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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

THE STUDY OF THE PROBABILISTIC CHARACTERISTICS OF THE SPARRE ANDERSEN TYPE INSURANCE RISK PROCESSES

Speciality:	1208.01 – Probability theory
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GENERAL CHARACTERISTICS OF THE WORK

Relevance and development of the topic. The theory of stochastic processes is one of the fastest growing branches of probability theory. The presented dissertation is dedicated to the study of the probabilistic characteristics of two classes of stochastic processes called insurance risk processes in the scientific literature.

Suppose that, there are given the sequences of the random variables $\{\xi_i\}$ and $\{\eta_i\}$, $i \geq 1$ in the probability space $(\Omega, \mathcal{F}, P)$. So that the random variables in each sequence are independent of each other and identically distributed. Let us construct a stochastic process $R(t) = R(\omega, t)$, $\omega \in \Omega$, $t \geq 0$, called the insurance risk process, using these sequences:

$$R(t) = u + ct - \sum_{i=1}^{\nu(t)} \eta_i.$$

where the counting process $\nu(t)$ is the number of insurance events that occurred before the moment of t:

$$\nu(t) = \max\{n: \xi_1 + \xi_2 + \cdots + \xi_n \leq t\}.$$

The stochastic process $R(t)$ represents the capital of the insurance company at time t. $u = R(0)$ represents the amount of initial capital of the insurance company, $c > 0$ represents the speed of insurance premiums, random variables η_i represent random insurance payments to be paid during insurance events, and random variables ξ_i represent the time between insurance events. Note that in the special case, when the random variables ξ_i have an exponential distribution, the process $R(t)$ is called the classical risk process or Cramer-Lundberg process and has been widely studied in the scientific literature. If the random variables ξ_i have an arbitrary distribution, the $R(t)$ process is called the Sparre Andersen insurance risk process. There are works of F.Lundberg, H.Cramér, E.S.Andersen, G.Alsmeyer, P.Embrechts, A.A.Borovkov, S.Asmussen, D.C.M.Dickson, H.U.Gerber, C.Klüppelberg, N.Veraverbeke, N.Gaoqin, J.Grandell, S.M.Ross, G.E.Willmot, A.Baltrunas, A.Aleškeviciene, T.H.Nasirova, H.M.Ahmedova, T.A.Khaniyev, R.T.Aliyev and other authors in this direction. However, the cases where the random variables indicating the time between insurance events and the amount of insurance payments have heavy-tailed distributions and the components of the insurance risk process are interdependent have been little studied. Since these cases more adequately express reality, the presented dissertation examines the important probabilistic characteristics of processes whose components are heavy-tailed distributions and interdependent random variables. Therefore, the topic of the dissertation is relevant. Another issue that determines the relevance of the topic of the dissertation is the study of the main probabilistic characteristics of the insurance risk process with a constant interest rate, as well as the Gerber-Shiu function, which has an important role in the modern theory of insurance risk.

Object and subject of research. The object of the dissertation work is the probabilistic characteristics of insurance risk processes. The subject of the research is obtaining exact formulas and asymptotic expansions for the probability characteristics of the considered stochastic processes.

Goals and objectives of the study. The main purpose of the dissertation is to address the following issues:

1. construction of integro-differential and ordinary differential equations, respectively, for the Gerber-Shiu function and its Laplace transform, which summarize many probabilistic characteristics of the insurance risk process, including the ruin probability;
2. obtaining the joint distribution function of the capital immediately prior to ruin and the deficit at ruin;
3. obtaining explicit statements for the probabilistic characteristics of the reinsurance risk process;
4. obtaining asymptotic results for mathematical expectation and variance of the reinsurance risk process;

5. obtaining asymptotic relations for the mathematical expectation of renewal-reward processes where the components are dependent random variables.

Research methods. The theoretical and methodological basis of the dissertation is the theory of probability, the theory of stochastic processes, the theory of differential and integral equations, as well as the theory of Laplace transform, asymptotic methods.

The main provisions of the defense. The following provisions are included in the defense:

1. the integro-differential equations for the Gerber-Shiu function and the ruin probability, as well as the differential equations for their Laplace transform, are extended to Erlang(n) insurance risk processes (**paragraphs 1.2** and **1.3**);
2. an explicit expression of the joint distribution function of the surplus immediately prior to ruin and the deficit at ruin was found for Erlang(n) insurance risk processes (**paragraph 1.4**);
3. the explicit expressions for the distribution function the first and second moments of insurance payments, have been obtained in the reinsurance risk process (**paragraph 2.1**);
4. the asymptotic relations with the reminder term $o(t^{-k})$ were obtained for the mathematical expectation and variance of the reinsurance risk process (**paragraphs 2.2** and **2.3**);
5. the estimations analogous to the existing for the mathematical expectation of renewal process were also given for the mathematical expectation of the renewal-reward process where the components are dependent random variables (**paragraph 2.4**);
6. an asymptotic relation under the Cramer condition with the reminder term $O(e^{-\varepsilon t})$ was obtained for the mathematical expectation of the renewal-reward process (**paragraph 2.5**);
7. an asymptotic relation was obtained for the mathematical expectation of renewal-reward process where the component has heavy-tailed distribution (**paragraph 2.6**);

8. an asymptotic relation was obtained for the mathematical expectation of renewal-reward process where the component has regularly varying distribution (**paragraph 2.7**);
9. the asymptotic relation was obtained for the mathematical expectation of renewal-reward process where the component has dominated varying distribution (**paragraph 2.8**).

Scientific novelty of the research. The following new results were obtained in the dissertation:

1. equations and obvious expressions for the Gerber-Shiu function in the Erlang(n) insurance risk process with constant interest rate are obtained;
2. the asymptotic relations for the mathematical expectation and variance of the reinsurance risk process;
3. some estimations were obtained for the mathematical expectation of the renewal-reward process where the components are dependent random variables;
4. the asymptotic relation under the Cramer condition were obtained for the mathematical expectation of the renewal-reward process where the components are dependent random variables;
5. the asymptotic relation were obtained for the mathematical expectation of the renewal-reward process where the component has distribution that belongs to the different classes of heavy-tailed distributions.

Theoretical and practical significance of the research. In the dissertation the important probabilistic characteristics of two classes of Sparre Andersen stochastic process type are studied. The research is of great theoretical importance for probability theory, as well as for the theory of stochastic processes. The results obtained in the dissertation for insurance risk and renewal-reward processes are also important for the application of mathematical insurance theory. These results can be used to model the surplus process in insurance portfolios, and to estimate both the ruin probability and the amount of capital of insurance companies for complex portfolios with smaller errors.

Approbation and application of research work. The main results of the dissertation were reflected in the scientific seminars of the Department of Operations Research and Probability Theory, Baku State University, as well as the laboratory of Applied Probabilistic Statistical Methods, Institute of Control Systems of ANAS, as well as in a number of reports made at the following national and international scientific conferences:

1. XXVII International Conference Problems of Decision Making Under Uncertainties. Tbilisi-Batumi, Georgia, 2016;
2. "Funksional analiz və onun tətbiqləri" adlı respublika elmi konfransın materialları, Bakı, 2016;
3. "Nəzəri və tətbiqi riyaziyyatın aktual məsələləri" respublika elmi konfransının materialları, Şəki, 2016;
4. XXIX International Conference Problems of Decision Making Under Uncertainties. Mukachevo, Ukraine, 2017;
5. Материалы III Международной научно-практической конференции, «Математическое моделирование в экономике,ууправлении, образовании», Финансовый университет, Калужский филиал, Москва, 2017;
6. XXXI International Conference Problems of Decision Making Under Uncertainties. Lankaran-Baku, Republic of Azerbaijan, 2018;
7. XXXII International Conference Problems of Decision Making Under Uncertainties. Prague, Czech Republic, 2018;
8. XXXIII International Conference Problems of Decision Making Under Uncertainties. Hurgada, Egypt, 2019;
9. XXXV International Conference Problems of Decision Making Under Uncertainties. Baku-Sheki, Republic of Azerbaijan, 2020;
10. 3rd International Conference on Mathematical Advances and Applications (ICOMAA-2020), Yildiz Technical

University, Istanbul, Turkey, Online Video Conferencing, 2020.

Name of the organization where the dissertation work is carried out. The dissertation work was carried out at the Department of Operations Research and Probability Theory, Faculty of Applied Mathematics and Cybernetics, Baku State University.

Publications. 18 scientific works on the topic of the dissertation were published. 8 of them are scientific articles. 2 of the articles were published in scientific journals included in international summarizing and indexing systems such as **“Web of Science”** and **“Zentralblatt MATH”**. The list of published articles is given at the end of the abstract.

The structure and scope of the dissertation. The dissertation consists of an introduction, 2 chapters, a conclusion and a list of references used in 147 titles. The volume of the work is 122 pages.

The title page of the dissertation consists of 315, table of contents – 2317, introduction – 21914, first chapter – 18295, second chapter – 38655, and conclusion – 1255 characters. The total volume of the dissertation is 82436 characters.

BRIEF CONTENT OF THE DISSERTATION

The introduction provides a summary of works related to the issues considered and a brief summary of the dissertation.

The first chapter of the dissertation consists of 4 paragraphs.

It is known that, most of the capital of the insurance company is formed on the basis of investment income, and the interest rate affects the management of the company. Therefore, it is important to study the problem of ruin, taking into account the interest rate. N.Gaogin, L.Cihua, Xu.Lixia studied the Gerber-Shiu function in the Erlang(2) risk process with constant interest rate and constructed the integro-differential equation for this function, and using this equation, the two-order differential equation for its Laplace transform[1]. In the first chapter of the dissertation, the more general Erlang(n) insurance risk process with constant interest rate is considered and the expected discounted penalty (Gerber-Shiu) function of this process is investigated.

Let's include the following insurance risk process:

$$R_\delta(t) = ue^{\delta t} + c\int_0^t e^{\delta(t-y)}dy - \sum_{i=1}^{\nu(t)} \eta_i e^{\delta(t-T_i)},$$

where $R_\delta(t)$, $t \geq 0$ represents the capital of the insurance company at t, $u = R_\delta(0) \geq 0$ is the initial capital of the insurance company, $c > 0$ is the rate of receipt of insurance premiums, δ is a constant interest rate, η_i, $i \geq 1$ are random variables defined in any probability space $(\Omega, \mathcal{F}, P)$, independent and have the common distribution, indicating positive insurance payments, $\nu(t)$, $t \geq 0$ is a counting process that shows the number of insurance events that occurred up to the moment t:

[1] Gaoqin, N., Cihua, L., Lixia, Xu. Expected discounted penalty function of Erlang(2) risk model with constant interest // – London: Appl. Math. J. Chinese Univ. Ser. B, – 2006. 21(3), –p. 243-251.

$$\nu(t) = \max\{n: T_n \le t\}, \qquad t > 0.$$

The variables $T_i = \sum_{j=1}^{i} \xi_j$, $i \ge 1$ represent the time of occurrence of the insurance event, positive random variables ξ_j, $j \ge 1$ represent the time between two consecutive insurance events.

This chapter assumes that the following conditions are met:

A1) random variables ξ_i, $i \ge 1$, are independent and identically distributed by the Erlang(n) distribution with the parameter $\beta > 0$;

A2) random variables η_i, $i \ge 1$ are independent, have the same distribution function $F_\eta(x)$, finite mathematical expectation $m_1 = E\eta_1$ and density function $f_\eta(x)$;

A3) the sequences $\{\xi_i\}$, $i \ge 1$ and $\{\eta_i\}$, $i \ge 1$ are independent of each other;

A4) $cE\xi_1 > E\eta_1$.

The random variable T_δ defined as follows is an important boundary functional of the process $R_\delta(t)$ and is called the moment of ruin of the insurance company:

$$T_\delta = \inf\{t: R_\delta(t) < 0\}.$$

If $R_\delta(t) \ge 0$ for any $t > 0$, then $T_\delta = \infty$.

The ruin probability in $R_\delta(t)$ process is defined as follows:

$$\Psi_{n;\delta}(u) = \mathrm{P}\{T_\delta < \infty | R_\delta(0) = u\}.$$

In the insurance risk theory, the random variables $R_\delta(T_\delta^-)$ representing the amount of capital immediately prior to ruin and $|R_\delta(T_\delta)|$ which shows deficit at ruin are of great importance. The function built with the help of these random variables and called the "expected discounted penalty function" or the Gerber-Shiu function in the scientific literature is one of the research objects of the presented dissertation and is defined as follows:

$$\Phi_{n;\delta,\alpha}(u) = E\left(e^{-\alpha T_\delta} \omega(R_\delta(T_\delta^-), |R_\delta(T_\delta)|) \mathbb{I}_{\{T_\delta < \infty\}} \middle| R_\delta(0) = u\right),$$

where $\alpha \geq 0$, $\mathbb{I}_B$ is a indicator function of any set B and $\omega(x_1, x_2)$, $0 < x_1, x_2 < \infty$, is non-negative function.

In paragraph 1.1 the Gerber-Shiu function is introduced and an auxiliary result is proved.

In paragraph 1.2 an integro-differential equation for the Gerber-Shiu expected discounted penalty function is constructed.

Theorem 1.2.1. Assume that conditions A1–A4 hold for the Erlang(n) insurance risk process with a constant interest rate. If

$$\int_0^\infty \int_0^\infty \omega(x_1, x_2) f_\eta(x_1 + x_2) dx_1\, dx_2 < \infty$$

then $\Phi_{n;\delta,\alpha}(u)$ satisfies the following integro-differential equation:

$$\beta^n \int_0^u \Phi_{n;\delta,\alpha}(u - x) f_\eta(x) dx + \beta^n \int_u^\infty \omega(u, x - u) f_\eta(x) dx =$$

$$= \mathbb{A}_\alpha^n \Phi_{n;\delta,\alpha}(u),$$

where

$$\mathbb{A}_\alpha = (\alpha + \beta)\mathbb{I} - (\delta u + c)\mathbb{D},$$

α, β, δ, c are constants, u is an operator variable, $\mathbb{I}$ and $\mathbb{D}$ denote the identity operator and differentiation operator, respectively, therefore for some function $h(u)$

$$\mathbb{I}h(u) \equiv h(u), \qquad \mathbb{D}h(u) \equiv \frac{d}{du} h(u),$$

$$\mathbb{A}_\alpha^n h(u) = \mathbb{A}_\alpha\left(\mathbb{A}_\alpha^{n-1} h(u)\right), \qquad n \geq 1, \qquad \mathbb{A}_\alpha^0 h(u) \equiv h(u).$$

If we take $\alpha = 0$ and $\omega(x_1, x_2) \equiv 1$, $\Phi_{n;\delta,\alpha}(u)$ reduces to the ruin probability $\Psi_{n;\delta}(u)$. Therefore, from Theorem 3.1 can be obtained the following corollary.

Corollary 1.2.1. Let the conditions of Theorem 3.1 be satisfied. Additionally, let $\alpha = 0$ and $\omega(x_1, x_2) \equiv 1$. Then the probability $\Psi_{n;\delta}(u)$ satisfies the following integro-differential equation:

$$\beta^n \int_0^u \Psi_{n;\delta}(u - x) f_\eta(x) dx + \beta^n \bar{F}_\eta(u) = A_0^n \Psi_{n;\delta}(u).$$

In paragraph 1.3 a differential equation for the Laplace transform of the Gerbe-Shiu function is constructed.

Theorem 1.3.1. Assume that conditions A1–A4 hold for the Erlang(n) insurance risk process with a constant interest rate. If $\omega(x_1, x_2)$ is bounded, then the Laplace transform of the Gerbe-Shiu function, denoted by $\widehat{\Phi}_{n;\delta,\alpha}(s)$ satisfies the following n-th order differential equation:

$$\sum_{i=0}^{n} \widehat{\Phi}_{n;\delta,\alpha}^{(i)}(s) K_{n,i}(s) - \beta^n \hat{f}_\eta(s) \widehat{\Phi}_{n;\delta,\alpha}(s) =$$

$$= \sum_{i=0}^{n-1} \Phi_{n;\delta,\alpha}^{(i)}(0) Q_{n,i}(s) + \beta^n \widehat{\omega}(s),$$

where $s > 0$ and $K_{n,i}(s)$ and $Q_{n,i}(s)$ are certain polynomials.

Corollary 1.3.1. Taking $\alpha = 0$ and $\omega(x_1, x_2) \equiv 1$ in Theorem 1.3.1 we can obtain the differential equation for the Laplace transform of the ruin probability $\Psi_{n;\delta}(u)$, denoted by $\widehat{\Psi}_{n;\delta}(s)$:

$$\sum_{i=0}^{n} \widehat{\Psi}_{n;\delta}^{(i)}(s) K_{n,i}(s) - \beta^n \hat{f}_\eta(s) \widehat{\Psi}_{n;\delta}(s) =$$

$$= \sum_{i=0}^{n-1} \Psi_{n;\delta}^{(i)}(0) Q_{n,i}(s) + \beta^n \hat{F}_\eta(s).$$

In the case of $\alpha = 0$ and $\omega(x_1, x_2) = \mathbb{I}_{(x_1 \le x, x_2 \le z)}$, $x > 0$, $z > 0$, $\Phi_{n;\delta,\alpha}(u)$ will represent the joint distribution of the surplus immediately prior to ruin and the deficit at ruin. N.Gaogin, L.Cihua, Xu.Lixia gave a recursive algorithm for the joint distribution function of the surplus immediately prior to ruin and the deficit at ruin in the Erlang(2) insurance risk process with constant interest rate[2]. **In paragraph 1.4** a recursive algorithm for the joint distribution function of the surplus immediately prior to ruin and the deficit at ruin in the Erlang(n) insurance risk process with constant interest rate is given by the similar method used by N.Gaoqin, L.Cihua, Xu.Lixia and Q.M.Lin, R.M.Wang.

Let us denote the joint distribution function of the surplus immediately before ruin and the deficit at ruin when ruin occurs at time T_k by $l_{n,k}(u, x, z)$:

$$l_{n,k}(u, x, z) =$$

$$= P_u \left\{ \bigcap_{i=1}^{k-1} \{R_\delta(T_i) \ge 0\}, 0 \le R_\delta(T_k^-) \le x, -z \le U_\delta(T_k) < 0 \right\},$$

where P_u indicates the conditional probability for the given event $R_\delta(0) = u$.

Theorem 1.4.1. Assume that conditions A1–A4 hold for the Erlang(n) insurance risk process with a constant interest rate, then the joint distribution of the surplus immediately prior to ruin and the deficit at ruin is

[2] Gaoqin, N., Cihua, L., Lixia, Xu. Expected discounted penalty function of Erlang(2) risk model with constant interest // – London: Appl. Math. J. Chinese Univ. Ser. B, – 2006. 21(3), –p. 243-251.

$$L_n(u,x,z) = \sum_{k=1}^{\infty} l_{n,k}(u,x,z),$$

where

$$l_{n,1}(u,x,z) = \frac{\beta^n}{(n-1)!} \times$$

$$\times \int_{t=0}^{[y(x,u)]_+} \left[F_\eta\big(z + v(t)\big) - F_\eta\big(v(t)\big)\right] t^{n-1} e^{-\beta t} dt,$$

$$l_{n,k}(u,x,z) = \frac{\beta^n}{(n-1)!} \int_0^{\infty} W_{n,k}(t) dt, \qquad k \geq 2,$$

$$W_{n,k}(t) = \int_0^{v(t)} l_{n,k-1}(v(t) - w, x, z) f_\eta(w) t^{n-1} e^{-\beta t} dw,$$

$$[y(x,u)]_+ = \max\left(0, \frac{1}{\delta} \ln\left(\frac{\delta x + c}{\delta u + c}\right)\right),$$

$$v(t) = u e^{\delta t} + c \frac{e^{\delta t} - 1}{\delta}.$$

The second chapter of the dissertation consists of 8 paragraphs.

The second chapter of the dissertation is devoted to the study of reinsurance risk processes. In this direction the works by T.Björk, J.Grandell, J.Cai, D.C.M.Dickson, J.Garrido, P.Embrechts, H.U.Gerber, E.S.W.Shiu, T.Jiang, C.Xu, V.Kalashnikov, C.Klüppelberg, H.S.Lin, G.E.Willmot, J.M.Reinhard, M.Snoussi, H.Schmidli, J.L.Teugels, H.Yang and other authors can be cited as examples. It is known that insurance companies enter into contracts to protect against financial losses associated with the uncertainty of the occurrence of insurance events. For this purpose, insurance companies also insure their risks in other companies. This form of insurance is called reinsurance, i.e. reinsurance is insurance for insurance companies. Reinsurance is one of the most important risk and capital management tools available to major insurance companies.

If the insurer effects reinsurance by paying a reinsurance premium continuously at a constant rate, then this process $R(t)$ becomes a net of reinsurance surplus process $R^*(t)$ given by

$$R^*(t) = u + c^*t - \sum_{i=1}^{\nu(t)} \eta_i^*,$$

where c^* denotes the insurer's premium income per unit time net of reinsurance, and η_i^* denotes the amount the insurer pays on the i-th claim, net of reinsurance

Basically, there are two types of reinsurance contracts: **proportional** and **non-proportional** reinsurance. Usually, there are two types of non-proportional reinsurance contracts: **excess of loss reinsurance** and **excess stop loss reinsurance**. If the insurer effects reinsurance, then the amount of claim paid by insurer is given by a function h in each type of reinsurance, so, if the amount of claim is x, then the insurer pays the amount of $h(x)$: $0 \leq h(x) \leq x$. It is clear that, in this case, the amount of claim that should be paid by reinsurance company is $x - h(x)$.

If the insurer effects proportional reinsurance, then the insurer pays some proportion α of each claim. In this case,

$$h(x) = \alpha x, \qquad 0 < \alpha \leq 1.$$

If the insurer effects excess of loss reinsurance with retention level $M > 0$, then the reinsurance company pays claims that exceed the level M. In this case,

$$h(x) = \min\{x; M\}, \qquad M > 0.$$

In non-proportional reinsurance, the reinsurance company can apply some upper bound L to insure itself against big losses, so, the maximum amount which can be paid by reinsurance company equals to $L > 0$. In this case, the insurer effects excess stop loss reinsurance with retention level $M > 0$ and upper bound $L > 0$. Then

$$h(x) = x - \min\{\max\{x - M; 0\}; L\} = \max\{\min\{x; M\}; x - L\}.$$

This chapter assumes that the following conditions are met for the reinsurance risk process:

B1) random variables ξ_i, $i \geq 1$, are independent and have the same distribution function F;
B2) random variables η_i, $i \geq 1$ are independent, have the same distribution function $F_\eta(x)$;
B3) the sequences $\{\xi_i\}$, $i \geq 1$ and $\{\eta_i\}$, $i \geq 1$ are independent of each other;
B4) $c^* E\xi_1 > E\eta_1^*$.

In paragraph 2.1 of the dissertation it is provided information on reinsurance risk contracts and obtained explicit expressions for the distribution function, first and second moments of random insurance payments η_1^* to be paid by the insurance company in each type of contract.

In paragraph 2.2 asymptotic expansions for the mathematical expectation $E\big(R^*(t)\big)$ of the reinsurance surplus process for each type of the reinsurance are obtained.

Theorem 2.2.1. Assume that conditions B1–B4 are satisfied for the reinsurance risk process. If F is a strongly non-lattice, $m_1^* = E\eta_1^*$ and $\mu_{k+2} = E\xi_1^{k+2}$, $k \geq 0$ exist, then the following asymptotic expansion as $t \to \infty$ can be written for the mathematical expectation of the reinsurance surplus process:

$$E\big(R^*(t)\big) = \left(c^* - \frac{m_1^*}{\mu_1}\right)t + u - \frac{m_1^*(\mu_2 - 2\mu_1^2)}{2\mu_1^2} + o(t^{-k}).$$

In paragraph 2.3 asymptotic expansions for the variance $\mathrm{Var}\big(R^*(t)\big)$ of the reinsurance surplus process for each type of the reinsurance are obtained.

Theorem 2.3.1. Assume that conditions B1–B4 are satisfied for the reinsurance risk process. If F belongs to the class ϑ and $m_2^* = E(\eta_1^*)^2$, $\mu_{k+3} = E\xi_1^{k+3}$, $k \geq 0$ exist, then the following asymptotic expansion as $t \to \infty$ can be written for the variance of the reinsurance surplus process:

$$\mathrm{Var}\big(R^*(t)\big) = A^*t + B^* + o(t^{-k}),$$

where

$$A^* = \frac{(m_1^*)^2\mu_2}{\mu_1^3} + \frac{m_2^* - 2(m_1^*)^2}{\mu_1},$$

$$B^* = \frac{5(m_1^*)^2\mu_2^2}{4\mu_1^4} - \frac{2(m_1^*)^2\mu_3}{3\mu_1^3} + \frac{(m_2^* - 2(m_1^*)^2)\mu_2}{2\mu_1^2} - m_2^* + (m_1^*)^2.$$

As it can be seen, the sum of a random number of random variables is involved in the expressions of the $R(t)$ insurance risk and $R^*(t)$ reinsurance risk processes:

$$\eta(t) = \sum_{i=1}^{\nu(t)} \eta_i\,, \qquad t \geq 0,$$

where $\nu(t) = \max\{n: T_n \leq t\}$, $t > 0$, and T_n is n-th renewal time:

$$T_n = \sum_{i=1}^{n} \xi_i\,, \qquad n = 1,2 \ldots$$

The process $\eta(t)$ is called **a renewal-reward process where the components are dependent random variables**, and is a generalization of a renewal process. Note that, in the renewal-reward process where the components are dependent random variables, η_i, $i \geq 1$ represents the claim at i-th renewal time (i.e. i-th claim) and in general, dependent from ξ_i. Therefore, the k-th initial moment of η_1 is defined as follows:

$$m_k \equiv E\eta_1^k = \int_0^\infty E\left(\eta_1^k \middle| \xi_1 = t\right) dF(t), \qquad k \geq 1,$$

where $E\left(\eta_1^k \middle| \xi_1 = t\right)$ is a conditional mathematical expectation of η_1^k given $\xi_1 = t$.

Thus, it is assumed that the following conditions are met for the renewal-reward process:

The mathematical expectation of the process $\eta(t)$ is called the reward function and is denoted by $D(t)$: $D(t) \equiv E\big(\eta(t)\big),\ t \geq 0$.

In paragraph 2.4 some estimations were obtained for the mathematical expectation of the renewal-reward process.

Theorem 2.4.1. Assume that conditions B1–B2 are satisfied for the renewal-reward process. If $\eta_1 \geq 0$ and $m_1 \equiv E\eta_1 < \infty$, then $D(t) < \infty$ for all $t \geq 0$.

Theorem 2.4.2. Assume that conditions B1–B2 are satisfied for the renewal-reward process. If $\eta_1 \geq 0$, $\mu_1 \equiv E\xi_1 < \infty$ and $m_1 \equiv E\eta_1 < \infty$, then for all $t \geq 0$

$$D(t) \leq m_1\left(2 + \frac{2t}{m_0}\right),$$

where m_0 is the median of distribution F.

Theorem 2.4.4. Assume that conditions B1–B2 are satisfied for the renewal-reward process. If $m_1 \equiv E\eta_1 < \infty$, then $|D(t)| \equiv \left|E\big(\eta(t)\big)\right| < \infty$ for all $t \geq 0$.

Theorem 2.4.5. Assume that conditions B1–B2 are satisfied for the renewal-reward process. If $\mu_1 \equiv E\xi_1 < \infty$ and $m_1 \equiv E\eta_1 < \infty$, then for all $t \geq 0$

$$|D(t)| \leq E|\eta_1|\left(2 + \frac{2t}{m_0}\right).$$

In paragraph 2.5 the asymptotic relation with reminder term $O(e^{-\varepsilon t})$ under the Cramer condition were obtained for the mathematical expectation of the renewal-reward process where the components are dependent random variables. For any random variable ξ (or its distribution), satisfying the Cramer condition means that $E(e^{\tau\xi}) < \infty$ for a certain $\tau > 0$.[3]

Theorem 2.5.1. Assume that conditions B1–B2 are satisfied for the renewal-reward process. Let F belong to the class ϑ, satisfy the Cramer condition and $q_{1,1} = E(\xi_1 e^{\varepsilon\xi_1}\eta_1)$ exists. Then the asymptotic expansion for the mathematical expectation of the renewal-reward process as $t \to \infty$ can be written in the following form:

$$D(t) = at + b + O(e^{-\varepsilon t}),$$

where $a = \mu_1^{-1} m_1$ və $b = \frac{1}{2}\mu_1^{-2}\mu_2 m_1 - \mu_1^{-1} n_{1,1}$.

In paragraph 2.6 the asymptotic expansion for the mathematical expectation of the renewal-reward process was derived when the time ξ_k between renewals has strong subexponential integrated tail function F_r and claims are non-negative random variables.

Let's introduce the following notation:

$$F^{+}(t) = \begin{cases} F(t), & t \geq 0 \\ 0, & t < 0 \end{cases}.$$

[3] Borovkov, A.A. Stochastic Processes in Queueing Theory / A.A.Borovkov. – Berlin: Springer, – 1976. – 279 p.

It is clear that if the random variable ξ has a distribution F, then the random variable $\xi^+ = \max(0, \xi)$ has a distribution F^+.

Definition 2.6.1. A distribution F on $\mathbb{R} = (-\infty, +\infty)$ is said to be **long-tailed**[4] ($F \in \mathcal{L}$) if $\bar{F}(t) > 0$ for all t and for any fixed $x > 0$

$$\lim_{t\to\infty} \frac{\bar{F}(t+x)}{\bar{F}(t)} = 1.$$

Definition 2.6.3. Let F be a distribution on $\mathbb{R} = (-\infty, +\infty)$ with $\bar{F}(t) > 0$ for all t and with finite mean on the positive half line. F is said to be **strong subexponential**[5] ($F \in \mathcal{S}^*$) if

$$\lim_{t\to\infty} \frac{\int_0^t \bar{F}(t-x)\bar{F}(x)dx}{\bar{F}(t)} = 2m,$$

where $m = E\xi^+$ and ξ has distribution F.

Theorem 2.6.3. Assume that conditions B1–B2 are satisfied for the renewal-reward process and

1) F is non-arithmetic;
2) $F_r \in \mathcal{S}^*$;
3) m_1 and $n_{1,1}$ exist;
4) $E(\eta_1 | \xi_1 = t) \le L$ as $t \to \infty$ for some $L > 0$.

Then the following asymptotic relation as $t \to \infty$ holds for the mathematical expectation of the renewal-reward process with non-negative claims:

$$D(t) = \frac{m_1}{\mu_1} t + \frac{\mu_2 m_1 - 2\mu_1 n_{1,1}}{2\mu_1^2} - \frac{m_1}{\mu_1} \int_t^\infty \bar{F}_r(x)dx + O\big(\bar{F}_r(t)\big).$$

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[4] Foss, S. An Introduction to Heavy-Tailed and Subexponential Distributions / Foss, S., Korshunov, D., Zachary, S.; –London: Springer Series in Operations Research and Financial Engineering, – 2011. – 123 p.

[5] Again, there

In paragraph 2.7 the asymptotic equivalence for the mathematical expectation of the renewal-reward process where the components are dependent random variables was derived when the time between renewals has regularly varying distribution.

Definition 2.7.1. A distribution F on $\mathbb{R} = (-\infty, +\infty)$ is called **regularly varying**[6] **with index $-\alpha$** ($F \in V_\alpha$) if for any fixed $c > 0$

$$\lim_{t\to\infty} \frac{\bar{F}(ct)}{\bar{F}(t)} = c^{-\alpha}.$$

Theroem 2.7.2. Assume that conditions B1–B2 are satisfied for the renewal-reward process. If $m_1 < \infty$ and $F \in V_\alpha \cap \mathcal{L}$, $0 \le \alpha \le 1$, then as $t \to \infty$

$$D(t) \sim \begin{cases} \dfrac{m_1}{\bar{F}(t)}, & \alpha = 0, \\ \dfrac{m_1 t^\alpha}{\bar{F}(t)} \dfrac{\sin \alpha\pi}{\alpha\pi}, & 0 < \alpha < 1, \\ \dfrac{m_1 t}{\int_0^t \bar{F}(x)dx}, & \alpha = 1. \end{cases}$$

In paragraph 2.8 the asymptotic expansion for the mathematical expectation of the renewal-reward process where the components are dependent random variables was derived when the time between renewals has dominated varying distribution.

Definition 2.8.1. A distribution F is called **dominated varying**[7] ($F \in \mathcal{D}$) if

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[6] Foss, S. An Introduction to Heavy-Tailed and Subexponential Distributions / Foss, S., Korshunov, D., Zachary, S.; –London: Springer Series in Operations Research and Financial Engineering, – 2011. – 123 p.

[7] Embrechts, P., Omey, E. A property of longtailed distributions // – Cambridge: J. Appl. Prob., – 1984. 21, – p. 80-87.

$$\limsup_{t\to\infty}\frac{\bar{F}(t/2)}{\bar{F}(t)}<\infty.$$

Theorem 2.8.1. Assume that conditions B1–B2 are satisfied for the renewal-reward process and

1) $F\in\mathcal{L}\cap\mathcal{D}$, F has a finite secont moment μ_2 and F is non-singular (or for some n, $F^{*(n)}$ has an absolutely continuous component);
2) m_1 and $n_{1,1}$ exist;
3) $E(\eta_1|\xi_1=t)\le L$ as $t\to\infty$ for some $L>0$.

Then the following asymptotic relation as $t\to\infty$ for the mathematical expectation of the renewal-reward process holds:

$$D(t)=\frac{m_1}{\mu_1}t-\frac{n_{1,1}}{\mu_1}+\frac{m_1}{\mu_1}\int_0^t\bar{F}_r(x)dx+o\big(t\bar{F}_r(t)\big).$$

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The main results of the dissertation:

1. the integro-differential equation for the Gerber-Shiu function and the differential equation for its Laplace transform, were obtained in Erlang(n) insurance risk process;
2. the explicit expression was found for the joint distribution of the surplus immediately prior to ruin and deficit at ruin in Erlang(n) insurance risk process;
3. the explicit expressions were obtained for the distribution function and the first and second moments of claims in the reinsurance risk process;
4. the asymptotic relations with the reminder term $o(t^{-k})$ were derived for the mathematical expectation and variance of the reinsurance risk process;
5. some estimations were obtained for the mathematical expectation of the renewal-reward process where the components are dependent random variables;
6. the asymptotic relation with reminder term $O(e^{-\varepsilon t})$ under the Cramer condition were obtained for the mathematical expectation of the renewal-reward process where the components are dependent random variables;
7. the asymptotic relation were obtained for the mathematical expectation of the renewal-reward process where the component has heavy-tailed distribution;
8. the asymptotic relation were obtained for the mathematical expectation of the renewal-reward process where the component has regularly varying distribution;
9. the asymptotic relation were obtained for the mathematical expectation of the renewal-reward process where the component has dominated varying distribution.

The main results of the dissertation are published in the following articles of the author:

1. Bayramov, V.A., Aliyev, R.T. On the asymptotic behaviour of the mathematical expectation of the renewal-reward process under the Cramer condition // –Baku: The Reports of National Academy of Sciences of Azerbaijan, – 2017. – Volume LXXIII, №1, – p.6-10.
2. Bayramov, V.A., Aliyev, R.T., Hasanova A.H. Constructing integro-differential equation for the Gerber-Shiu function in Erlang(n) insurance risk model with constant interest rate // – Baku:. Transaction of Azerbaijan National Academy of Sciences, Series of Physical-Technical and Mathematical Sciences: Informatics and Control Problems, – 2017. – Vol. XXXVII, No. 3, – p.64-68.
3. Aliyev, R.T., Bayramov, V.A. On the asymptotic behaviour of the covariance function of the rewards of a multivariate renewal-reward process // – Amsterdam: Statistics and Probability Letters, – 2017. 127, –p.138–149. (**Web of Science, SCI IF: 0.87)**
4. Bayramov, V.A., Aliyev, R.T. On the joint distiıbution of the surplus immediately prior to ruin and the deficit at ruin // – Baku: Advanced Mathematical Models & Applications, – 2018. – Vol.3, No.2, – p.137-141.
5. Bayramov, V.A. On the Laplace transform of the Gerber-Shiu function in a Sparre Andersen risk model with constant interest rate // – Baku: Transaction of Azerbaijan National Academy of Sciences, Series of Physical-Technical and Mathematical Sciences: Informatics and Control Problems, – 2018. – Vol. XXXVIII, No. 6, – p.53-58.
6. Bayramov, V.A., Abdullayeva, A.E., Aliyev, R.T. On the Asymptotic Expansions for the Expected Value and Variance of the Reinsurance Surplus Process // – Istanbul: Conference Proceeding Science and Technology, – 2020. – 3(1), – p.207–214.
7. Bayramov, V. On the mathematical expectation of the renewal-reward process // – Baku: Journal of Contemporary Applied

Mathematics, – 2019. – V. 9, No 2, – p.93-98. **(Zentralblatt MATH)**

8. Bayramov, V.A., Aliyev, R.T. On the asymptotic behavior of the mathematical expectation of the renewal-reward process // Seventeenth International Scientific Mykhailo Kravchuk Conference, Volume 3, – Kyiv: – 19-20 May, – 2016, – p. 19-22.
9. Bayramov, V.A., Aliyev, R.T. Asymptotic behaviors of the expectation and the variance of the renewal-reward process // XXVII International Conference Problems of Decision Making Under Uncertainties, – Tbilisi-Batumi, Georgia: – 23-27 May, – 2016, –p. 23.
10. Bayramov, V.Ə., Əliyev, R.T. Erlanq(n) risk modelində Gerber-Shiu funksiyası üçün inteqro-diferensial tənliyin qurulması // Əməkdar elm xadimi, professor Həbibzadənin anadan olmasının 100-cü ildönümünə həsr olunmuş "Funksional analiz və onun tətbiqləri" adlı respublika elmi konfransın materialları, – Bakı: "Bakı Universiteti" nəşriyyatı, – 2016, – s. 51-53.
11. Bayramov, V.Ə., Əliyev, R.T. Erlanq(n) sığorta riski modelində Gerber-Shiu funksiyasının Laplas çevirməsi üçün diferensial tənliyin qurulması // Azərbaycan Milli Elmlər Akademiyasının həqiqi üzvü, əməkdar elm xadimi, professor Rəsulovun 100 illik yubileyinə həsr olunmuş "Nəzəri və tətbiqi riyaziyyatın aktual məsələləri" respublika elmi konfransının materialları, – Şəki: – 28-29 oktyabr, – 2016, – s. 91-93.
12. Bayramov, V.A., Aliyev, R.T. Joint distribution of the surplus immediately prior to ruin and the deficit at ruin of Erlang(n) risk model with constant interest // XXIX International Conference Problems of Decision Making Under Uncertainties, – Mukachevo, Ukraine: – May 10-13, – 2017, – p. 23.
13. Bayramov, V.A., Aliyev, R.T., Hasanova A.H. Construction of integro-differential equation for the Gerber-Shiu function in Gamma insurance risk model with constant interest rate // Материалы III Международной научно-практической конференции, «Математическое моделирование в

экономике, управлении, образовании», Финансовый университет, Калужский филиал, – Москва: –2017, с. 3-5.
14. Bayramov, V.A., Aliyev, R.T. On the stochastic process with subexponential distributed and dependent components // XXXI International Conference Problems of Decision Making Under Uncertainties, – Lankaran-Baku, Republic of Azerbaijan: – July 27-31, – 2018, – p. 28-29.
15. Bayramov, V.A., Aliyev, R.T. On the renewal-reward process with regularly varying component // XXXII International Conference Problems of Decision Making Under Uncertainties, – Prague, Czech Republic: – August 27-31, – 2018, – p. 17.
16. Bayramov, V.A., Aliyev, R.T. On the renewal-reward process with dependent components and interclaim times having a distribution which belongs to the class $\mathcal{L}\cap\mathcal{D}$ // XXXIII International Conference Problems of Decision Making Under Uncertainties, – Hurgada, Egypt: – January 24 - February 1, – 2019, – p. 19.
17. Bayramov, V.A., Abdullayeva, A.E., Aliyev, R.T. On the mathematical expectation of the reinsurance surplus process // 3rd International Conference on Mathematical Advances and Applications (ICOMAA-2020), Yildiz Technical University, – Istanbul, Turkey, Online Video Conferencing: –2020, – p. 201.
18. Bayramov, V.A., Aliyev, R.T. On the mathematical expectation of the reinsurance surplus process with dependent components // XXXV International Conference Problems of Decision Making Under Uncertainties, – Baku-Sheki, Republic of Azerbaijan: May 11-15, – 2020, – p. 16-17.

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