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ABSTRACT

of the dissertation for the degree of Doctor of Science

BOUNDEDNESS CRITERIA OF THE FRACTIONAL MAXIMAL, FRACTIONAL INTEGRAL OPERATORS AND THEIR COMMUTATORS IN ORLICZ SPACES AND IN GENERALIZED ORLICZ-MORREY SPACES

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GENERAL DESCRIPTION OF WORK

Relevance of the topic and the degree of development. The dissertation is devoted to the following important sections of harmonic analysis such as: theory of function spaces, theory of maximal operators, theory of fractional maximal operators, theory of Riesz potential, which are closely interconnected and successfully complement each other. The development of functional analysis and the needs of the theory of differential equations led to the study of functional spaces. The above said quite well emphasizes the relevance of the subject matter of this dissertation and, as we see, has both theoretical and practical significance.

One of the main achievements of recent decades, influencing the appearance of harmonic analysis, cosists in successfully attracting the ideas and techniques of the theory of maximal operators, the theory of fractional maximal operatos and integral oppperators of the type of potential. These ideas and methods are applied in the theory of partial differential equations, function theory, functional analysis, probablity theory, problems of approximation theory, harmonic analysis on homogeneous groups and other sections of mathematics. A sistematic study of the maximal operator, fractional-maximal operator, and Riesz potential in function spaces originates from the classical Hardy-Littlewood-Sobolev theorem. Around the 1970s, the Hardy-Littlewood-Sobolev inequality was extended from Lebegue spaces to the Morrey spaces by J. Peetre and D.R. Adams. The Morrey space was introduced in 1938 by C.B.Morrey, in connection with some problems of elliptic equations and calculus of variations. Further study of these operators and their commutators in some functional spaces was continued by W. Orlicz, R.Coifman, R.Rochberg, G.Weiss, E.Stein, C.Fefferman, S.Chanillo, A. Cianchi, H.Kita, E.Nakai, S.Janson, Y.Sawano, V.Kokilashvilli, D.Yang, M.A.Ragusa, X.Fu, E.Sawano, V.T.Burenkov, L.Softova, V.S.Guliyev, A.Gogatishvili, R.Mustafaev, P.Zhang, A.M. Nadjafov et al.

Application of the potential theory method to solving many boundary value problems encountered in the theory of differential equations with partial derivatives, in problems of the theory of analytic functions, as well as in problems of mechanics, has a rich history and successful practice. The boundedness of fractional-maximal, fractional-integral operators and their commutators can be applied to the study of the regularity of solutions of elliptic equations with VMO coefficients in some function spaces. Various properties of the Riesz potential were investigated in the works of M. Riesz, G.Kh. Hardy, J.E. Littlewood, S.L. Sobolev, I. Stein and others. Among Azerbaijani mathematicians in this direction, we note the works of A.D. Gadjiev, S.K. Abdullayev, R.K.Seyfullayev, V.S. Guliyev, I.A. Aliyev, R.M. Rzayev, R. Bandaliyev, J.J.Hasanov, E.J.Ibrahimov, Y.Y.Mamedov, M.G.Gadjibekov and others.

A criterion for the boundedness of fractional-maximal, fractional-integral operators in Orlicz spaces in terms of capacity type was obtained by A.Cianchi[1]. For these operators to be bounded, we have proved necessary and sufficient conditions in Orlicz spaces in terms of Sobolev[15,19]. These results are different characteristics for the indicated operators. Moreover, we have studied a criterion for the boundedness of commutators of fractional-maximal, fractional-integral operators in Orlicz spaces. In the work of V.S.Guliyev and F.Deringoz[2] proved sufficient conditions for Spanne type boundedness of the Riesz potential and its commutator in generalized Orlicz-Morrey spaces. We have proved the necessary conditions. Moreover, we have proved necessary and sufficient conditions for Adams type boundedness of fractional-maximal, fractional-integral operators and their commutators in generalized Orlicz-Morrey spaces[9,14,17]. In the work of R.Zhang[3], boundedness of the commutator $[b, M]$ of the maximal operator in the Lebegue space, when b belongs to the Lipschitz space, is considered. We have proved boundedness criteria for commutators of fractional-maximal operators in Orlicz space when b belongs to Lipschitz and BMO spaces, respectively[15,18,19].

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[1] Cianchi, A. Strong and weak type inequalities for some classical operators in Orlicz spaces // Journal of London Mathematical Society -1999, 60(1), p. 247-286.

[2] Guliyev, V.S., Deringoz, F. On the Riesz potential and its commutators on generalized Orlicz-Morrey spaces // Journal of Functional Spaces. Article ID 617414, -2014, -11 p.

[3] Zhang, P. Characterization of Lipscitz spaces via commutators of the Hardy-Littlewood maximal function // C.R. Acad. Sci., -2017, v.355, -p. 336-344.

Object and subject of research. The object of the dissertation is the study of the boundedness criteria of the fractional maximal, fractional integral operators and their commutators in Orlicz spaces and in generalized Orlicz-Morrey spaces.

Purpose and objectives of research.

- obtaining the necessary and sufficient conditions for the boundedness of the fractional maximal operator, the Riesz potential and their commutators on the Orlicz spaces and the generalized Orlicz-Morrey spaces, respectively.
- characterization of the BMO space and the Lipschitz space using commutators of the fractional maximal operator and the Riesz potential in the Orlicz spaces and generalized Orlicz-Morrey spaces, respectively.
- obtaining the necessary and sufficient conditions for the boundedness of the maximal operator and its commutator in the generalized weighted Orlicz-Morrey spaces.
- the proof of the boundedness of the maximal operator in the vector-valued generalized weighted Orlicz-Morrey spaces and the proof of the two-weight inequality for the Riesz potential in the modular *p*-convex weighted Banach function spaces.
- obtaining the necessary and sufficient conditions for the boundedness of the multisublinear fractional maximal operator on the product modified Morrey spaces, as well as for the boundedness of the multilinear fractional integral operator on the product Morrey spaces and on the product modified Morrey spaces.
- the study of the boundedness of the parametric Marcinkiewicz integral in the generalized Orlicz-Morrey spaces.

Research methods. In the dissertation, methods of the theory of functions and functional analysis, the theory of integral operators are used.

The main provisions to be defended.

1. Necessary and sufficient conditions for the boundedness of the fractional-maximal, fractional-integral operators and their commutators in Orlicz spaces and in generalized Orlicz- Morrey spaces, respectively.

2. Criteria for the boundedness of the maximal operator and its commutator in generalized weighted Orlicz-Morrey spaces and two-weighted inequalities for the Riesz potential in p-convex Banach function spaces.

3. Criteria for the boundedness of the multisublinear fractional-maximal and multilinear fractional-integral operators on the product Morrey spaces and on the product modified Morrey spaces, respectively.

Scientific novelty of the research. The following main results were obtained in the dissertation:

1. Necessary and sufficient conditions are proved for the boundedness of the fractional maximal operators and their commutators on the Orlicz spaces.

2. Necessary and sufficient conditions are found for the boundedness of the Riesz potential and its commutator on the Orlicz spaces.

3. Criteria for the boundedness of fractional-maximal and fractional-integral operators and their commutators in generalized Orlicz-Morrey spaces are proved.

4. Necessary and sufficient conditions are proved for the boundedness of the maximal operator and its commutator on the generalized weighted Orlicz-Morrey spaces.

5. The boundedness of the maximal operator in the vector-valued generalized weighted Orlicz-Morrey spaces is proved.

6. Two-weighted inequalities for the Riesz potential in p-convex modular weighted Banach function spaces are proved.

7. Necessary and sufficient conditions are found for the boundedness of the multisublinear fractional maximal operator on the product modified Morrey spaces.

8. Necessary and sufficient conditions are proved for the boundedness of the fractional integral operator on the product Morrey spaces and on the product modified Morrey spaces.

9. The boundedness of the parametric Marcinkiewicz integral on the generalized Orlicz-Morrey spaces is proved.

Theoretical and practical value of the research. The dissertation work is theoretical in nature and contributes to the development of the theory of operators in Orlicz spaces and generalized Orlicz-Morrey spaces. The results obtained may be of interest not only for specialists in functional analysis and operator theory, on which it is primarily designed, but also, for example, in partial differential equations and variational problems, in view of the opening applications of new spaces.

Approbation and testing. The results of the dissertation were reported at the seminars of the department "Mathematical Analysis" of the IMM of the National Academy of Sciences of Azerbaijan (head. Cor. Mem. of NASA, Prof. V.S. Guliyev); at the seminars of the department "Differential Equations " (head. Doc. of Physics and Math., Prof. A.B. Aliyev); at the Institute-wide seminar of IMM NASA; at the seminars of the department "Mathematical Analysis " of the BSU (head. Doc. of Physics and Math., Prof. S.S. Mirzoev), at the seminars of the department "Theory of Functions and Functional Analysis " of the BSU (head. Doc. of Physics and Math., Prof. A.M. Akhmedov). The main results of the dissertation were also reported at the international conference on mathematics "Math. Anal., Differ. Equations" (Baku, 2015); at the international conference on mathematics "Nonharmonic Anal. and Differ. Oper." (Baku, 2016); at the international conference dedicated to the 80th anniversary of Acad. A. Gadjiev (Baku, 2017), at the international conference on mathematics "Operators in Morrey type spaces and Appl." (Turkey, 2017, 2019); at the international conference on mathematics "Modern Problems of Mathematics and Mechanics"(Baku, 2019).

The results can be applied in functional analysis and operator theory. Using the results obtained, one can investigate the qualitive properties of boundary value problems for elliptic operators. Estimates for fractional-integral operators and their commutators can be applied to the study of the regularity of solutions of elliptic equations with VMO cofficients in the function spaces under consideration. And also the boundedness of fractional-maximal operators and their commutators can be applied to the study of the boundedness of the Schrodinger operator in the function spaces under consideation.

Personal contribution of the author. All the main results to be defended were obtained by the author personally using the methods of functional analysis and operator theory. The scientific consultant took part in choosing the direction of research and analysis of the results.

Publications of the author.

- Publications in publications recommended by the HAC under the President of the Republic of Azerbaijan -16.
- Abstracts -6.

The name of the institution where the dissertation work was performed. The work was performed at "Mathematical Analysis" department of the Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan.

The structure and scope of the dissertation(in signs). Title page -419 signs, contents -2660 signs, introduction -83708 signs, contents of the dissertation -320244 signs(chapter I -83283 signs, chapter II -101442 signs, chapter III -77654 signs, chapter IV -57865 signs), conclusions -2101 sign, literature -26280 signs, total -435412 signs.

CONTENT OF THE DISSERTATION.

The main content of the dissertation consists of four chapters. In the first chapter of the dissertation, the characterization of the fractional maximal operator, fractional integral operator and their commutators in Orlicz spaces is defined and investigated. In this chapter necessary and sufficient conditions are proved for the boundedness of the fractional maximal operator M_α, Riesz potential I_α and their commutators in Orlicz spaces. As an application of these results, we study the boundedness of the fractional maximal commutator $M_{b,\alpha}$, nonlinear commutator $[b,M_\alpha]$, and commutator $[b,I_\alpha]$ when b belongs to the BMO space and the Lipschitz space and obtain new characteristics of the BMO and Lipschitz spaces, respectively. The first paragraph of the first charter sets out the necessary facts about the Orlicz space.

Definition 1. A function $\Phi:[0,+\infty)\to[0,\infty]$ is called a Young function if Φ is convex and continuous, $\lim\limits_{r\to+0}\Phi(r)=\Phi(0)=0$ and $\lim\limits_{r\to+\infty}\Phi(r)=\infty.$

The set of Young functions such that $0<\Phi(r)<+\infty$, $0<r<+\infty$ is denoted by $\mathcal{Y}$.

Definition 2. (Orlicz space). For a Young function Φ, the set

$$L^\Phi(\mathbb{R}^n)=\left\{f\in L^1_{\rm loc}(\mathbb{R}^n):\int\limits_{\mathbb{R}^n}\Phi(k\,|\,f(x)\,|)dx<+\infty \text{ for some } k>0\right\}$$

is called Orlicz space. The space $L^\Phi_{\rm loc}(\mathbb{R}^n)$ is defined as the set of all functions f such that $f\chi_B\in L^\Phi(\mathbb{R}^n)$ for all balls $B\subset\mathbb{R}^n$.

$L^\Phi(\mathbb{R}^n)$ is a Banach space with respect to the norm

$$\|f\|_{L^\Phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\}.$$

For a measurable set $\Omega \subset \mathbb{R}^n$, a measurable function f and $t>0$, let

$$m(\Omega, f, t) = |\{x \in \Omega : |f(x)| > t\}|.$$

In the case $\Omega = \mathbb{R}^n$, we for brevity denote by $m(f,t)$.

Definition 3. The weak Orlicz space

$$WL^\Phi(\mathbb{R}^n) := \{f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_{WL^\Phi} < +\infty\}$$

is defined by the norm

$$\|f\|_{WL^\Phi} = \inf\left\{\lambda > 0 : \sup_{t>0} \Phi(t) m\left(\frac{f}{\lambda}, t\right) \le 1\right\}.$$

For a Young function Φ and $0 \le s \le +\infty$, let

$$\Phi^{-1}(s) = \inf\{r \ge 0 : \Phi(r) > s\} \quad (\inf \varnothing = +\infty).$$

If $\Phi \in \mathrm{Y}$, then Φ^{-1} is the usual inverse function of Φ. It is well known that

$$\Phi(\Phi^{-1}(r)) \le r \le \Phi^{-1}(\Phi(r)) \;\text{ for }\; 0 \le r < +\infty.$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \le k\Phi(r),\; r>0,$$

for some $k>1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathrm{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \le \frac{1}{2k}\Phi(kr), \quad r \ge 0,$$

for some $k > 1$.

For a Young function Φ, the complementary function $\tilde{\Phi}(r)$ is defined by

$$\tilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\}, & r \in [0,\infty) \\ +\infty, & r = +\infty. \end{cases}$$

The complementary function $\tilde{\Phi}$ is also a Young function and it satisfies $\tilde{\tilde{\Phi}} = \Phi$.

Recall that the fractional maximal operator M_α is given by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)|\,dy, \quad 0 \le \alpha < n,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$ containing *x*, and the Riesz potential operator I_α is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \quad 0 < \alpha < n.$$

If α=0, then $M \equiv M_0$ is the Hardy-Littlewood maximal operator.

The fractional maximal commutator with a locally integrable function *b* is defined by

$$M_{b,\alpha} f(x) = \sup_{x\in B} |B|^{-1+\frac{\alpha}{n}} \int_B |b(x)-b(y)|\,|f(y)|\,dy,$$

where the supremum is taken over all balls B containing *x*.

If α=0, then $M_b \equiv M_{b,0}$ is the maximal commutator.

We can define the (nonlinear) commutator of the fractional maximal operator M_α with a locally integrable function by

$$[b,M_\alpha](f)(x) = b(x)M_\alpha(f)(x) - M_\alpha(bf)(x),$$

and the commutator $[b,I_\alpha]$ and the operator $|b,I_\alpha|$ are defined by (respectively)

$$[b,I_\alpha] = I_\alpha(bf)(x) - b(x)I_\alpha(f)(x), \quad |b,I_\alpha| = \int_{\mathbb{R}^n} \frac{|b(x)-b(y)|}{|x-y|^{n-\alpha}} f(y)\,dy.$$

In the second section we obtain necessary and sufficient conditions for the boundedness of M_α in Orlicz spaces and weak Orlicz spaces.

The main result of section 1.2 is

Theorem 1. Let $0 < \alpha < n$, Φ, Ψ be Young functions and $\Phi \in Y$. The condition

$$r^{-\frac{\alpha}{n}}\Phi^{-1}(r) \le C\Psi^{-1}(r) \qquad (1)$$

for all $r>0$, where $C>0$ does not depend on r, is necessary and sufficient for the boundedness of M_α from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$. Moreover, if $\Phi\in\nabla_2$, the condition (1) is necessary and sufficient for the boundedness of M_α from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

In the paragraph 1.3 of the first chapter of the dissertation work we were proved necessary and sufficient conditions for the boudedness of fractional maximal commutator $M_{b,\alpha}$ and nonlinear commutator of fractional maximal operator $[b,M_\alpha]$ on Orlicz spaces.

Definition 4. Suppose that $b\in L^1_{\rm loc}(\mathbb{R}^n)$, let

$$\|f\|_* = \sup_{x\in\mathbb{R}^n, r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)-f_{B(x,r)}|\,dy,$$

where $f_{B(x,r)} = \dfrac{1}{|B(x,r)|}\displaystyle\int_{B(x,r)} f(y)dy$.

Define

$$BMO(\mathbb{R}^n)=\{f\in L^1_{loc}(\mathbb{R}^n)\setminus\{\text{const}\}: \|f\|_*<\infty\}.$$

Modulo constants, the space $BMO(\mathbb{R}^n)$ is a Banach space with respect to the norm $\|\cdot\|_*$

The following theorem gives necessary and sufficient conditions for the boundedness of the operator $M_{b,\alpha}$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Theorem 2. Let $0<\alpha<n$, $b\in BMO(\mathbb{R}^n)$ and Φ, Ψ Young functions such that $\Phi\in Y$.

1. If $\Phi\in\nabla_2$ and $\Psi\in\Delta_2$, then the condition

$$r^\alpha\Phi^{-1}(r^{-n}) + \sup_{r<t<\infty}(1+\ln\frac{t}{r})\Phi^{-1}(t^{-n})t^\alpha \le C\Psi^{-1}(r^{-n})$$

for all $r>0$, where $C>0$ does not depend on r, is sufficient for the boundedness of $M_{b,\alpha}$ as an operator from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

2. If $\Psi\in\Delta_2$, then condition

$$r^{\alpha}\Phi^{-1}(r^{-n}) \le C\Psi^{-1}(r^{-n}) \qquad (2)$$

is necessary for the boundedness of $M_{b,\alpha}$ as an operator from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.

3. If $\Phi \in \nabla_2$, $\Psi \in \Delta_2$, and the condition

$$\sup_{r<t<\infty}(1+\ln\frac{t}{r})\Phi^{-1}(t^{-n})t^{\alpha} \le Cr^{\alpha}\Phi^{-1}(r^{-n}) \qquad (3)$$

holds for all $r>0$, where $C>0$ does not depend on r, then condition (2) is necessary and sufficient for the boundedness of $M_{b,\alpha}$ as an operator from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.

Definition 5. Let $0<\beta<1$. We say a function b belongs to the Lipschitz space $\dot{\Lambda}_{\beta}(\mathbb{R}^n)$ if there exists a constant C such that for all $x, y \in \mathbb{R}^n$

$$|b(x)-b(y)| \le C|x-y|^{\beta}.$$

The smallest such constant C is called the $\dot{\Lambda}_{\beta}(\mathbb{R}^n)$ norm of b and is denoted by $\|b\|_{\dot{\Lambda}_{\beta}(\mathbb{R}^n)}$.

Theorem 3. Let $0<\beta<1$, $0\le\alpha<n$, $0<\alpha+\beta<n$, $b\in L^1_{\mathrm{loc}}(\mathbb{R}^n)\setminus\{\mathrm{const}\}$, Φ, Ψ be Young functions and $\Phi \in \mathcal{Y}$.

1. If $\Phi\in\nabla_2$ and the condition

$$t^{-\frac{\alpha+\beta}{n}}\Phi^{-1}(t) \le C\Psi^{-1}(t), \qquad (4)$$

holds for all $t>0$, where $C>0$ does not depend on t, then the condition $b\in\dot{\Lambda}_{\beta}(\mathbb{R}^n)$ is sufficient for the boundedness of $M_{b,\alpha}$ from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.

2. If the condition

$$\Psi^{-1}(t) \le C\Phi^{-1}(t)t^{-\frac{\alpha+\beta}{n}}, \qquad (5)$$

holds for all $t>0$, where $C>0$ does not depend on t, then the condition $b\in\dot{\Lambda}_{\beta}(\mathbb{R}^n)$ is necessary for the boundedness of $M_{b,\alpha}$ from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.

3. If $\Phi \in \nabla_2$ and $\Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\frac{\alpha+\beta}{n}}$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Theorem 4. Let $0 < \beta < 1$, $0 \le \alpha < n$, $0 < \alpha + \beta < n$, $b \in L^1_{\rm loc}(\mathbb{R}^n) \setminus \{\rm const\}$, Φ, Ψ be Young functions and $\Phi \in Y$.

1. If condition (4) holds, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is sufficient for the boundedness of $M_{b,\alpha}$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.

2. If condition (5) holds and $\frac{t^{1+\varepsilon}}{\Psi(t)}$ is almost decreasing for some $\varepsilon > 0$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary for the boundedness of $M_{b,\alpha}$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.

3. If $\Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\frac{\alpha+\beta}{n}}$ and $\frac{t^{1+\varepsilon}}{\Psi(t)}$ is almost decreasing for some $\varepsilon > 0$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.

For a fixed ball B_0, the fractional maximal function with respect to B_0 of a function f is given by

$$M_{\alpha,B_0}(f)(x) = \sup_{B_0 \supseteq B \ni x} \frac{1}{|B|^{1-\frac{\alpha}{n}}} \int_B |f(y)|\, dy, \quad 0 \le \alpha < n,$$

where the supremum is taken over all the balls B with $B \subseteq B_0$ and $x \in B$.

Theorem 5. Let $0 < \beta < 1$, $0 \le \alpha < n$, $0 < \alpha + \beta < n$ and $b \in L^1_{\rm loc}(\mathbb{R}^n) \setminus \{\rm const\}$ be a locally integable non-negative function. Suppose that Φ, Ψ be Young functions, $\Phi \in Y \cap \nabla_2$ and $\Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\frac{\alpha+\beta}{n}}$. Then the following statements are equivalent:

1. $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$.

2. $[b, M_\alpha]$ is bounded from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

3. There exists a constant $C>0$ such that

$$\sup_{B} |B|^{-\beta/n}\,\Psi^{-1}(|B|^{-1})\,\| b(\cdot)-|B|^{-\alpha/n} M_{\alpha,B}(b)(\cdot)\|_{L^{\Psi}(B)} \le C.$$

In the paragraph 1.4 of the first chapter of the dissertation was obtained necessary and sufficient conditions for the boundedness of the Riesz potential on Orlicz spaces.

The following theorem gives necessary and sufficient conditions for the boundedness of the operator I_α from $L^{\Phi}(\mathbb{R}^n)$ to $WL^{\Psi}(\mathbb{R}^n)$ and from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.

Theorem 6. Let $0<\alpha<n$ and $\Phi,\Psi\in \mathcal{Y}$.

1. The condition

$$r^{\alpha}\Phi^{-1}(r^{-n})+\int_r^{\infty}\Phi^{-1}(t^{-n})t^{\alpha}\frac{dt}{t}\le C\Psi^{-1}(r^{-n}) \qquad (6)$$

for all $r>0$, where $C>0$ does not depend on r, is sufficient for the boundedness of I_α from $L^{\Phi}(\mathbb{R}^n)$ to $WL^{\Psi}(\mathbb{R}^n)$. Moreover, if $\Phi\in\nabla_2$, the condition (6) is sufficient for the boundedness of I_α from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.

2. The condition

$$r^{\alpha}\Phi^{-1}(r^{-n})\le C\Psi^{-1}(r^{-n}) \qquad (7)$$

for all $r>0$, where $C>0$ does not depend on r, is necessary for the boundedness of I_α from $L^{\Phi}(\mathbb{R}^n)$ to $WL^{\Psi}(\mathbb{R}^n)$ and from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.

3. If the regularity condition

$$\int_r^{\infty}\Phi^{-1}(t^{-n})t^{\alpha}\frac{dt}{t}\le Cr^{\alpha}\Phi^{-1}(r^{-n})$$

holds for all $r>0$, where $C>0$ does not depend on r, then the condition (7) is necessary and sufficient for the boundedness of I_α from $L^{\Phi}(\mathbb{R}^n)$ to $WL^{\Psi}(\mathbb{R}^n)$. Moreover, if $\Phi\in\nabla_2$, the condition (7) is necessary and sufficient for the boundedness of I_α from $L^{\Phi}(\mathbb{R}^n)$ to $L^{\Psi}(\mathbb{R}^n)$.

In the paragraph 1.5 of first chapter of the dissertation was proved necessary and sufficient conditions for the boundedness of the commutator of the Riesz potential on Orlicz spaces.

The following theorem gives necessary and sufficient conditions for the boundedness of the operator $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Theoreм 7. Let $0<\alpha<n$, $b \in BMO(\mathbb{R}^n)$ and $\Phi, \Psi \in \Upsilon$.

1. If $\Phi \in \nabla_2$ and $\Psi \in \Delta_2$, then the condition

$$r^\alpha \Phi^{-1}(r^{-n}) + \int_r^\infty \left(1+\ln\frac{t}{r}\right)\Phi^{-1}(t^{-n}) t^\alpha \frac{dt}{t} \le C\Psi^{-1}(r^{-n}) \qquad (8)$$

for all $r>0$, where $C>0$ does not depend on r, is sufficient for the boundedness of $[b, I_\alpha]$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

2. If $\Psi \in \Delta_2$, then the condition (2) is necessary for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

3. Let $\Phi \in \nabla_2$ and $\Psi \in \Delta_2$. If the condition

$$\int_r^\infty \left(1+\ln\frac{t}{r}\right)\Phi^{-1}(t^{-n}) t^\alpha \frac{dt}{t} \le C r^\alpha \Phi^{-1}(r^{-n}) \qquad (9)$$

holds for all $r>0$, where $C>0$ does not depend on r, then the condition (2) is necessary and sufficient for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Theoreм 8. Let $0<\alpha<n$, $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \setminus \{\mathrm{const}\}$ and $\Phi, \Psi \in \Upsilon$.

1. If $\Phi \in \nabla_2$, $\Psi \in \Delta_2$ and the condition (8) holds, then the condition $b \in BMO(\mathbb{R}^n)$ is sufficient for the boundedness of $[b, I_\alpha]$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

2. If $\Psi^{-1}(t) \lesssim \Phi^{-1}(t) t^{-\alpha/n}$, then the condition $b \in BMO(\mathbb{R}^n)$ is necessary for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

3. If $\Phi \in \nabla_2$, $\Psi \in \Delta_2$, $\Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\alpha/n}$ and the condition (9) holds, then the condition $b \in BMO(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $|b, I_\alpha|$ из $L^\Phi(\mathbb{R}^n)$ в $L^\Psi(\mathbb{R}^n)$.

As an application of the Theorem 6, we consider the boundedness of the commutators of Riesz potential operator $[b, I_\alpha]$ on Orlicz spaces when b belongs to the Lipschitz spaces by which some new characterizations of the Lipschitz spaces are given.

Theorem 9. Let $0<\beta<1$, $0<\alpha<n$, $0<\alpha+\beta<n$, $b\in L^1_{\mathrm{loc}}(\mathbb{R}^n)\setminus\{\mathrm{const}\}$, $\Phi,\Psi\in\mathcal{Y}$.

1. If $\Phi\in\nabla_2$ and the conditions

$$\int_t^\infty \Phi^{-1}(r^{-n})r^{\alpha+\beta}\frac{dr}{r}\le Ct^{\alpha+\beta}\Phi^{-1}(t^{-n}), \tag{10}$$

$$t^{-\frac{\alpha+\beta}{n}}\Phi^{-1}(t)\le C\Psi^{-1}(t), \tag{11}$$

hold for all $t>0$, where $C>0$ does not depend on t, then the condition $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ is sufficient for the boundedness of $[b,I_\alpha]$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

2. If the condition

$$\Psi^{-1}(t)\le C\Phi^{-1}(t)t^{-\frac{\alpha+\beta}{n}}, \tag{12}$$

holds for all $t>0$, where $C>0$ does not depend on t, then the condition $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary for the boundedness of $|b,I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

3. If $\Phi\in\nabla_2$, condition (10) holds and $\Psi^{-1}(t)\approx\Phi^{-1}(t)t^{-\frac{\alpha+\beta}{n}}$, then the condition $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $|b,I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $L^\Psi(\mathbb{R}^n)$.

Theorem 10. Let $0<\beta<1$, $0<\alpha<n$, $0<\alpha+\beta<n$, $b\in L^1_{\mathrm{loc}}(\mathbb{R}^n)\setminus\{\mathrm{const}\}$, $\Phi,\Psi\in\mathcal{Y}$.

1. If the conditions (10) and (11) are satisfied, then the condition $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ is sufficient for the boundedness of $[b,I_\alpha]$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.

2. If the condition (12) holds and $\dfrac{t^{1+\varepsilon}}{\Psi(t)}$ is almost decreasing for some $\varepsilon>0$, then the condition $b\in\dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary for the boundedness of $|b,I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.

3. If $\Psi^{-1}(t) \approx \Phi^{-1}(t) t^{-\frac{\alpha+\beta}{n}}$, condition (10) holds and $\frac{t^{1+\varepsilon}}{\Psi(t)}$ is almost decreasing for some $\varepsilon > 0$, then the condition $b \in \dot{\Lambda}_\beta(\mathbb{R}^n)$ is necessary and sufficient for the boundedness of $|b, I_\alpha|$ from $L^\Phi(\mathbb{R}^n)$ to $WL^\Psi(\mathbb{R}^n)$.

The second chapter is devoted to the boundedness of the fractional maximal, fractional integral operators and their commutators in generalized Orlicz-Morrey spaces.

Section 2.1 provides the necessary facts about Orlicz-Morrey space and generalized Orlicz-Morrey space.

Definition 6. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and Φ any Young function. We denote by $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ the generalized Orlicz-Morrey space, the space of all functions $f \in L^\Phi_{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{\mathcal{M}^{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} \Phi^{-1}(|B(x,r)|^{-1}) \|f\|_{L^\Phi(B(x,r))} < \infty.$$

A function $\varphi:(0,\infty)\to(0,\infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C>0$, such that

$$\varphi(r) \le C\varphi(s) \quad (\text{respectively } \varphi(r) \ge C\varphi(s)) \text{ to } r \le s.$$

For a Young function Φ, we denote by $\mathcal{G}_\Phi$ the set of all almost decreasing functions $\varphi:(0,\infty)\to(0,\infty)$ such that $\frac{\varphi(t)}{\Phi^{-1}(\upsilon_n^{-1} t^{-n})}, t\in(0,\infty)$ is almost increasing.

In section 2.2, necessary and suffisient conditions are proved for the Adams type strong (weak) boundedness of the fractional maximal operator in generalized Orlicz-Morrey spaces.

Theorem 11. Let $\Phi \in \nabla_2$ and $0 < \alpha < n$. Let $\varphi \in \Omega_\Phi$ satisfy the conditions

$$\sup_{r<t<\infty} \Phi^{-1}(|B(x,t)|^{-1}) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi(x,s)}{\Phi^{-1}(|B(x,s)|^{-1})} \le C\varphi(x,r), \tag{13}$$

and

$$r^\alpha \varphi(x,r) + \sup_{r<t<\infty} t^\alpha \varphi(x,t) \le C\varphi(x,r)^\beta, \tag{14}$$

for some $\beta \in (0,1)$ and for every $x \in \mathbb{R}^n$ and $r > 0$. Define $\eta(x,r) \equiv \varphi(x,r)^\beta$ and $\Psi(r) \equiv \Phi(r^{1/\beta})$. Then the operator M_α is bounded from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\eta}(\mathbb{R}^n)$.

The following theorem is one of our main results.

Theorem 12 (Adams type result). Let $0<\alpha<n$, Φ be any Young function, $\varphi \in \Omega_\Phi$, $\beta \in (0,1)$, $\eta(t) \equiv \varphi(t)^\beta$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$.

1. If $\Phi \in \nabla_2$ and $\varphi(t)$ satisfies (13) then the condition

$$t^\alpha \varphi(t) + \sup_{t<r<\infty} r^\alpha \varphi(r) \le C\varphi(t)^\beta, \tag{15}$$

for all $t>0$, where $C>0$ does not depend on t, is sufficient for the boundedness of M_α from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\eta}(\mathbb{R}^n)$.

2. If $\varphi \in \mathcal{G}_\Phi$, then the condition

$$t^\alpha \varphi(t) \le C\varphi(t)^\beta, \tag{16}$$

for all $t>0$, where $C>0$ does not depend on t, is necessary for the boundedness of M_α from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\eta}(\mathbb{R}^n)$.

3. Let $\Phi \in \nabla_2$ and $\varphi \in \mathcal{G}_\Phi$ satisfies condition

$$\sup_{t<r<\infty} r^\alpha \varphi(r) \le C t^\alpha \varphi(t), \tag{17}$$

for all $t>0$, where $C>0$ does not depend on t, then the condition (16) is necessary and sufficient for the boundedness of M_α from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\eta}(\mathbb{R}^n)$.

If we take $\Phi(t) = t^p$, $p \in [1,\infty)$ and $\beta = \frac{p}{q}$ with $p<q<\infty$ at Theorem 12 we get the following new result for the generalized Morrey spaces.

Corollary 1. Let $0<\alpha<n$, $1<p<q<\infty$ and $\varphi \in \Omega_p \equiv \Omega_{t^p}$.

1. If $\varphi(t)$ satisfies

$$\sup_{r<t<\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi(s) s^{\frac{n}{p}}}{t^{\frac{n}{p}}} \le C\varphi(r), \tag{18}$$

then the condition

$$t^{\alpha}\varphi(t)+\sup_{t<r<\infty} r^{\alpha}\varphi(r)\leq C\varphi(t)^{\frac{p}{q}},$$

for all $t>0$, where $C>0$ does not depend on *t*, is sufficient for the boundedness of M_{α} from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

2. If $\varphi\in\mathcal{G}_p\equiv\mathcal{G}_{t^p}$, then the condition

$$t^{\alpha}\varphi(t)\leq C\varphi(t)^{\frac{p}{q}}, \tag{19}$$

for all $t>0$, where $C>0$ does not depend on *t*, is necessary for the boundedness of M_{α} from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

3. If $\varphi\in\mathcal{G}_{\Phi}$ satisfies (17), then the condition (19) is necessary and sufficient for the boundedness of M_{α} from $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

If we take

$$\varphi(t)=\frac{\Phi^{-1}(v_n^{-1}t^{-n})}{\Phi^{-1}(v_n^{-1}t^{-\lambda})},\ 0\leq\lambda\leq n,\ \Psi(t)\equiv\Phi(t^{1/\beta}),\ \beta\in(0,1],$$

$$\eta(t)\equiv\varphi(t)^{\beta}=\left(\frac{\Phi^{-1}(v_n^{-1}t^{-n})}{\Phi^{-1}(v_n^{-1}t^{-\lambda})}\right)^{\beta}=\frac{\Psi^{-1}(v_n^{-1}t^{-n})}{\Psi^{-1}(v_n^{-1}t^{-\lambda})},$$

at Theorem 12, we get the following new result for Orlicz-Morrey spaces.

Corollary 2. Let $\Phi\in\nabla_2$, $\Psi(t)\equiv\Phi(t^{1/\beta})$ and $\beta\in(0,1)$. If

$$\sup_{t<r<\infty} r^{\alpha}\frac{\Phi^{-1}(v_n^{-1}r^{-n})}{\Phi^{-1}(v_n^{-1}r^{-\lambda})}\leq Ct^{\alpha}\frac{\Phi^{-1}(v_n^{-1}t^{-n})}{\Phi^{-1}(v_n^{-1}t^{-\lambda})},$$

for all $t>0$, where $C>0$ does not depend on *t*, then the condition

$$t^{\alpha}\leq C\left[\frac{\Phi^{-1}(v_n^{-1}t^{-n})}{\Phi^{-1}(v_n^{-1}t^{-\lambda})}\right]^{\beta-1}$$

for all $t>0$, where $C>0$ does not depend on *t*, is necessary and sufficient for the boundedness of M_{α} from $\mathcal{M}^{\Phi,\lambda}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\lambda}(\mathbb{R}^n)$.

Definition 7. For a Young function Φ and $\lambda \in \mathbb{R}$, we denote by $W\mathcal{M}^{\Phi,\lambda}(\mathbb{R}^n)$ the weak Orlicz-Morrey space, defined as the cpace of all functions $f \in WL^{\Phi}_{\rm loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{W\mathcal{M}^{\Phi,\lambda}} = \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}(|B(x,r)|^{-\frac{\lambda}{n}}) \|f\chi_{B(x,r)}\|_{WL^{\Phi}}.$$

Definition 8. Let $\varphi(x,r)$ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and Φ any Young function. We denote by $W\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ the weak generalized Orlicz-Morrey space, the space of all functions $f \in WL^{\Phi}_{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathcal{M}^{\Phi,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\Phi^{-1}(|B(x,r)|^{-1}) \|f\|_{WL^{\Phi}(B(x,r))} < \infty.$$

Theorem 13. Let Φ Young function and $0<\alpha<n$. Let $\varphi \in \Omega_\Phi$ satisfy the conditions (13) and (14). Define $\eta(x,t) \equiv \varphi(x,t)^{\beta}$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$ $\beta \in (0,1)$. Then the operator M_α is bounded from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $W\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

Theorem 14 (Weak version of Adams type result). Let $0<\alpha<n$, Φ Young function, $\varphi \in \Omega_\Phi$, $\beta \in (0,1)$ и $\eta(t) \equiv \varphi(t)^{\beta}$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$.

1. If $\varphi(t)$ satisfies (13), then the condition (15) is sufficient for the boundedness of M_α from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $W\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

2. If $\varphi \in \mathcal{G}_\Phi$, then the condition (16) is necessary for the boundedness of M_α from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $W\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

3. If $\varphi \in \mathcal{G}_\Phi$ satisfies condition (17), then the condition (16) is necessary and sufficient for the boundedness of M_α from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $W\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

In the Section 2.3 of the second chapter we obtain necessary and sufficient conditions for the Spanne type and Adams type boundedness of the commutator of the fractional maximal operator in generalized Orlicz-Morrey spaces, respectively..

The following theorem is one of our main results.

Theorem 15 (Spanne type result). Let $0 \le \alpha < n$, $\varphi_1 \in \Omega_\Phi$, $\varphi_2 \in \Omega_\Psi$ and $b \in BMO(\mathbb{R}^n)$.

1. Let $\Psi^{-1}(t)=\Phi^{-1}(t)t^{-\alpha/n}$ and $\Phi,\Psi\in\Delta_2\cap\nabla_2$, then the condition

$$\sup_{r<t<\infty}\left(1+\ln\frac{t}{r}\right)\Psi^{-1}(t^{-n})\operatorname*{ess\,inf}_{t<s<\infty}\frac{\varphi_1(s)}{\Phi^{-1}(s^{-n})}\le C\varphi_2(r) \tag{20}$$

for all $r>0$, where $C>0$ does not depend on *r*, is sufficient for the boundedness of $M_{b,\alpha}$ from $M^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M^{\Psi,\varphi_2}(\mathbb{R}^n)$.

2. If $\varphi_1\in G_\Phi$ and $\Psi\in\Delta_2$, then the condition

$$t^{\alpha}\varphi_1(t)\le C\varphi_2(t) \tag{21}$$

for all $t>0$, where $C>0$ does not depend on *t*, is necessary for the boundedness of $M_{b,\alpha}$ from $M^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M^{\Psi,\varphi_2}(\mathbb{R}^n)$.

3. Let $\Psi^{-1}(t)=\Phi^{-1}(t)t^{-\alpha/n}$ and $\Phi,\Psi\in\Delta_2\cap\nabla_2$. If $\varphi_1\in G_\Phi$ satisfies the condition

$$\sup_{r<t<\infty}\left(1+\ln\frac{t}{r}\right)t^{\alpha}\varphi_1(t)\le Cr^{\alpha}\varphi_1(r) \tag{22}$$

for all $r>0$, where $C>0$ does not depend on *r*, then the condition (21) is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $M^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M^{\Psi,\varphi_2}(\mathbb{R}^n)$.

If we take $\Phi(t)=t^p$, $\Psi(t)=t^q$, $p,q\in[1,\infty)$ at Theoreм 15, we get the following new result for generlized Morrey spces.

Corollary 3. Let $p,q\in[1,\infty)$, $0\le\alpha<n$, $\varphi_1\in\Omega_p\equiv\Omega_{t^p}$, $\varphi_2\in\Omega_q$ and $b\in BMO(\mathbb{R}^n)$.

1. Let $1<p<\dfrac{n}{\alpha}$, $\dfrac{1}{q}=\dfrac{1}{p}-\dfrac{\alpha}{n}$, then the condition

$$\sup_{r<t<\infty}\left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(s)s^{\frac{n}{p}}}{t^{\frac{n}{q}}}\le C\varphi_2(r)$$

for all $r>0$, where $C>0$ does not depend on *r*, is sufficient for the boundedness of $M_{b,\alpha}$ from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{q,\varphi_2}(\mathbb{R}^n)$.

2. If $\varphi_1\in G_\Phi$, then the condition (21) is necessary for the boundedness of $M_{b,\alpha}$ from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{q,\varphi_2}(\mathbb{R}^n)$.

3. Let $1 < p < \frac{n}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $\varphi_1 \in \mathcal{G}_p$ satisfies the condition (22), then the condition (21) is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{p,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi_2}(\mathbb{R}^n)$.

Theorem 16. Let $0 < \alpha < n$, $b \in BMO(\mathbb{R}^n)$, Φ Young function with $\Phi \in \Delta_2 \cap \nabla_2$. Let $\varphi \in \Omega_\Phi$ satisfy the conditions (20) and

$$r^\alpha \varphi(x,r) + \sup_{r<t<\infty} \left(1 + \ln\frac{t}{r}\right) t^\alpha \varphi(x,t) \le C\varphi(x,r)^\beta$$

for some $\beta \in (0,1)$ and for every $x \in \mathbb{R}^n$ and $r>0$. Define $\eta(x,r) \equiv \varphi(x,r)^\beta$ and $\Psi(r) \equiv \Phi(r^{1/\beta})$. Then, the operator $M_{b,\alpha}$ is bounded from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

The following theorem is one of our main results.

Theorem 17 (Adams type result). Let $0 < \alpha < n, \Phi \in \Delta_2$, $\varphi \in \Omega_\Phi$, $b \in BMO(\mathbb{R}^n)$, $\beta \in (0,1)$, $\eta(r) \equiv \varphi(r)^\beta$ and $\Psi(r) \equiv \Phi(r^{1/\beta})$.

1. If $\Phi \in \nabla_2$ and $\varphi(t)$ satisfies (20), then the condition

$$r^\alpha \varphi(r) + \sup_{r<t<\infty} \left(1 + \ln\frac{t}{r}\right) \varphi(t) t^\alpha \le C\varphi(r)^\beta$$

for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

2. If $\varphi \in \mathcal{G}_\Phi$, then the condition

$$r^\alpha \varphi(r) \le C\varphi(r)^\beta \tag{23}$$

for all $r > 0$, where $C > 0$ does not depen on r, is necessary for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

3. Let $\Phi \in \nabla_2$. If $\varphi \in \mathcal{G}_\Phi$ satisfies the condition

$$\sup_{r<t<\infty} \left(1 + \ln\frac{t}{r}\right) \varphi(t) t^\alpha \le C r^\alpha \varphi(r)$$

for all $r > 0$, where $C > 0$ does not depen on r, then the condition (23) is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\eta}(\mathbb{R}^n)$.

If we take $\Phi(t)=t^p$, $p\in[1,\infty)$ and $\beta=\frac{p}{q}$ with $p<q<\infty$ at Theorем 17 we get the following new result for generalized Morrey spaces.

Corollary 4. Let $0<\alpha<n$, $1\le p<q<\infty$, $\varphi\in\Omega_p$ and $b\in BMO(\mathbb{R}^n)$.

1. If $1<p<\infty$ and $\varphi(t)$ satisfies

$$\sup_{r<t<\infty}\left(1+\ln\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi(s)s^{\frac{n}{p}}}{t^{\frac{n}{p}}}\le C\varphi(r), \tag{24}$$

then the condition

$$r^{\alpha}\varphi(r)+\sup_{r<t<\infty}\left(1+\ln\frac{t}{r}\right)\varphi(t)t^{\alpha}\le C\varphi(r)^{\frac{p}{q}}$$

for all $r>0$ and $C>0$ does not depend on r, is sufficient for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

2. If $\varphi\in\mathcal{G}_\Phi$, then the condition

$$r^{\alpha}\varphi(r)\le C\varphi(r)^{\frac{p}{q}} \tag{25}$$

for all $r>0$ and $C>0$ does not depend on r, is necessary for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

3. Let $1<p<\infty$. If $\varphi\in\mathcal{G}_\Phi$ satisfies the condition

$$\sup_{r<t<\infty}\left(1+\ln\frac{t}{r}\right)\varphi(t)t^{\alpha}\le C\varphi(r)^{\frac{p}{q}}$$

for all $r>0$ and $C>0$ does not depend on r, then the condition (23) is necessary and sufficient for the boundedness of $M_{b,\alpha}$ from $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

In the paragraph 2.4 of the second chapter we get necessary and sufficient conditions for the Spanne type and Adams type boundedness of the Riesz potential I_α on the generalized spaces $\mathcal{M}^{\Phi,\varphi}$, respectively.

Theorем 18 (Spanne type result). Let Φ,Ψ be Young functions,

$\Phi,\Psi \in \mathcal{Y}$ and $0<\alpha<n$.

1.If the functions (Φ,Ψ) satisfy the conditions $\Phi \in \nabla_2$ and (6), then the condition

$$\int_t^\infty \operatorname*{ess\,inf}_{r<s<\infty} \frac{\varphi_1(s)}{\Phi^{-1}(\upsilon_n^{-1}s^{-n})} \Psi^{-1}(\upsilon_n^{-1}r^{-n}) \frac{dr}{r} \le C\varphi_2(t) \tag{26}$$

for all $t>0$, where $C>0$ does not depend on *t*, is sufficient for the boundedness of I_α from $M^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M^{\Psi,\varphi_2}(\mathbb{R}^n)$.

2. If the function $\varphi_1 \in \mathcal{G}_\Phi$, then the condition

$$t^\alpha \varphi_1(t) \le C\varphi_2(t) \tag{27}$$

for all $t>0$, where $C>0$ does not depend on *t*, is necessary for the boundedness of I_α from $M^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M^{\Psi,\varphi_2}(\mathbb{R}^n)$.

3. Let the functions (Φ,Ψ) satisfy the conditions $\Phi \in \nabla_2$ and (6). If $\varphi_1 \in \mathcal{G}_\Phi$ satisfies the regularity type condition

$$\int_t^\infty \frac{\Psi^{-1}(\upsilon_n^{-1}r^{-n})}{\Phi^{-1}(\upsilon_n^{-1}r^{-n})} \varphi_1(r) \frac{dr}{r} \le Ct^\alpha \varphi_1(t), \tag{28}$$

for all $t>0$, where $C>0$ does not depend on *t*, then the condition (27) is necessary and sufficient for the boundedness of I_α from $M^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M^{\Psi,\varphi_2}(\mathbb{R}^n)$.

If we take $\Phi(t)=t^p$, $\Psi(t)=t^q$, $p,q\in[1,\infty)$ at Theorem 18 we get the following new result for generalized Morrey spaces

Corollary 5. Let $0<\alpha<n$ and $p,q\in[1,\infty)$.

1. If $1<p<\frac{n}{\alpha}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, then the condition

$$\int_t^\infty \frac{\operatorname*{ess\,inf}_{r<s<\infty} \varphi_1(s) s^{\frac{n}{p}}}{r^{\frac{n}{q}}} \frac{dr}{r} \le C\varphi_2(t), \tag{29}$$

for all $t>0$, where $C>0$ does not depend on *t*, is necessary for the boundedness of I_α from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{q,\varphi_2}(\mathbb{R}^n)$.

2. If the condition $\varphi_1 \in \mathcal{G}_p \equiv \mathcal{G}_{t^p}$ holds, then the condition (27) is necessary for the boundedness of I_α from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{q,\varphi_2}(\mathbb{R}^n)$.

3. Let $1<p<\frac{n}{\alpha}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. If $\varphi_1 \in \mathcal{G}_p$ satisfies the regularity condition

$$\int_t^\infty r^\alpha \varphi_1(r)\frac{dr}{r} \le C t^\alpha \varphi_1(t), \tag{30}$$

for all $t>0$, where $C>0$ does not depend on *t*, then the condition (27) is necessary and sufficient for the boundedness of I_α from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{q,\varphi_2}(\mathbb{R}^n)$.

The following theorem is one of our main results.

Theorem 19 (Adams type result for I_α). Let $0<\alpha<n$, $\Phi \in \mathcal{Y}$, $\beta \in (0,1)$ and $\eta(t) \equiv \varphi(t)^\beta$, $\Psi(t) \equiv \Phi(t^{1/\beta})$.

1. If $\Phi \in \nabla_2$ and $\varphi(t)$ satisfies (13), then the condition

$$t^\alpha \varphi(t) + \int_t^\infty r^\alpha \varphi(r)\frac{dr}{r} \le C\varphi(t)^\beta, \tag{31}$$

for all $t>0$, where $C>0$ does not depend on *t*, is sufficient for the boundedness of I_α from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\eta}(\mathbb{R}^n)$.

2. If $\varphi \in \mathcal{G}_\Phi$, then the condition

$$t^\alpha \varphi(t) \le C\varphi(t)^\beta, \tag{32}$$

for all $t>0$, where $C>0$ does not depend on *t*, is necessary for the boundedness of I_α from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\eta}(\mathbb{R}^n)$.

3. Let $\Phi \in \nabla_2$. If $\varphi \in \mathcal{G}_\Phi$ satisfies the regularity condition

$$\int_t^\infty r^\alpha \varphi(r)\frac{dr}{r} \le C t^\alpha \varphi(t) \tag{33}$$

for all $t>0$, where $C>0$ does not depend on *t*, then the condition (32) is necessary and sufficient for the boundedness of I_α from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\eta}(\mathbb{R}^n)$.

If we take $\Phi(t)=t^p$, $p\in[1,\infty)$ and $\beta=\frac{p}{q}$ with $p<q<\infty$ at Theorem 19 we get the following new result for generalized Morrey spaces.

Corollary 6. Let $1<p<q<\infty$.

1. If $\varphi(t)$ satisfies (18), then the condition

$$t^{\alpha}\varphi(t)+\int_t^{\infty} r^{\alpha}\varphi(r)\frac{dr}{r}\le C\varphi(t)^{\frac{p}{q}}, \tag{34}$$

for all $t>0$, where $C>0$ does not depend on t, is sufficient for the boundedness of I_α from $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

2. If $\varphi\in\mathcal{G}_p$, then the condition

$$t^{\alpha}\varphi(t)\le C\varphi(t)^{\frac{p}{q}}, \tag{35}$$

for all $t>0$, where $C>0$ does not depend on t, is necessary for the boundedness of I_α from $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

3. If $\varphi\in\mathcal{G}_p$ satisfies the regularity condition (33), then the condition (35) is necessary and suffisient for the boundedness of I_α from $\mathcal{M}^{p,\varphi}(\mathbb{R}^n)$ to $\mathcal{M}^{q,\varphi^{\frac{p}{q}}}(\mathbb{R}^n)$.

In the paragraph 2.5 of the second chapte we get necessary and sufficient conditions for the Spanne type and Adams type boundedness of the commutator $[b,I_\alpha]$ on generalized Orlicz-Morrey spaces, respectively.

The following theorem is one of our main results.

Theorem 20 (Spanne type result for $[b,I_\alpha]$). Let $0<\alpha<n$ and $b\in BMO(\mathbb{R}^n)$.

1. Let Φ be Young function and Ψ defined, via its inverse, by setting, for all t: $\Psi^{-1}(t):=\Phi^{-1}(t)t^{-\alpha/n}, t\in(0,\infty)$. If $\Phi,\Psi\in\Delta_2\cap\nabla_2$, then the condition

$$\int_r^{\infty}\left(1+\ln\frac{t}{r}\right)\operatorname*{ess\,inf}_{t<s<\infty}\frac{\varphi_1(s)}{\Phi^{-1}(s^{-n})}\Psi^{-1}(t^{-n})\frac{dt}{t}\le C\varphi_2(r),$$

for all $r>0$, where $C>0$ does not depend on r, is sufficient for the boundedness of $|b,I_\alpha|$ from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

2. Let Φ,Ψ be Young functions. If $\Psi\in\Delta_2$ and $\varphi_1\in\mathcal{G}_\Phi$, then the condition (27) is necessary for the boundedness of $|b,I_\alpha|$ from $\mathcal{M}^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $\mathcal{M}^{\Psi,\varphi_2}(\mathbb{R}^n)$.

3. Let Φ be a Young function and Ψ defined, via its inverse, by setting, for all $t\in(0,\infty), \Psi^{-1}(t):=\Phi^{-1}(t)t^{-\alpha/n}$ and $\Phi,\Psi\in\Delta_2\cap\nabla_2$. If $\varphi_1\in\mathcal{G}_\Phi$ satisfies the regularity type condition

$$\int_r^\infty\left(1+\ln\frac{t}{r}\right)t^\alpha\varphi_1(t)\frac{dt}{t}\le Cr^\alpha\varphi_1(r), \tag{36}$$

for all $r>0$, where $C>0$ does not depend on r, then the condition (27) is necessary and sufficient for the boundedness of $|b,I_\alpha|$ from $M^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M^{\Psi,\varphi_2}(\mathbb{R}^n)$.

Theorem 21 (Adams type result for $[b,I_\alpha]$). Let $0<\alpha<n$, $\Phi\in Y$, $b\in BMO(\mathbb{R}^n)$, $\beta\in(0,1)$ and $\eta(t)\equiv\varphi(t)^\beta$ and $\Psi(t)\equiv\Phi(t^{1/\beta})$.

1. If $\Phi\in\Delta_2\cap\nabla_2$ and $\varphi(t)$ satisfies the condition

$$\sup_{r<t<\infty}\left(1+\ln\frac{t}{r}\right)\Phi^{-1}(|B(x,t)|^{-1})\operatorname*{ess\,inf}_{t<s<\infty}\frac{\varphi(x,s)}{\Phi^{-1}(|B(x,s)|^{-1})}\le C\varphi(x,r),$$

where C does not depend on x and r, then the condition

$$r^\alpha\varphi(r)+\int_r^\infty\left(1+\ln\frac{t}{r}\right)\varphi(t)t^\alpha\frac{dt}{t}\le C\varphi(r)^\beta,$$

for all $r>0$, where $C>0$ does not depend on r, is sufficient for the boundedness of $|b,I_\alpha|$ from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\eta}(\mathbb{R}^n)$.

2. If $\Phi\in\Delta_2$ and $\varphi\in\mathcal{G}_\Phi$, then the condition (32) is necessary for the boundedness of $|b,I_\alpha|$ from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\eta}(\mathbb{R}^n)$.

3. Let $\Phi\in\Delta_2\cap\nabla_2$. If $\varphi\in\mathcal{G}_\Phi$ satisfies the conditions

$$\sup_{r<t<\infty}\left(1+\ln\frac{t}{r}\right)\varphi(t)\le C\varphi(r), \tag{37}$$

and

$$\int_r^\infty\left(1+\ln\frac{t}{r}\right)\varphi(t)t^\alpha\frac{dt}{t}\le Cr^\alpha\varphi(r), \tag{38}$$

for all $r>0$, where $C>0$ does not depend on r, then the condition (32) is necessary and sufficient for the boundedness of $|b,I_\alpha|$ from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $M^{\Psi,\eta}(\mathbb{R}^n)$.

In the section 2.6 of the second chapter was obtained necessary and sufficient conditions for the Spanne type and Adams type weak boundedness of Riesz potential I_α in weak generalized spaces $WM^{\Phi,\varphi}(\mathbb{R}^n)$, respectively.

Theorem 22 (Weak version of Spanne type result). Let $0<\alpha<n$ and Φ,Ψ are Young functions, $\Phi,\Psi\in Y$.

1. If the functions (Φ,Ψ) satisfy the condition (6), then the condition (26) is sufficient for the boundedness of I_α from $M^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM^{\Psi,\varphi_2}(\mathbb{R}^n)$.

2. If the function $\varphi_1\in G_\Phi$, then the condition (27) is necessary for the boundedness of I_α from $M^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM^{\Psi,\varphi_2}(\mathbb{R}^n)$.

3. Let the functions (Φ,Ψ) satisfy the condition (6). If $\varphi_1\in G_\Phi$ satisfies the condition (28), then the condition (27) is necessary and sufficient for the boundedness of I_α from $M^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM^{\Psi,\varphi_2}(\mathbb{R}^n)$.

The following theorem is one of our main results.

Theorem 23 (Weak version of Adams type result). Let $0<\alpha<n$, $\Phi\in Y$, $\beta\in(0,1)$ и $\eta(t)\equiv\varphi(t)^\beta$, $\Psi(t)\equiv\Phi(t^{1/\beta})$.

1. If $\varphi(t)$ satisfies (13), then the condition (31) is sufficient for the boundedness of I_α from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $WM^{\Psi,\eta}(\mathbb{R}^n)$.

2. If $\varphi\in G_\Phi$, then the condition (32) is necessary for the boundedness of I_α from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $WM^{\Psi,\eta}(\mathbb{R}^n)$.

3. If $\varphi\in G_\Phi$ satisfies the regularity condition (33), then the condition (32) is necessary and sufficient for the boundedness of I_α from $M^{\Phi,\varphi}(\mathbb{R}^n)$ to $WM^{\Psi,\eta}(\mathbb{R}^n)$.

The third chapter is devoted to the boundedness of the maximal operators and integral operators in some modular weighted spaces.

In section 3.1, we obtain necessary and sufficient conditions for the boundedness of the maximal operator and its commutator in generalized weighted Orlicz-Morrey spaces.

Definition 9. For, $1<p<\infty$, a locally integrable function $w:\mathbb{R}^n\to[0,\infty)$ is said to be an A_p weight if

$$\sup_{B\in\mathcal{B}}\left(\frac{1}{|B|}\int_B w(x)dx\right)\left(\frac{1}{|B|}\int_B w(x)^{-\frac{p'}{p}}dx\right)^{\frac{p'}{p}}<\infty.$$

A locally integrable function $w:\mathbb{R}^n\to[0,\infty)$ is said to be A_1 weight if

$$\frac{1}{|B|}\int_B w(y)dy\le Cw(x),\ \text{a.e. } x\in B$$

for some constant $C>0$. We define $A_\infty=\bigcup_{p\ge1}A_p$.

For any $w\in A_\infty$ and any Lebesgue measurable set E, we write $w(E)=\int_E w(x)dx$.

Definition 10. Let φ be a positive measurable function on $\mathbb{R}^n\times(0,\infty)$, let w be non-negative function on $\mathbb{R}^n$ and Φ any Young function. Denote by $M_w^{\Phi,\varphi}(\mathbb{R}^n)$ the generalized weighted Orlicz-Morrey space, the space of all function $f\in L_w^{\Phi,loc}(\mathbb{R}^n)$ such that

$$\|f\|_{M_w^{\Phi,\varphi}}=\sup_{x\in\mathbb{R}^n,r>0}\varphi(x,r)^{-1}\Phi^{-1}(w(B(x,r))^{-1})\|f\|_{L_w^\Phi(B(x,r))}\equiv$$

$$\equiv\sup_{B\in\mathcal{B}}\varphi(B)^{-1}\Phi^{-1}(w(B)^{-1})\|f\|_{L_w^\Phi(B)}<\infty.$$

The following theorem is one of our main results.

Theorem 24. Let Φ be a Young function and φ_1,φ_2 positive measurable functions on $\mathbb{R}^n\times(0,\infty)$.

1. If $\Phi\in\nabla_2$ and $w\in A_{i_\Phi}$, then the condition

$$\sup_{r<t<\infty}\left(\operatorname*{ess\,inf}_{t<s<\infty}\frac{\varphi_1(x,s)}{\Phi^{-1}(w(B(x,s))^{-1})}\right)\Phi^{-1}(w(B(x,t))^{-1})\le C\varphi_2(x,r),\quad(39)$$

where C does not depend on x and r, is sufficient for the boundedness of M from $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.

2. If the function $\varphi_1\in\mathcal{G}_w^\Phi$, then the condition

$$\varphi_1(x,r)\le C\varphi_2(x,r),\quad(40)$$

where C does not depend on x and r, is necessary for the boundedness of M from $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.

3. Let $\Phi \in \nabla_2$ and $w \in A_{i_\Phi}$. If $\varphi_1 \in \mathcal{G}_w^\Phi$, then the condition (40) is necessary and sufficient for the boundedness of M from $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.

Denote by $WM_w^{\Phi,\varphi}(\mathbb{R}^n)$ the weak generalized weighted Orlicz-Morrey space,the space of all functions $f \in WL_w^{\Phi,loc}(\mathbb{R}^n)$ such that

$$\| f \|_{WM_w^{\Phi,\varphi}} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\Phi^{-1}(w(B(x,r))^{-1}) \| f \|_{WL_w^\Phi(B(x,r))} < \infty.$$

Теорема 25. Let Φ be a Young function and φ_1, φ_2 positive measurable functions on $\mathbb{R}^n \times (0,\infty)$.

1. If $w \in A_{i_\Phi}$, then the condition (39) is sufficient for the boundedness of M from $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.

2. If the function $\varphi_1 \in \mathcal{G}_w^\Phi$, then the condition (40) is necessary for the boundedness of M from $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.

3. Let $w \in A_{i_\Phi}$. If $\varphi_1 \in \mathcal{G}_w^\Phi$, then the condition (40) is necessary and sufficient for the boundedness of M from $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $WM_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.

For the boundedness of the commutator M_b is valid

Theorem 26. Let $b \in BMO(\mathbb{R}^n)$, Φ be a Young function and φ_1, φ_2 positive measurable functions on $\mathbb{R}^n \times (0,\infty)$.

1. Let $\Phi \in \Delta_2 \cap \nabla_2$ and $w \in A_1$, then the condition

$$\sup_{r<t<\infty} \left(1+\ln\frac{t}{r}\right) \Phi^{-1}(w(B(x,t))^{-1}) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}(w(B(x,s))^{-1})} \le C\varphi_2(x,r),$$

where C does not depend on x and r, is sufficient for the boundedness of M_b from $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.

2. If $\Phi \in \Delta_2$, $\varphi_1 \in \mathcal{G}_w^\Phi$ and $w \in A_1$, then the condition (40) is necessary for the boundedness of M_b from $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.

3. Let $\Phi \in \Delta_2 \cap \nabla_2$ and $w \in A_1$. If $\varphi_1 \in \mathcal{G}_w^\Phi$ satisfies the condition

$$\sup_{r<t<\infty} \left(1+\ln\frac{t}{r}\right) \varphi_1(x,t) \le C\varphi_1(x,r),$$

where C does not depend on x and r, then the condition (40) is necessary and sufficient for the boundedness of M_b from $M_w^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M_w^{\Phi,\varphi_2}(\mathbb{R}^n)$.

In the paragraph 3.2 of third chapter we obtain bondedness of maximal operator on the vector-valued generalized weighted Otlicz-Morrey spaces.

Definition 11. Let φ be positive measurable function $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$, Φ be any Young function and $1 \le q \le \infty$. The vector-valued generalized weighted Orlicz-Morrey spaces $M_w^{\Phi,\varphi}(l_q) = M_w^{\Phi,\varphi}(l_q,\mathbb{R}^n)$ is defined as the set of all sequences $F = \{f_j\}_{j=1}^{\infty}$ of Lebesgue measurable functions on $\mathbb{R}^n$ such that

$$\| F \|_{M_w^{\Phi,\varphi}(l_q)} = \| \{f_j\}_{j=1}^{\infty} \|_{M_w^{\Phi,\varphi}(l_q)} := \Big\| \| \{f_j(\cdot)\}_{j=1}^{\infty} \|_{l_q} \Big\|_{M_w^{\Phi,\varphi}} < \infty.$$

The main result of the paragraph is

Теорема 27. Let $1 < q < \infty$, Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $w \in A_1$ and $(\Phi,\varphi_1,\varphi_2)$ sarisfies the condition

$$\int_r^{\infty} \left(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}(w(B(x,s))^{-1})} \right) \Phi^{-1}(w(B(x,t))^{-1}) \frac{dt}{t} \le C\varphi_2(x,r),$$

where C does not depend on x and r. Then the maximal operator M is bounded from $M_w^{\Phi,\varphi_1}(l_q)$ to $M_w^{\Phi,\varphi_2}(l_q)$, i.e., there is a constant $C > 0$ such that

$$\| MF \|_{M_w^{\Phi,\varphi_2}(l_q)} \le C \| F \|_{M_w^{\Phi,\varphi_1}(l_q)}$$

holds for all $F = \{f_j\}_{j=1}^{\infty} \in M_w^{\Phi,\varphi_1}(l_q)$.

Note that for $q = \infty$, we have the following more general result.

Theorem 28. Let $w \in A_{i_\Phi}$, $\Phi \in \nabla_2$ and $(\Phi,\varphi_1,\varphi_2)$ satisfies the condition (39). Then the maximal operator M is bounded from $M_w^{\Phi,\varphi_1}(l_\infty)$ to $M_w^{\Phi,\varphi_2}(l_\infty)$, i.e., there is a constant C>0 such that

$$\| MF \|_{M_w^{\Phi,\varphi_2}(l_\infty)} \le C \| F \|_{M_w^{\Phi,\varphi_1}(l_\infty)}$$

holds for all $F=\{f_j\}_{j=1}^{\infty}\in M_w^{\Phi,\varphi_1}(l_\infty)$.

Section 3.3 of the third chapter gives conditions of an integral-type on weights that ensure the boundedness of the Riesz operator from one modular p-convex weighted BFS to another and prove the two-weight inequality for the potential. In particular, sufficient conditions are given on weight functions that ensure the boundedness of the Riesz potential in the Musilak-Orlicz space.

Let (Ω,μ) be a complete σ-finite measure space. By $L_0=L_0(\Omega,\mu)$ we denote the collection of all realvalued μ-measurable functions on Ω.

Definition 12. We say that a real normed space X is a Banach function space (BFS) provided that:

P1) the norm $\|f\|_X$ is defined for any μ-measurable function f and, moreover, $f\in X$ if and only if $\|f\|_X<\infty$ and $\|f\|_X=0$ if and only if $f=0$ a.e.,

P2) $\|f\|_X=\||f|\|_X$ for all $f\in X$,

P3) if $0\le f_n\uparrow f\le g$ a.e., then $\|f_n\|_X\uparrow\|f\|_X$ (Fatou property),

P4) if E is a measurable subset of Ω such that $\mu(E)<\infty$, then $\|\chi_E\|<\infty$, where χ_E is the characteristic function of the set E,

P5) for any measurable set $E\subset\Omega$ with $\mu(E)<\infty$, there is a constant $C_E>0$ such that

$$\int_E f(x)dx\le C_E\|f\|_X.$$

Let $Z=\{0,\pm1,\pm2,...\}$. For $k\in Z$ we define

$E_k=\{x\in\mathbb{R}^n:2^k<|x|\le2^{k+1}\}$, $E_{k,2}=\{x\in\mathbb{R}^n:2^{k-1}<|x|\le2^{k+2}\}$,

$E_{k,1}=\{x\in\mathbb{R}^n:|x|\le2^{k-1}\}$, $\quad E_{k,3}=\{x\in\mathbb{R}^n:|x|>2^{k+2}\}$.

The main result of the section is

Theorem 29. Let $v(x)$ and $w(x)$ be weight functions on $\mathbb{R}^n$. Let X and Y be Banach function spaces of functions on $\mathbb{R}^n$ with the Lebesgue measure and norms $\|.\|_{X(\mathbb{R}^n)}$ and $\|.\|_{Y(\mathbb{R}^n)}$, respectively. Suppose that $X_v(\mathbb{R}^n)$ and $Y_w(\mathbb{R}^n)$ is corresponding weighted spaces and there exists $p>1$ such that $Y_w(\mathbb{R}^n)$ is a p-convex Banach function space. Let $I_s \in [L_p(\mathbb{R}^n); Y(\mathbb{R}^n)]$, $X_v(\mathbb{R}^n) \hookrightarrow L_{p,v}(\mathbb{R}^n)$ and let satisfy the following conditions:

1) $$A = \sup_{t>0} \left(\int_{|y|<t} v(y)^{-p'} dy \right)^{\alpha/p'} \left\| \frac{\chi_{\{|x|>t\}}}{|x|^{n-s}} \left(\int_{|y|<|x|} v(y)^{-p'} dy \right)^{(1-\alpha')/p'} \right\|_{Y_w} < \infty ,$$

2) $$B = \sup_{t>0} \left(\int_{|y|>t} (v(y)|y|^{n-s})^{-p'} dy \right)^{\beta/p'} \left\| \chi_{\{|x|<t\}} \left(\int_{|y|>x} (v(y)|y|^{n-s})^{-p'} dy \right)^{(1-\beta)/p'} \right\|_{Y_w} < \infty,$$

where $0<\alpha,\beta<1$;

3) $\exists C>0$, that

$$\operatorname*{ess\,sup}_{y\in E_k} w(y) \le C \operatorname*{ess\,inf}_{y\in E_{k,2}} v(y), \quad \forall k \in Z,$$

4) $\exists C>0$, that

$$\left\| \sum_k |g_k| \chi_{E_k} \right\|^p_{Y_w(\mathbb{R}^n)} \le C \sum_k \left\| |g_k| \chi_{E_k} \right\|^p_{Y_w(\mathbb{R}^n)} .$$

Then $I_s \in [X_v(\mathbb{R}^n); Y_w(\mathbb{R}^n)]$.

In the fourth chapter, integral operators are studied in some function spaces of Morrey type. The first paragraph of the fourth chapter sets the necessary and sufficient conditions for the boundedness of the multisublinear fractional maximal operator $\mathrm{M}_{\Omega,\alpha,m}$ on the product of modified Morrey spaces.

Definition 13. Let $1\le p<\infty$, $0\le\lambda\le n$. We denote by $L^{p,\lambda}(\mathbb{R}^n)$ the

Morrey space, and by $WL^{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space, the set of locally integrable functions $f(x)$, $x\in\mathbb{R}^n$, with the finite norms

$$\|f\|_{L^{p,\lambda}}=\sup_{x\in\mathbb{R}^n,t>0} r^{-\frac{\lambda}{p}}\|f\|_{L^p(B(x,r))},\ \|f\|_{WL^{p,\lambda}}=\sup_{x\in\mathbb{R}^n,t>0} r^{-\frac{\lambda}{p}}\|f\|_{WL^p(B(x,r))}$$

respectively.

Definition 14. Let $1\le p<\infty,\ 0\le\lambda\le n,\ [r]=\min\{1,r\}$. We denote by $\tilde{L}^{p,\lambda}(\mathbb{R}^n)$ the modified Morrey space, and by $W\tilde{L}^{p,\lambda}(\mathbb{R}^n)$ the weak modified Morrey space, the set of locally integrable functions $f(x),\ x\in\mathbb{R}^n$, with the finite norms

$$\|f\|_{\tilde{L}^{p,\lambda}}=\sup_{x\in\mathbb{R}^n,r>0}[r]^{-\frac{\lambda}{p}}\|f\|_{\tilde{L}^p(B(x,r))},\ \|f\|_{W\tilde{L}^{p,\lambda}}=\sup_{x\in\mathbb{R}^n,r>0}[r]^{-\frac{\lambda}{p}}\|f\|_{W\tilde{L}^p(B(x,r))}$$

respectively.

Let $1<s\le\infty,\ \Omega\in L^s(\mathrm{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$. The multisublinear fractional operator $\mathrm{M}_{\Omega,\alpha,m}$ with rough kernels Ω is defined by

$$\mathrm{M}_{\Omega,\alpha,m}(\vec{f})(x)=\sup_{r>0}\frac{1}{r^{nm-\alpha}}\int_{B(\vec{0},r)}|\Omega(\vec{y})|\prod_{j=1}^{m}|f_j(x-y_j)|\,d\vec{y},\ \ 0\le\alpha<nm.$$

The following theorem is one of our main results.

Theorem 30. Let $0<\alpha<mn,\ \ 1\le s'<\frac{mn}{\alpha}$ and $\Omega\in L^s(\mathrm{S}^{mn-1})$. Suppose that $\sum_{j=1}^{m}\frac{\lambda_j}{p_j}=\frac{\lambda}{p},\ \sum_{j=1}^{m}\frac{\lambda_j}{q_j}=\frac{\lambda}{q}$.

(i) If $p>s',\ \frac{\alpha}{mn}\le\frac{1}{p_j}-\frac{1}{q_j}\le\frac{\alpha}{m(n-\lambda_j)}$ and $0\le\lambda_j<n-\frac{\alpha p_j}{m}$, then the condition $\frac{\alpha}{n}\le\frac{1}{p}-\frac{1}{q}\le\frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator $\mathrm{M}_{\Omega,\alpha,m}$ from product modified Morrey space $\tilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times\cdots\times\tilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to the modified Morrey space $\tilde{L}^{q,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $f\in\tilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times\cdots\times\tilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\| \mathrm{M}_{\Omega,\alpha,m} f \|_{\tilde{L}^{q,\lambda}} \le C \prod_{j=1}^{m} \| f_j \|_{\tilde{L}^{p_j,\lambda_j}}.$$

(ii) If $p = s'$, $\frac{\alpha}{mn} \le \frac{1}{p_j} - \frac{1}{p_j q_j} \le \frac{\alpha}{m(n-\lambda_j)}$ and $0 \le \lambda_j < n - \frac{\alpha p_j}{m}$, then the condition $\frac{\alpha}{n} \le \frac{1}{s'} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator $\mathrm{M}_{\Omega,\alpha,m}$ from product modified Morrey space $\tilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \cdots \times \tilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to the weak modified Morrey space $W\tilde{L}^{q,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $f \in \tilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \cdots \times \tilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\| \mathrm{M}_{\Omega,\alpha,m} f \|_{W\tilde{L}^{q,\lambda}} \le C \prod_{j=1}^{m} \| f_j \|_{\tilde{L}^{p_j,\lambda_j}}.$$

(iii) If $s' \le \frac{mn-\lambda}{\alpha} \le p \le \frac{mn}{\alpha}$, then the operator $\mathrm{M}_{\Omega,\alpha,m}$ is bounded from $\tilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \cdots \times \tilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{\infty}(\mathbb{R}^n)$.

Section 4.2 of the fourth charter describes the necessary and sufficient conditions for the boundedness of the multilinear fractional integral operator $I_{\Omega,\alpha,m}$ on the product of Morrey spaces and on the product of modified Morrey spaces.

The multilinear integral operator with rough kernels Ω is defined by

$$I_{\Omega,\alpha,m}(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \Omega(\vec{y}) \frac{f_1(y_1)\cdots f_m(y_m)}{|(x-y_1,\dots,x-y_m)|^{nm-\alpha}} d\vec{y}.$$

Theoreм 31. Let $0 < \alpha < mn$, $1 < s \le \infty$ and $\Omega \in L^s(\mathrm{S}^{mn-1})$. Let also $\sum_{j=1}^{m} \frac{\lambda_j}{p_j} = \frac{\lambda}{p}$, $\frac{1}{p_j} - \frac{1}{q_j} = \frac{\alpha}{m(n-\lambda_j)}$ and $0 \le \lambda_j < n - \frac{\alpha p_j}{m}$, $j = 1,\dots,m$.

(i) If $p > s'$ and $\sum_{j=1}^{m} \frac{\lambda_j}{q_j} = \frac{\lambda}{q}$, then the condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator $I_{\Omega,\alpha,m}$

from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\cdots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $f\in L^{p_1,\lambda_1}(\mathbb{R}^n)\times\cdots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\| I_{\Omega,\alpha,m}f\|_{L^{q,\lambda}}\le C\prod_{j=1}^{m}\| f_j\|_{L^{p_j,\lambda_j}}.$$

(ii) If $p=s'$ and $\lambda\sum_{j=1}^{m}\frac{1}{p_jq_i}=\sum_{j=1}^{m}\frac{\lambda_j}{p_jq_j}$, then the condition $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator $I_{\Omega,\alpha,m}$ from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\cdots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $WL^{q,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $f\in L^{p_1,\lambda_1}(\mathbb{R}^n)\times\cdots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\| I_{\Omega,\alpha,m}f\|_{WL^{q,\lambda}}\le C\prod_{j=1}^{m}\| f_j\|_{L^{p_j,\lambda_j}}.$$

The following theorem is one of our main results.

Theorem 32. Let $0<\alpha<mn,\ 1<s\le\infty$ and $\Omega\in L^s(S^{mn-1})$. Let also $\sum_{j=1}^{m}\frac{\lambda_j}{p_j}=\frac{\lambda}{p},\ \frac{1}{p_j}-\frac{1}{q_j}=\frac{\alpha}{m(n-\lambda_j)}$ and $0\le\lambda_j<n-\frac{\alpha p_j}{m}, j=1,\dots,m$.

(i) If $p>s'$ and $\sum_{j=1}^{m}\frac{\lambda_j}{q_j}=\frac{\lambda}{q}$, then the condition $\frac{\alpha}{n}\le\frac{1}{p}-\frac{1}{q}\le\frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator $I_{\Omega,\alpha,m}$ from product modified Morrey spaces $\tilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times\cdots\times\tilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to $\tilde{L}^{q,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $f\in\tilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times\cdots\times\tilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\| I_{\Omega,\alpha,m}f\|_{\tilde{L}^{q,\lambda}}\le C\prod_{j=1}^{m}\| f_j\|_{\tilde{L}^{p_j,\lambda_j}}.$$

(ii) If $p=s'$ and $\lambda\sum_{j=1}^{m}\frac{1}{p_jq_j}=\sum_{j=1}^{m}\frac{\lambda_j}{p_jq_j}$, then the condition

$\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ is necessary and sufficient for the boundedness of the operator $I_{\Omega,\alpha,m}$ from product of modified Morrey spaces $\tilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times\cdots\times\tilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to weak modified Morrey space $W\tilde{L}^{q,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $f \in \tilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times\cdots\times\tilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\| I_{\Omega,\alpha,m} f \|_{W\tilde{L}^{q,\lambda}} \le C \prod_{j=1}^{m} \| f_j \|_{\tilde{L}^{p_j,\lambda_j}}.$$

In the section 4.3, sufficient conditions are found for $(\varphi_1, \varphi_2, \Phi)$, that ensure the boundedness of the parametric Marcinkiewicz integral from one generalized Orlicz–Morrey space to another. As an application of this result, the boundedness of the Marcinkiewicz integral associated with the Schredinger operator in the generalized Orlicz-Morrey space is proved.

The parametric Marcinkiewicz integral operator of higher dimension as follows:

$$\mu_\Omega^\rho(f)(x) = \left(\int_0^\infty \left| \frac{1}{t^\rho} \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-p}} f(y)dy \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

where $0 < \rho < n$.

We will also use the numerical characteristics of Young functions:

$$a_\Phi := \inf_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}, \quad b_\Phi := \sup_{t\in(0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}.$$

Theorem 33. Let $0 < \rho < n$, Φ any Young function, φ_1, φ_2 and Φ satisfy the condition

$$\int_r^\infty \left(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}(|B(x_0,s)|^{-1})} \right) \Phi^{-1}(|B(x_0,t)|^{-1}) \frac{dt}{t} \le C\varphi_2(x,r), \quad (41)$$

where C does not depend on x and r. Let also $\Omega \in L^\infty(S^{n-1})$. If Φ satisfies the condition $1 < a_\Phi \le b_\Phi < \infty$, then the operator μ_Ω^ρ is bounded from $M^{\Phi,\varphi_1}(\mathbb{R}^n)$ to $M^{\Phi,\varphi_2}(\mathbb{R}^n)$.

Similar to the classical Marsinkiewicz function, we define the Marsinkiewicz functions μ_j associated with the Schredinger operator L by

$$\mu_j^L f(x) = \left(\int_0^\infty \left| \int_{|x-y|\le t} K_j^L(x,y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{\frac{1}{2}},$$

where $K_j^L(x,y) = \tilde{K}_j^L(x,y)\,|x-y|$ and $\tilde{K}_j^L(x,y)$ is the kernel of $R_j = \dfrac{\partial}{\partial x_j} L^{-\frac{1}{2}},\ j = 1,\ldots,n.$

Theorem 34. Let $V \in B_n$, Φ be a Young function and (φ_1, φ_2) satisfies the condition (41). If $1 < a_\Phi \le b_\Phi < \infty$, then the operator μ_j^L is bounded from M^{Φ,φ_1} to M^{Φ,φ_2}.

COLCLUSIONS

The dissertation is devoted to obtaining criteria for the boundedness of maximal operators and integral operators in some function spaces.

One of the main achievements of recent decades, affecting the appearance of harmonic analysis, consists in successfully attracting ideas and techniques from the theory of maximal operators and integral operators such as potentials. These ideas and methods are applied in the theory of partial differential equations, function theory, functional analysis, probability theory, problem of approximation theory, harmonic analysis and other branches of mathematics. Therefore, the topic of the dissertation is relevant and is of special scientific interest.

In the dissertation, the following results were obtained.

1. Necessary and sufficient conditions for the boundedness of fractional maximal operators and their commutators in Orlicz spaces are proved.

2. Necessary and sufficient conditions for the boundedness of the Riesz potential and its commutator in Orlicz spaces are proved.

3. Necessary and sufficient conditions for the boundednes of fractional maximal operators and their commutators in generalized Orlicz-Morrey spaces are proved.

4. Necessary and sufficient conditions are found for the boundedness of the Riesz potential and its commutator in generalized Orlicz-Morrey spaces.

5. Necessary and sufficient condition for the boundedness of the Riesz potential is proved.

6. Necessary and sufficient conditions for the boundedness of the maximal operator and its commutator in the generalized weighted Orlicz-Morrey spaces are proved.

7. The boundedness of the maximal operator in generalized vector-valued weighted Orlicz-Morrey spaces is studied.

8. Two-weighted inequalities for the Riesz potential in p-convex weighted modular Banach function spaces are proved.

9. Necessary and sufficient conditions are proved for the boundedness of multisublinear fractional maximal operator on product Morrey spaces.

10. Necessary and sufficient conditions are proved for the boundedness of multilinear fractional integral operator on product Morrey spaces and on product modified Morrey spaces.

11. The boundedness of the Marcinkiewicz parametric integral on generalized Orlicz-Morrey spaces in studied.

Note that the results obtained on the dissertation are theoretical.

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