REPUBLIC OF AZERBAIJAN

On the rights of the manuscript

ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

OPTIMALITY CONDITIONS IN GOURSAT-DARBOUX SYSTEMS CONTROLLED BY THE BOUNDARY CONDITIONS

Specialty: 1203.01 – Computer sciences

Field of science: Mathematics

Applicant: **Vusala Abdulla kizi Suleymanova**

Baku - 2022

The work was performed at the chair of "Mathemtical analysis and function theory" of Sumgait State University.

Scientific supervisor: Doctor of Physics-Mathematical Sciences, Professor.
Kamil Bayramali Mansimov

Official opponents: Doctor of Sciences in Mathematics, Professor
Rafig Galandar Tagiyev

Doctor of Philosopy in Mathematics, Assoc. Professor
Rashad Oktay Mastaliyev

Candidate of Physical and Mathematical Sciences, Assoc. Professor
Rashad Siraj Mammadov

Dissertation council ED 1.19 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at the Institute of Control Systems of the Azerbaijan National Academy of Sciences.

Chairman of the Dissertation council:
Dr. Sc.(Phys.-Math.) Professor
Qalina Yurievna Mehdieva

Scientific secretary of the Dissertation council:
Cand. Sc.(Phys.-Math.) ass. Professor
Elkhan Nariman Sabziev

Chairman of the scientific seminar: corr.-member of ANAS, Dr. Sc. (Phys.-Math.) Professor
Vagif Rza Ibrahimov

GENERAL CHARACTERISTICS OF THE WORK

Rationale and development degree of the topic: Starting from the middle of the XX century, due to intensive development of science, engineering and also economy, there was a need to solve optimal control problems of various control processes.

During these years, the main results generalizing the classic results of the classic variation calculus were obtained. In the first turn, the Pontryagin's maximum principle, Bellman's dynamic programming method, Milyutin-Dubovitski's general extremum theory were worked out. In optimal control problems considered in these theories, in many cases various constraints were taken into account. Therefore, alongside with optimal control problems described by the system of ordinary differential equations, more complex optimal control problems described by partial differential equations (distributed parameter optimal control problems) began to be studied.

The works of V.M.Abdullayev, K.R.Aydazadeh, K.R. Aydazadeh and V.M.Abdullayev, S.S.Akhiev, A.G.Butkovskii, A.I. Egorov and L.N. Znamenskoye, A.V.Fursikov, F.D. Iskenderov, H.F. Kuliyev, I.G. Mamedov, K.B.Mansimov, K.B. Mansimov and M.J.Mardanov, T.K.Melikov, A.A.Niftiyev, M.A.Sadigov, Ya.A.Sharifov, R.G. Tagiyev, F.P.Vasil'ev, M.H. Yagubov and other works were devoted to various parameter optimal control problems.

One of the problems of great interest and attention of specialists are optimal control problems described by the system of second order hyperbolic equations and Goursat boundary conditions. Some aspects (control, observation, necessary and sufficient conditions for optimality, the methods of numerical solution and others) of optimal control problems described by Goursat-Darboux system are intensively developing.

F.Sh.Ahmedov, S.S.Akhiev and G.T.Ahmedov, S.S.Akhiev, E.P. Bokmelder and V.A. Dikhtan's, A.I.Egorov, K.Q.Hasanov, I.V.Lisachenko, K.B.Mansimov, M.J.Mardanov, A.S.Matveev, A.S.Matveev and V.A.Yacubovich, T.K.Melikov, V.I.Plotnikov, N.I.Pogodayev, A.R.Safari, V.A.Srochko, M.I.Sumin, V.I.Sumin, Ya.A.Sharifov, O.V. Vasiliev, F.P.Vasiliev and others scientist's works are in this direction.

At present the greatest attention is paid to the derivation of various necessary optimaliry conditions in the Goursat-Darboux systems. Theory of necessary optimality conditions of first order in the problems of optimal control by Goursat-Darboux systems was completely developed mainly for control problems with distributed parameters. But the problems of control of the Goursat-Darboux systems with boundary controls have not been studied enough. Furthermore, the cases of degreneration of first order optimality conditions are not uncommon. Such cases are called (following L.I.Rozoner) singular cases and appropriate controls singular controls. To present, theory of necessary conditions of optimality of higher order, in particular theory of singular controls for the Goursat-Darboux systems with boundary controls was little developed get. Proceeding from what was said above, the topic of the dissertation work devoted to the derivation of various necessary optimality conditions of first order and to the study of the cases of their degenaration in the boundary value problem of optimal control of the Goursat-Darboux systems can be considered urgent.

Object and subject of the study. The main object of the dissertation work are boundary value problems of optimal control of the processes described by the system of second order hyperbolic equations with Goursat boundary conditions. This choice was stipulated by numerical applications of such optimal control problems. The subject of the study is to establish various necessary optimality condition of first order and to study the cases of their degeneration.

The goal and objectives of the study. To prove necessary optimality conditions of first order and to study the cases of their degenaration for the problems of optimal control of the Goursat-Darboux systems with boundary controls.

Research methods. In the dissertation work when solving the stated problems, the methods of variational calculus, quality theory of optimal control of the systems with concentrated and distributed parameters and also one variant of the increment method.

The basic theses to be defended.

- Necessary and sufficient condition for optimality in a linear boundary value problem of optimal control of Goursat-Darboux systems;
- Pontryagin maximum principle type necessary optimality condition for the linearized maximum condition and analogue of the Euler equation;
- Various necessary optimality conditions of first order in the cases of nonsmooth quality criterion and also inequality type nonsmooth functional contraints;
- Necessary optimality condition of first and second orders in the case openness of the control domain;
- Analogue of the linearized maximum condition and necessary condition for optimality of singular controls;
- Analogue of the Pontryagin maximum principle and necessary conditions for the optimality of singular controls.

Scientific novelty of the study. All the results of the dissertation work representing different necessary optimality conditions of first and second orders for the considered optimal control boundary value problems are new.

Theoretical and practical value of the study.

The dissertation work is of theoretical character. It can be used in further development of theory of second order necessary optimality conditions for boundary value problems of optimal control of Goursat-Darboux systems. The obtained results can be used for solving specific practical problems occuring in applications.

Approbation and applications. The main results of the dissertation work were reported in the seminars of the chair of "Mathematical analysis and function theory" of Sumgait State University, in the seminars of the laboratory "Control in complex dynamical systems" of the Institute of Control Systems of ANAS, at the republican conference dedicated to the 100-th anniversary of prof. G.Sh.Gabibzade (Baku, 2016), at the III Republican conference “Mathematical applied problems and mew information technology” (Sumgait 2016), at the International conference “Theoretical and applied problems of mathematics” (Sumgait, 2018), at the International conference “Dynamics systems: stability, control, optimization” (Minsk, 2018), at the International conference “Information systems and achievement technologies and prospectives” (Sumgait, 2018), at the International conference “Dynamical systems, optimal control and mathematical modelling” (Irkutsk, 2019).

Author's persional contribution. All the results of the dissertation were obtained by the author. In the papers coauthored with the supervisor, only the problem statement belongs to the supervisor.

Author's publications. The main results of the work were published in authors 16 works the list of which is at the end of the abstract's.

Name of the organization where the work was performed.

The work was performed in the chair of “Mathematical analysis and functions theory” of Sumgait State University.

Structure and volume of the dissertation work.

The work consists of introduction, II chapters, conclusion, list of references consisting of 93 names and a list of denotations. General volume of the dissertation consists of 144 pages of typewritten text (96412 signs), the basic volume 133 pages (82041 signs). Chapter I consists of – 31389, chapter II - 28021 signs.

THE CONTENT OF THE DISSERTATION WORK

Introduction of the dissertation work contains the justification of the chosen direction of studies, review of papers devoted to the problems considered in the dissertation, the goal of the work, basic theses to be defended and brief content of chapters.

Chapter I consists of six sections.

In chapter I we consider a problem of optimal control by the Goursat-Darboux systems, in which one of the boundary conditions is determined from the controlled system of ordinary differential equations.

In section 1.1 we consider a linear boundary value optimal control problem with a linear multipoint quality functional. L.S.Pontryagin maximum principle type necessary and sufficients optimality condition is proved.

Let the control process in a given area $D = [t_0, t_1] \times [x_0, x_1]$ be described by a system of linear hyperbolic equations

$$z_{tx} = A(t,x)z + B(t,x)z_t + C(t,x)z_x + f(t,x), \qquad (1)$$

with Goursat boundary conditions

$$\begin{gathered} z(t, x_0) = a(t), \quad t \in [t_0, t_1], \\ z(t_0, x) = b(x), \quad x \in [x_0, x_1], \\ a(t_0) = b(x_0) = a_0 . \end{gathered} \qquad (2)$$

described by boundary conditions.

Here $A(t,x)$, $B(t,x)$, $C(t,x)$ – is a given measurable and bounded $(n \times n)$ matrix-function, $f(t,x)$ – is a given n dimensional vector-function, continuous in totality of variables together with partial derivatives, a_0 – is the given constant vector, $b(x)$ – is the given absolutely continuous n dimensional vector-function, $a(t)$ - is a given n dimensional vector-function

$$\dot{a} = D(t)a(t) + g(t,u), \quad t \in [t_0, t_1], \qquad 3)$$

$$a(t_0) = a_0,$$

Defined as a solution to the Cauchy problem.

Here $D(t)$ – is a given measurable and bounded $(n \times n)$ matrix function, $g(t,u)$ – is a given n dimensional vector-function, continuous in totality of variables together with partial derivatives, $u(t)$ - is r dimensional measurable and bounded vector-function of control actions with values from a given non-empty and bounded set $U \subset R^r$,

$$u(t) \in U \subset R^r, \quad t \in [t_0, t_1] \tag{4}$$

We call every function $u(t)$ with such properties an admissible control.

It is assumed that for each possible $u(t)$ control (1) - (2) the only and absolutely uninterrupted $z(t,x)$ solution of the Goursat problem, and (3) the only, absolutely uninterrupted $a(t)$ solution of the Cauchy problem.

Let (T_i, X_i), $i = \overline{1,k}$ $(t_0 < T_1 < T_2 < \ldots < T_k \le t_1;\ x_0 < X_1 < X_2 < \ldots < X_k \le x_1)$ – are the given points, $c_i, d_i, i = \overline{1,k}$ - is the given n-dimensional constant vectors. The solutions of the above (1) - (2) boundary problem and (3) the Cauchy problem are determined according to all admissible control

$$I(u) = \sum_{i=1}^{k} \left[c_i' z(T_i, X_i) + d_i' a(T_i) \right], \tag{5}$$

look at the problem of finding the minimum of functionality.

The admissible control $u(t)$ that affords a minimum value to the functional (5) under the contraints (1) - (4) is said to be an optimal control, the appropriate process $(u(t), a(t), z(t,x))$ an optimal process.

Let $(u(t), a(t), z(t,x))$ be a fixed admissible process. $\alpha_i(t,x), i = \overline{1,k}$ is a characteristic functions of the rectangles

$\left[t_0, T_i\right] \times \left[x_0, X_i\right], i = \overline{1, k}$, while $\beta_i(t), i = \overline{1, k}$ is a characetrisatical function of the segment $\left[t_0, T_i\right]$, $i = \overline{1, k}$.

Here $\psi(t, x)$ and $p(t)$ are n dimensional vector functions being the solutions of the linear Volterra type integral equations

$$\psi(t, x) = -\sum_{i=1}^{k} \alpha_i(t, x) c_i + \int_{t_0}^{t_1}\int_{x_0}^{x_1} A'(\tau, s)\psi(\tau, s)ds\,d\tau + \int_{x}^{x_1} B'(t, s)\psi(t, s)ds +$$

$$+\int_{t}^{t_1} C'(\tau, x)\psi(\tau, x)\,d\tau,$$

$$p(t) = -\sum_{i=1}^{k} \beta_i(t)d_i - \sum_{i=1}^{k} \beta_i(t)c_i + \int_{t}^{t_1} D'(\tau)p(\tau)d\tau + \int_{x_0}^{x_1}\int_{t}^{t_1} (A'(\tau, x)\psi(\tau, x)d\tau)dx +$$

$$+\int_{x_0}^{x_1} B'(t, x)\psi(t, x)dx$$

Let us include the analogue of the Hamilton-Pontryagin function in the form.

$$H(t, u, p) = p'g(t, u)$$

The following condemnation has been proved by the method of growth.

Theorem 1. For optimality of the admissible control $u(t)$ in the problem (1) - (5) it is necessary and sufficient that the relation

$$H(\theta, v, p(\theta)) - H(\theta, u(\theta), p(\theta)) \le 0,$$

of inequality, to be fulfilled for all $\theta \in [t_0, t_1)$ and $v \in U$.

Here and in the sequal $\theta \in [t_0, t_1)$ is an arbitrary point (Lebesque point) of the control $u(t)$.

In the case of nonlinear convex quality functional the Pontryagin condition’s sufficiency is proved.

In section 1.2 we consider a problem on minimum of the multipoint functional

$$S(u) = \varphi(z(T_1, X_1),\ldots, z(T_k, X_k)) + G(a(T_1),\ldots, a(T_k)), \qquad (6)$$

constraints with

$$u(t) \in U \subset R^r, \quad t \in [t_0, t_1] \qquad (7)$$

$$z_{tx} = B(t,x)z_t + f(t,x,z,z_x),(t,x) \in D = [t_0,t_1] \times [x_0,x_1], \quad (8)$$

$$z(t,x_0) = a(t), \quad t \in [t_0,t_1],$$
$$z(t_0,x) = b(x), \quad x \in [x_0,x_1], \quad (9)$$

$$a(t_0) = b(x_0) = a_0,$$

$$\dot{a} = g(t,a,u), \quad t \in [t_0,t_1], \quad (10)$$

$$a(t_0) = a_0. \quad (11)$$

Here $B(t,x)-$ is a given measurable and bounded $(n \times n)$ matrix function, $f(t,x,z,z_x)$, ($g(t,a,u)$)- is a given n -dimensional vector-function continuous by the totality of variables together with partial derivatives with respect to $(z,z_x)(a), \varphi(z_1,z_2,\ldots,z_k)$, $G(a_1,a_2,\ldots,a_k)$ - are the given continuously differentiable scalar functions, (T_i, X_i), $i = \overline{1,k}$ $(t_0 < T_1 < T_2 < \ldots < T_k \le t_1;\ x_0 < X_1 < X_2 < \ldots < X_k \le x_1)$ - are the given points, $b(x)$ is the given absolutely continuous vector-function, $t_0, t_1, x_0, x_1 - (t_0 < t_1;\ x_0 < x_1)$ are given, a_0 is the given constant vector, $u(t)$- is r - dimensional and bounded vector of control actions, U - is the given non-empty and boundaed set. We call every function $u(t)$ with such properties an admissible control.

It is assumed that admissible control $u(t)$ there correponds a unigue absolutely continuous solution $z(t,x)$, ($a(t)$) of boundary value problem (8) – (9) (of the Cauchy problem (10) - (11)).

The admissible control $u(t)$ that affords a minimum value to the functional (6) under the contraints (8) - (11) is said to be an optimal control, the appropriate process $(u(t), a(t), z(t,x))$ an optimal process. Let $(u(t), a(t), z(t,x))$ be a fixed admissible process. We introduce the denotation

$$H(t,x,z,z_t,z_x,\psi) = \psi' f(t,x,z,z_x)$$

$$M(t,a,u,p) = p' g(t,a,u)$$

Here $\psi(t,x)$ and $p(t)$ are n dimensional vector functions being the solutions of the Volterra type integral equations

$$\psi(t,x) = -\sum_{i=1}^{k} \alpha_i(t,x) \frac{\partial \varphi\left(z(T_1, X_1),\dots, z\left(T_k, X_k\right)\right)}{\partial z_i} +$$

$$+\int_t^{t_1}\int_x^{x_1} H_z(\tau, s, z(\tau,s), z_s(\tau,s), \psi(\tau,s))dsd\tau +$$

$$+\int_t^{t_1} H_{z_x}(\tau, x, z(\tau,x), z_x(\tau,x), \psi(\tau,x))d\tau + \int_x^{x_1} B'(t,s)\psi(t,s)ds,$$

$$p(t) = -\sum_{i=1}^{k} \beta_i(t) \frac{\partial G(a(T_1),\dots, a(T_k))}{\partial a_i} - \sum_{i=1}^{k} \beta_i(t) \frac{\partial \varphi(z(T_1, X_1),\dots, z(T_k, X_k))}{\partial z_i} +$$

$$+\int_t^{t_1} M_a(\tau, a(\tau), u(\tau), p(\tau))d\tau + \int_{x_0}^{x_1} B'(t,x)\psi(t,x)dx +$$

$$+\int_{x_0}^{x_1}\int_t^{t_1} H_z(\tau, x, z(\tau,x), z_x(\tau,x), \psi(\tau,x))d\tau dx$$

Here and in the sequal $\alpha_i(t,x)$ is a characteristic function of the rectangle $[t_0, T_i]\times[x_0, X_i]$, while $\beta_i(t)$ is a characetrisatical function of the segment $[t_0, T_i]$

Theorem 2. For optimality of the admissible control $u(t)$ in the problem (6) - (11) it is necessary that the relation (condition of maximum)

$$\max_{v\in U} M(\theta, a(\theta), v, p(\theta)) = M(\theta, a(\theta), u(\theta), p(\theta))$$

to be fulfilled for all $\theta \in [t_0, t_1)$.

Here and in the sequal $\theta \in [t_0, t_1)$ is an arbitrary point (Lebesque point) of the control $u(t)$.

Then we consider the cases of convexity, and then openness of control domains. The analogues of the linearized maximum condition and Euler equation are proved.

In section 1.3 we consider a problem on the minimum of the functional

$$S(u) = G(z(t_1, x_1)) + \varphi(a(t_1)) + \int_{t_0}^{t_1}\int_{x_0}^{x_1} [D'(t,x)z_t + E(t,x,z,z_x)]dx\ dt \qquad (12)$$

with the constraints

$$u(t) \in U \subset R^r, \qquad t \in T = [t_0, t_1], \qquad (13)$$

$$z_{tx} = B(t,x)\, z_t + f(t,x,z,z_x), \quad (t,x) \in D = [t_0, t_1] \times [x_0, x_1],$$

(14)

$$z(t, x_0) = a(t), \qquad t \in T = [t_0, t_1],$$

$$z(t_0, x) = b(x), \qquad x \in X = [x_0, x_1], \qquad (15)$$

$$a(t_0) = b(x_0) = a_0,$$

$$\dot{a} = g(t,a,u), \quad t \in T, \qquad (16)$$

$$a(t_0) = a_0.$$

(17)

Here $B(t,x)$ is a given $(n \times n)$ matrix function, $\varphi(z)$ and $\varphi(a)$ are the given differnetiable scalar functions, $D(t,x)$ is the given measurable and bounded *n*-dimensional vector-function, $E(t,x,z,z_x)$ is a given scalar function differentiable in totality of variables together with partial derivatives with respect to z, z_x, $f(t,x,z,z_x)$ is the given n - dimensional vector-function continuous in totality of variables together with partial derivatives with respect to z, z_x, $b(x)$ is a given absolutely continuous vector-function, $g(t,a,u)$ is a given n - dimensional vector-function, continuous in totality of variables together with partial derivatives with respect to a, U is a given non-empty and bounded set, $u(t)$ is an r dimensional measurable and bounded control vector-function[1].

It is assument that for each given admissible control $u(t)$ the Cauchy problem (16) - (17) and the Goursat boundary value problem (14) - (15) have unique abslutely continuous solutions $a(t)$ and $z(t,x)$ respectively.

In his work S.S.Akhiev proved necessary optimality condition in the form of L.S.Pontryagin's maximum principle in the problem of optimal control by the distributed parameter Goursat-Darboux systems. By means of this method called the method of division of Lagrange multipliers by the summands, in this work we used analogue of the method of division of Lagrange multipliers by the summands, and proved necessary condition for optimality in the form of L.S.Pontryagin's maximum principle in the case of boundary controls, i.e. in problem (12) - (17).

In section 1.4 was devoted to the study of a boundary value optimal control problem of the form

$$u(t) \in U \subset R^r, \quad t \in T = [t_0, t_1], \tag{18}$$

$$z_{tx} = B(t,x) z_t + f(t,x,z,z_x) \; (t,x) \in D = [t_0,t_1] \times [x_0,x_1], \tag{19}$$

$$z(t,x_0) = a(t), \quad t \in T = [t_0,t_1],$$

$$z(t_0,x) = b(x), \quad x \in X = [x_0,x_1], \tag{20}$$

$$a(t_0) = b(x_0),$$

$$\dot{a} = F(t,a,u), \quad t \in T, \tag{21}$$

$$a(t_0) = a_0. \tag{22}$$

It is assumed that the quality criterion is of the form

$$S(u) = \Phi_1\left(z(T_1,X_1),\ldots,z(T_k,X_k)\right) + \Phi_2\left(a(T_1),\ldots,a(T_k)\right), \tag{23}$$

where $\Phi_i \; i = \overline{1,2}$ are the given scalar functions with derivatives in

---

[1]Мансимов К.Б., Сулейманова В.А. Аналог способа разделения множителя Лагранжа на слагаемые в одной граничной задаче оптимального управления системами Гурса-Дарбу // - Пермь: Прикладная математика и вопросы управления. - 2019. № 2, - с. 7-22.

any directions and satisfying the Lipschits condition[2]. Necessary condition for optimality in the terms of directional derivatives is proved.

In section 1.5 we study a minimax problem.

It is required to find minimum value of the maximum type terminal functional

$$S(u) = \max_{y \in Y} \varphi_1(z(t_1, x_1), y) + \max_{\alpha \in A} \varphi_2(a(t_1), \alpha), \qquad (24)$$

with the constraints

$$u(t) \in U \subset R^r, \quad t \in T, \qquad (25)$$

$$\dot{a} = F(t, a, u), \quad t \in T, \qquad (26)$$

$$a(t_0) = a_0. \qquad (27)$$

$$z_{tx} = B(t,x) z_t + f(t, x, z, z_x), \quad (t, x) \in D = [t_0, t_1] \times [x_0, x_1], \qquad (28)$$

$$\begin{aligned} z(t, x_0) &= a(t), \quad t \in T, \\ z(t_0, x) &= b(x), \quad x \in X. \end{aligned} \qquad (29)$$

$$a(t_0) = b(x_0) = a_0$$

Here Y and A are finite sets of m and q dimensional vectors y and α, respectively, $\varphi_1(z, y), \varphi_2(a, \alpha)$ are the given scalar functions continuously differentiable with respect to z or $y(\alpha)$ and (a) for all, the other data of the problem on the minimum of the functional (24) under the constraints (25) - (29) satisfy the smoothness conditions similar from the previous sections.

Such optimal control problems with functional of type (18) are called minimax problems.

Pontryagin’s maximum principle type necessary optimality condition is proved by means of the scheme based on explicit linearization of the initial system.

In section 1.6 we study a boundary value optimal control problem with a smooth terminal type quality functional involving

[2]Мансимов К.Б., Сулейманова В.А. Необходимые условия оптимальности в одной негладкой граничной задаче управления системами Гурса-Дарбу // - Bakı: Bakı Universitetinin xəbərləri, fizika-riyaziyyat elmləri seriyası, - 2016. №2, - s. 29-36.

terminal type inequalities type nonsmooth functional constraints assuming that the functions giving the quality criterion and functional constraints satisfy the Lipschits condition and have directional derivatives. Applying one variant of the increament method a necessary optimality condition is proved in terms of directional derivatives.

Chapter II of the dissertation work consists of four sections and is devoted to the derivation of Pontryagin maximum principle type necessary optimality conditions of first order, linearized maximum principle, analogue of the Euler equation and of the studied singular controls, i.e. controls for which the Pontryagin maximum principle and linearized maximum principle degenerate.

In sections 2.1 we consider a problem on finding the minimum of the functional

$$S(u) = \varphi\left(a(t_1)\right) + G\left(z(t_1, x_1)\right), \tag{30}$$

with the constraints

$$u(t) \in U \subset R^r,\ t \in T = [t_0, t_1], \tag{31}$$

$$z_{tx} = B(t,x) z_t + f(t, x, z, z_x),\ (t,x) \in D = [t_0, t_1] \times [x_0, x_1], \tag{32}$$

$$\begin{aligned} z(t, x_0) &= a(t), \quad t \in T = [t_0, t_1], \\ z(t_0, x) &= b(x), \quad x \in X = [x_0, x_1], \end{aligned} \tag{33}$$

$$a(t_0) = b(x_0) = a_0,$$

$$\dot{a} = g(t, a, u), \quad t \in T \tag{34}$$

$$a(t_0) = a_0. \tag{35}$$

Here $f(t, x, z, z_x)$ is a given n - dimensional vector function continuous by totality of variables together with partial derivatives with respect to (z, z_x) to the second order inclusively, $g(t, a, u)$ is a given n - dimensional vector-function continuous by totality of variables together with partial derivatives with respect to (a, u) to the second order incluisively, $\varphi(a)$ and $G(z)$ are the given twice differentiable scalar functions, U is a given non-empty, bounded and open set, the other data of the problem satisfy the smooth conditions introduced in the previous sections of chapter I, Let $(u(t), a(t), z(t,x))$ be a fixed admissible process. Introduce the denotation

$$H(t,x,z,z_x,\psi) = \psi' f(t,x,z,z_x)$$

$$M(t,a,u,q) = q' g(t,a,u)$$

$$M_a(t) \equiv M_a(t, a(t), u(t), q(t)),$$

$$M_{aa}(t) \equiv M_{aa}(t, a(t), u(t), q(t)),$$

$$M_{ua}(t) \equiv M_{ua}(t, a(t), u(t), q(t)),$$

$$M_{uu}(t) \equiv M_{uu}(t, a(t), u(t), q(t)),$$

$$H_z(t,x) \equiv H_z(t,x,z(t,x), z_x(t,x), \psi(t,x)),$$

$$H_{zz}(t,x) \equiv H_{zz}(t,x,z(t,x), z_x(t,x), \psi(t,x)),$$

$$H_{zz_x}(t,x) \equiv H_{zz_x}(t,x,z(t,x), z_x(t,x), \psi(t,x)),$$

$$g_a(t) \equiv g_a(t, a(t), u(t)),$$

$$g_u(t) \equiv g_u(t, a(t), u(t))$$

$$f_z(t,x) \equiv f_z(t,x,z(t,x), z_x(t,x)),$$

$$f_{z_x}(t,x) \equiv f_{z_x}(t,x,z(t,x), z_x(t,x)),$$

where $\psi = \psi(t,x)$ and $q = q(t)$ are n -dimensional vector-functions being the solutions of the adjoint system of equations

$$\psi(t,x) = -G_z(z(t_1,x_1)) + \int_s^{x_1} B'(t,s)\psi(t,s)\,ds + \int_t^{t_1}\int_x^{x_1} H_z(\tau,s)\,ds\,d\tau + \int_t^{t_1} H_{z_x}(\tau,x)\,d\tau, \tag{36}$$

$$q(t) = -\varphi_a(a(t_1)) - G_z(z(t_1,x_1)) + \int_t^{t_1} M_{aa}(\tau)\,d\tau - \int_{x_0}^{x_1}\int_t^{t_1} H_z(\tau,x)\,dx\,d\tau + \int_{x_0}^{x_1} B'(t,x)\psi(t,x)\,dx, \tag{37}$$

In this section we calculate the first and second variation of the quality functional and prove first and second order necessary optimality conditions[3].

Theorem 3. For the optimality of the admissible control $u(t)$ in the problem (30) – (35) it is necessary that for all $\theta \in [t_0, t_1)$ the relation

$$M_u(\theta) = 0, \tag{38}$$

to be fulfilled.

The relation (38) is the analogy of the Euler equation for the considered problem. We call each admissible control $u(t)$, being the solution of the Euler equation (38) a classic extremum.

Note that for the classic extremums suspicious for optimality can be rather great. Therefore, we have to deal with second order necessary optimality conditions for narrowing the set of classic extremums suspicious for optimality.

Let $F(t,\tau)$ and $R(t,x;\tau,s)$ be $(n \times n)$ dimensional matrix functions being the solutions of the following problems

$$F_\tau(t,\tau) = -F(t,\tau)\, g_u(\tau), \tag{39}$$

$$F(t,t) = E,$$

$$\begin{gathered} R(t,x;\tau,s) = E + \int_\tau^t \int_s^x R(t,x;\alpha,\beta)\, f_z(\alpha,\beta)\, d\alpha\, d\beta + \\ + \int_\tau^t R(t,x;\alpha,s)\, f_{z_x}(\alpha,s)\, d\alpha + \int_s^x R(t,x;\tau,\beta) B(\tau,\beta)\, d\beta . \end{gathered} \tag{40}$$

(E $(n \times n)$ is a unit matrix).

We introduce the denotations

[3] Мансимов К.Б., Сулейманова В.А. Необходимые условия оптимальности в одной граничной задаче управления системами Гурса-Дарбу // - Томск: Вестник Томского Государственного университета, сер. управления математика и механика, - 2017. № 49, - с. 26-42.

$$Q(t,x,\tau)=\int_{\tau}^{t} L(t,x,s)F(s,\tau)ds + R(t,x;\tau,x),$$

$$L(t,x,\tau)= R(t,x;\tau,x_0)\left[g_a(\tau)- f_{z_x}(\tau,x_0)\right]$$

$$K(\tau,s) = -F'(t_1,\tau)\varphi_{aa}(a(t_1))F(t_1,s)-$$

$$-Q'(t_1,x_1,\tau)G_{zz}(z(t_1,x_1))Q(t_1,x_1,s)+\int_{\max(\tau,s)}^{t_1} F'(t,\tau)M_{aa}(t)F(t,s)dt +$$

$$+\int_{x_0}^{x_1}\int_{\max(\tau,s)}^{t_1}\left[Q'(t,x,\tau)H_{zz}(t,x)Q(t,x,s)+Q'(t,x,\tau)H_{zz_x}(t,x)Q_x(t,x,s)+\right.$$

$$\left.+Q'_x(t,x,\tau)H_{z_x z}(t,x)Q(t,x,s)\right]dx\ dt .$$

(41)

Introduction of the matirix function (41) allowed to set structurally verifiable necessary optimality condition of second order using the condition of non-negativeness of the second variation of the quality functional along the optimal control.

We prove the following theorem

Theorem 4. For the optimality of the classic extremum $u(t)$ in the problem (30) – (35) it is necessary that inequality

$$\int_{t_0}^{t_1}\int_{t_0}^{t_1}\delta'u(\tau)g'_u(\tau)K(\tau,s)g_u(s)\delta u(s)ds\,d\tau +$$

$$+2\int_{t_0}^{t_1}\left[\int_{t_0}^{t_1}\delta'u(\tau)M_{ua}(\tau)F(\tau,t)d\tau\right]g_u(t)\delta u(t)dt + \tag{42}$$

$$+\int_{t_0}^{t_1}\delta'u(t)M_{uu}(t)\delta u(t)dt \le 0 .$$

be fulfilled for all $\delta u(t)\in R^r$, $t\in T$.

Inequality (42), being a necessary optimality condition of second order is of rather general character. Using the arbitrariness of the admissible variation $\delta u(t)$ of the control $u(t)$ one can obtain a second order optimality condition.

In the work, expecially the analogue of the Legendre-Klebsh condition was proved and a number of necessary optimality conditions are proved for classic singular controls, i.e. the controls for which the analogue of the Legendre-Klebesh condition degenerates.

In section 2.2. we study the case of multipoint quality criterion of the form

$$S(u) = \varphi_1(z(t_1, x_1)) + \varphi_2(a(X_1), \dots, a(X_k)),$$

with the constraints

$$u(x) \in U \subset R^r, \; x \in X,$$

$$\dot{a} = g(x, a, u), \quad x \in X,$$

$$a(x_0) = a_0,$$

$$z_{tx} = A(t,x)z_t + B(t,x)z_x + f(t,x,z), \; (t,x) \in D,$$

$$z(t_0, x) = a(x), \qquad x \in X,$$

$$z(t, x_0) = b(t), \qquad t \in T,$$

$$a(x_0) = b(t_0) = a_0.$$

Under the assumption that the control domain U is open, we prove the analogue of the Euler equation and derive necessary order optimality condition of second order[4.]

In section 2.3 we consider an optimal control problem from section 1 under the assumption that the control domain U is open. The analogue of the linearized maximum condition is proved and the case of its degeneration is studied (the case of quasisingular controls).

In section 2.4 the study of problem (30) – (35), is continued assuming that U is an arbitrary non-empty bounded set, while $g(t, a, u)$ has a continuous derivative of second order only with

---

4 Сулейманова В.А. О многоточечных необходимых условиях оптимальности особых, в классическом смысле, управлений в одной граничной задаче управления системами Гурса-Дарбу // - Baku: Journal of Baku Engineering University Mathematics and computer science, - 2019. Volume 3, Number 1, - p. 36-48.

respect to a. Under these assumptions in the problem (30) – (35) the analugue of the Pontryagin maximum principle was proved and the case of its degeneration was studied.

The author expresses her gratitute to his supervisor prof. K.B.Mansimov for his constant attention to the work.

CONCLUSIONS

The dissertation work was devoted to the quality study of a class of optimal control problems with concentrated controls for the Goursat-Darboux system.

The work consists of introduction, II chapters, a list of main denotations and a list of the used references. Chapter I consists of six sections.

In the linear case, Pontryagin's maximum principle type necessary and sufficient condition for optimality was proved, in the nonlinear case a number of necessary optimality conditions under various assumptions were proved in the form of L.S.Pontryagin maximum principle, linearized maximum condition and the analogue of the Euler equation. The case of nonsmooth quality criterion and nonsmooth functional constants of inequality type were also studied. The minimax problem was also studied.

Chapter II of the dissertation consists of four sections.

The so-called singular controls for optimality, i.e. the cases of degeneration of the analogue of the linearized maximum principle were studied. In the case of openness of the control domain, a second order necessary optimality condition was proved.

The basic results of the dissertation work were published in the following scientific works

1. Мансимов К.Б., Сулейманова В.А. Необходимые условия оптимальности второго порядка в одной граничной задаче оптимального управления системами Гурса-Дарбу // Bakı Dövlət Universiteti, əməkdar elm xadimi, professor Əmir Şamil oğlu Həbibzadənin anadan olmasının 100-cü ildönümünə həsr olunmuş Funksional analiz və onun tətbiqləri adlı respublika elmi konfransı, BDU, - Bakı: - 2016, - s.173.
2. Мансимов К.Б., Сулейманова В.А. Необходимые условия оптимальности в одной негладкой граничной задаче управления системами Гурса-Дарбу // - Bakı: Bakı Universitetinin xəbərləri, fizika-riyaziyyat elmləri seriyası, - 2016. №2, - s. 29-36.
3. Мансимов К.Б., Сулейманова В.А. Об оптимальности особых управлений в одной граничной задаче оптимального управления системами Гурса-Дарбу // - Bakı: Bakı Universitetinin xəbərləri, fizika-riyaziyyat elmləri seriyası, - 2016. №4, - s. 5-19.
4. Мансимов К.Б., Сулейманова В.А. Необходимые условия оптимальности второго порядка в одной граничной задаче управления системами Гурса-Дарбу // - Sumqayıt: Sumqayıt Dövlət Universiteti, Azərbaycan Milli Elmlər Akademiyası İnformasiya Texnologiyaları İnstitutu, Riyaziyyatın tətbiqi məsələləri və yeni informasiya texnologiyaları, III Respublika elmi konfransının materialları, 15-16 dekabr, - 2016, - s. 34-35.
5. Сулейманова В.А. Задача на минимакс в одной граничной задаче управления системами Гурса-Дарбу // Riyaziyyatın nəzəri və tətbiqi problemləri Beynəlxalq elmi konfransının materialları. – Sumqayıt: 2017, - s. 245-246. https:// www. sdu. edu.az/ serfiles/file/ conferences/math_01022016.pdf

6. Мансимов К.Б., Сулейманова В.А. Необходимые условия оптимальности в одной граничной задаче управления системами Гурса-Дарбу // - Томск: Вестник Томского Государственного университета, сер. управления математика и механика, - 2017. № 49, - с. 26-42. http://journals.tsu.ru/mathematics/&journal_page=archive&id=1632
7. Мансимов К.Б., Сулейманова В.А. Линеаризованное квадратичное необходимое условие оптимальности в одной граничной задаче оптимального управления системами Гурса-Дарбу // - Пермь: Прикл. матем. и вопросы управления, – 2017. № 4, - с. 7-28. http://vestnik.pstu.ru/matmech/archives/?id=&folderid=7167
8. Мансимов К.Б., Сулейманова В.А. Об одной граничной задаче оптимального управления системами Гурса-Дарбу // Journal Of Baku Engineering University - Mathematics And Computer Science, - 2017. Volume 1, Number 2, - p. 107-117.http://journal.beu.edu.az/journal-details/30/
9. Мансимов К.Б., Сулейманова В.А. Необходимое условие оптимальности граничных управлений в системах Гурса-Дарбу при функциональных ограничениях типа неравенств // В еб. Матер. Международной научной конференций. Динамические системы: устойчивость, управление, оптимизация. – Минск: - 24-29 сентября, - 2018, - с. 159-160.
10. Сулейманова В.А. Необходимые условие оптимальности граничной управлений в системах Гурса-Дарбу // İnformasiya sistemləri və texnologiyaları: nailiyyətlər və perspektivlər. Beynəlxalq elmi konfransının materialları, - Sumqayıt: 15-16 noyabr, - 2018, - s. 176-177. https://sdu.edu.az/userfiles/file/conferences/15-16% 20 noyabr%202018-ci_il%20Beyn %C9% 99 lxalq %20 elmi% 20 konfrans% C4%B1n %C4%B1n% 20 mate rial%C4%B1.pdf

11. Сулейманова В.А. Достаточное условие оптимальности типа принципа максимума Понтрягина в одной граничной задаче управления системами Гурса-Дарбу // SDU, Elmi xəbərlər, Təbiət və texniki elmlər bölməsi, cild 19, - Sumqayıt: – 2019. № 1, s.12-15. https://www.ssu-scientificnews.edu.az/pdf/T19-1.pdf
12. Мансимов К.Б., Сулейманова В.А. Необходимое условие оптимальности в одной граничной задаче управления системами Гурса-Дарбу при наличии негладких функциональных ограничений типа неравенств // - Bakı: Bakı Universitetinin xəbərləri, fizika-riyaziyyat elmləri seriyası, - 2019. №1, - с. 13-20.http://static.bsu.az/w1/elmi_jurnallar/2019-1-fizika-riyaziyyat.pdf
13. Мансимов К.Б., Сулейманова В.А. Аналог способа разделения множителя Лагранжа на слагаемые в одной граничной задаче оптимального управления системами Гурса-Дарбу // - Пермь: Прикладная математика и вопросы управления. - 2019. № 2, - с. 7-22. http://vestnik.pstu.ru/matmech/archives/?id=&folder_id=8640
14. Сулейманова В.А. Необходимое условие оптимальности второго порядка в одной граничной задаче управления квазилинейными системами Гурса-Дарбу // Материалы международного симпозиум. Динамические системы, оптимальное управление и математическое моделирование. – Иркутск: - 7-11 октября, - 2019, - с. 282-285. https://elibrary.ru/item.asp?id=40928303&pff=1
15. Сулейманова В.А. О многоточечных необходимых условиях оптимальности особых, в классическом смысле, управлений в одной граничной задаче управления системами Гурса-Дарбу // - Baku: Journal of Baku Engineering University Mathematics and computer science, - 2019. Volume 3, Number 1, - p. 36-48. http://journal.beu.edu.az/journal-details/15/

16. Сулейманова В.А. Необходимое условие оптимальности в задаче на минимакс для одной граничной задачи управления системами Гурса-Дарбу // - Baku: Journal of Baku Engineering University Mathematics and computer science, - 2019. Volume 3, Number 1, - p. 49-55. http://journal.beu.edu.az/journal-details/15/

The defense will be held on 17 June 2022 year at 14^{00} at the meeting of the Dissertation council ED 1.19 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at Institute of Control Systems of the Azerbaijan National Academy of Sciences.

Address: AZ 1141, Baku city, B.Vahabzade str, 9.

Dissertation is accessible at the library of the Institute of Control Systems of the Azerbaijan National Academy of Sciences library.

Electronic versions of the dissertation and its abstract are available on the official website (http://www.isi.az) of the Institute of Control Systems of the Azerbaijan National Academy of Sciences.

Abstract was sent to the required addresses on 12 May 2022.

Signed for print: 15.04.2022

Paper format: A5

Volume: 17447

Number of hard copies: 20