourier series (31) of the function* $f(x)$ *in the system* $\{y_k\}_{k=1, k\neq r}^{\infty}$ *uniformly converges on the interval* $[0, 1]$. *If* $(f, \vartheta_r) \neq 0$ *and* $d \neq 0$, *then series (31) uniformly converges on the interval* $[0, \chi]$ *for any* $\chi \in (0, 1)$.

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Conclusion

The presented dissertation work is devoted to the study of the uniform convergence of spectral expansions in subsystems of systems of root functions of ordinary differential operators of fourth order with a spectral parameter in boundary conditions.

The following main results were obtained in the dissertation work: for the ordinary differential operators of fourth order with spectral parameter in the third of the boundary conditions

– the structure of rooted subspaces is studied;

– the general characteristic of the location of eigenvalues on the real axis is given;

– the number of zeros contained in the interval of all eigenfunctions is defined;

– refined asymptotic formulas for eigenvalues and eigenfunctions were obtained;

– conditions have been determined under which the system of root functions, after removing one function, forms a basis in the space $L_p, 1 < p < \infty$;

– conditions were found for the uniform convergence of spectral expansions in the system of root functions after removing one function;

for the ordinary differential operators of fourth order with spectral parameter in the fourth of the boundary conditions

– refined asymptotic formulas for eigenvalues and eigenfunctions were obtained;

– conditions were found for the uniform convergence of spectral expansions in the system of root functions after removing one function.

The main results of the dissertation are published in the following works:

1. Abdullayeva, K.F. Asymptotic formulas for eigenvalues and eigenfunctions of some ordinary differential operators of fourth order // Transactions National Academy of Sciences of Azerbaijan Series of Physical-Technical and Mathematical Sciences, – 2018. v. 38, no. 4, – p. 8–16.
2. Kerimov N.B., Abdullayeva K.F. Uniform convergence of the spectral expansions in the terms of root functions of a spectral problem for the equation of vibrating rod // Materials of the international seminar "Spectral theory and its applications" dedicated to the 80th anniversary of the outstanding mathematician, academician Mirabbas Gasimov, – Bakı: – 07-08 june, – 2019, – s. 98–99.
3. Abdullayeva, K.F. Uniform convergence of the Fourier series expansions in the subsystems of root functions of some spectral problem for ordinary differential equations of fourth order // Transactions National Academy of Sciences of Azerbaijan Series of

Physical-Technical and Mathematical Sciences, – 2020. v. 40, no. 4, – p. 4–12.
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5. Aliyev, Z.S., Abdullayeva, K.F. On the uniform convergence of expansions by eigenfunctions of a fourth-order differential operator with a spectral parameter in the boundary condition // Materials of the International Conference of the Voronezh Winter Mathematical School "Modern methods of the theory of functions and related problems", – Voronezh: Russia, –28 january-2 february, – 2021, – p. 34–35.
6. Abdullaeva K.F. Basis properties of root functions of one spectral problem with a boundary condition depending on the spectral parameter // Proceedings of the scientific conference "Some current problems of modern mathematics and mathematical education. Herzen readings – 2021", - St. Petersburg: Russia, – 5-10 april, - 2021, - p. 29–30.
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9. Abdullayeva, K.F. Uniform convergence of spectral expansions for a boundary value problem with a boundary condition depending on the spectral parameter // Caspian Journal of Applied Mathematics, Ecology and Economics, - 2022. v. 10, no. 1, - p. 3-14

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