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**ABSTRACT**

of the dissertation for the degree of  
Doctor of Philosophy

**The existence of global solutions for systems semi-linear hyperbolic fourth order equations with anisotropic elliptical**

**Specialty: 1211.01 – Differential equations**

**Field of science: Mathematics**

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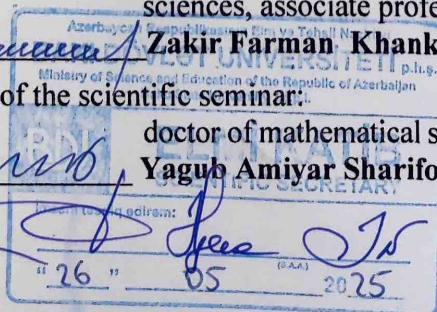
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## GENERAL CHARACTERISTICS OF THE WORK

**Actuality of the research theme and degree of processing.** The solution of different problems of natural science reduce to the study of various problems for partial nonlinear equations. Hyperbolic type semilinear equations are of special interest among these equations. The interest to this field of theory of differential equations is related to the solution of important applied problems and also to natural interest of mathematicians to difficult mathematical problems. Difficulty of investigation of various problems is due to the fact that unlike elliptic and parabolic operators there is no maximum principle for hyperbolic operators and hyperbolic operators are not monotone. For this reason, known powerful principles used when investigating elliptic and parabolic equations are not suitable for studying hyperbolic equations. In spite of numerous studies of various authors, there is no completed theory in this field. Each problem considered for nonlinear hyperbolic equations, requires a special approach. The main problem for nonlinear hyperbolic equations is global solvability of the Cauchy problem. The principal problem of theory of differential equations is finding such a class of differential equations and appropriate initial data that the appropriate Cauchy problem or a mixed problem has a global solution and also finding a class of nonlinear equations and initial data for which the appropriate Cauchy problem or a mixed problem has no global solution.

In 1976 A. Matsumura studied global solvability of the Cauchy problem for quasilinear hyperbolic equations with a dissipation. He proved that when the growth order of the nonlinear part satisfies certain conditions, the Cauchy problem has a global solution for sufficiently small initial data. Among nonlinear hyperbolic equations, the global solvability was

studied in detail for semilinear wave equations. The main problem for semilinear hyperbolic equations is in finding the exponent  $p^*$  characterized by the property that if the growth order of the nonlinearity is in the right vicinity of  $p^*$  then for sufficiently “small” initial data for which the appropriate Cauchy problem has a global solution, and if the growth order of nonlinearity is in the left vicinity of  $p^*$ , then there exist sufficiently small and sufficiently smooth initial data for which the appropriate Cauchy problem has no global solution.

In the case when  $f(h, x, u) = f(u) \approx |u|^p$  it was proved that  $p^* = 1 + \frac{2}{n}$  i.e. in this case the critical exponent coincides with the Fudjita-Hayakawa exponent known for semilinearparabolic equations. There are numerous studies in this direction and among them we can mention the work of E.Mitederi, S.I.Pohozhaev, S.Sattah, A.O.Milani, G.Todorov, Jordana, R.Ikehata, Y.Miyaoka, T.V.Nakatake, A.V.Aliyev, V.N.Lichaei, A.A.Kazimov, A.Pashaev, M.Abbico, M.R.Ebert, Y.Yamada, N.Nakao, E.L.Kaikina, P.Naumkin, Y.Sugitani, S.Kawashima, Y.Shibata, K.Nishihara, Y.Q.Lin, M.Reissiy, J.Lin, S.Klainman, Ponce, etc. In all these works the proof of theorems on global solvability was based on the behavior of solution of the resolving operator of the Cauchy problem as  $t \rightarrow \infty$  – for appropriate linear hyperbolic equation.

Recently, the existence or absence of global solutions of the Cauchy problem for hyperbolic and pseudohyperbolic equations with structural dissipation is intensively studied (see for example the works of M.Abbico, M.R.Ebert, A.B.Aliev, A.F.Pashaev, R.Ikehata).

The study of mixed problems for nonlinear hyperbolic equations is also of great importance. The Fudjita-Hayakava effect does not hold for mixed problems in a bounded domain,

but when studying mixed problems for nonlinear hyperbolic equations the existence of global solutions and finding such a class of initial data for which appropriate mixed problem has no global solution, is the main problem.

In this direction we can note the works of V.Georgiev, G.Todorov, S.A.Messaoudi, B.Said-Houari, M.Kirane, H.Yang, A.Milani, Q.S.Zhang where the following mixed problem is considered for a semi-linear hyperbolic equation with nonlinear dissipation

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{q-1} u_t = |u|^{p-1} u & t > 0, \quad x \in \Omega \\ u(t, x) = 0, & t > 0, \quad x \in \partial\Omega \\ u(t, x) = \varphi(x), \quad u_t(0, x) = \Psi(x), & x \in \Omega \end{cases}$$

where  $\Omega \subset R^n$  in a bounded domain with smooth boundary  $\partial\Omega$ .

The main issue of the given problem is that when some relations between  $p$  and  $q$  are fulfilled, the given problem has a global solution for any initial data from some functional space and when these relations are not fulfilled, there exist such initial data (even sufficiently smooth) for which appropriate local solution can not be continued on the interval, in fact at some time blow-up of solutions happens. In this work we study existence or absence of global solutions for the Cauchy problem or a mixed problem for a class of semilinear hyperbolic equations of fourth-order. The considered systems of equations are typical by the fact that each equation contains fourth order and second order derivatives with respect to various variables. Among this class of semilinear systems we find such semilinear fourth order systems for which the Cauchy problem has a global solution for sufficiently small and sufficiently smooth initial data. We find also a class of fourth order semilinear systems for which the

Cauchy problem has no global solution even for sufficiently small and sufficiently smooth initial data.

From the above mentioned ones we consider that the topic of the dissertation work is urgent.

**Goal and objectives of the study.** Using the  $L_p \rightarrow L_q$  type estimations for fourth order linear hyperbolic equations with an anisotropic elliptic part, to obtain theorem on global solvability of the Cauchy problem for a class of a system of semilinear hyperbolic equations of fourth order. To find conditions on the absence of solutions for the Cauchy problem and to get Fudjita type criteria for a class of systems of nonlinear hyperbolic equations of fourth order. To prove a theorem on local and global solvability of a mixed problem for a class of systems of semilinear hyperbolic equations of fourth order with nonlinear dissipation. To find a class of semilinear hyperbolic equations of fourth order with an anisotropic elliptic part and a class of initial data for which appropriate Cauchy problem has no global solution.

**Research methods.**In the work we use the Furior transform, methods of harmonic analysis, method of a priori estimates, method of operator-differential equations and the Galerkin method.

**Scientific novelty of the study.** In the work we obtained the following main results:

- Obtained  $L_2 \rightarrow L_m \cap W_2^\chi$  type estimations for linear hyperbolic equations of fourth order with an anisotropic elliptic part;
- Proved a theorem on local solvability of the Cauchy problem for a class of systems of semilinear hyperbolic equations of fourth order and established decreasing order of appropriate solutions at infinity with respect to time;

- Proved a theorem on local solvability of a mixed problem: for a class of fourth order semilinear hyperbolic equations with nonlinear dissipation;
- Proved a theorem on global solvability of a mixed problem for a class of system of semilinear fourth order hyperbolic equations with defocusing nonlinear part and with nonlinear dissipation;
- Proved a theorem on global solvability of a mixed problem for a class of system of fourth order semilinear hyperbolic equations with focusing nonlinear part and with nonlinear dissipation;
- Proved a theorem on the absence of global solution of the Cauchy problem for a class of systems of fourth order semilinear hyperbolic equations.

**Theoretical and practical value of the study.** The dissertation work is of theoretical character. The results of the dissertation work can be applied in numerous problems of modern physics, mechanics, geophysics, oceanology, hydrodynamics and so on.

**Approbation and application.** The results of the dissertation work were reported at the XVIII Republican conference of young scientists and doctoral students (Baku, 2013), at the international conference “Actual problems of Mathematics” devoted to 90-th jubilee of H.Aliyev (Baku, 2013), at the international conference “Mathematical analysis, differential equations and their applications” MADEA-7 (Baku, 2015) held together with Azerbaijan-Turkey-Ukraine Mathematicians, at the scientific seminars of the department of “Differential equations” of IMM ANAS (head:A.B.Aliyev).

**Authors personal contribution.** All the results on scientific novelties belong to the author.

**Published scientific works.** The full content of the dissertation work was reflected in 7 scientific works of the author given at the end of the work. Publications in the editions

recommended by Higher Certification Commission under President of the Republic of Azerbaijan is 6; (2 of them in WOS, 2 of them in Scopus), conference materials-4. Abstracts of papers-1.

The work was executed at the Institute of Mathematics and Mechanics of ANAS.

**Total volume of the dissertation work indicating separate structural units of the work in signs.** The work consists of introduction, three chapters and list of references. The volume of the work is 135 pages. References consists of 85 titles.

### **THE CONTENT OF THE DISSERTATION WORK**

In chapter I we study global solvability of the Cauchy problem for a system consisting of semilinear hyperbolic equations with an anisotropic elliptic part when the linear main part of each equation has fourth order derivatives with respect to some phase variables and second order derivatives with respect to other variable. At first we study local solvability of the appropriate problem and get a condition subject of which the local solution can be extended to the whole domain. The given condition is expressed in the form of priori estimation. As a result, using the asymptotics of fundamental solutions of linear problems, we get a priori estimations for solving appropriate nonlinear problems that allow to prove a global solvability theorem.

In 1.1 we give the problem statement and the main result of chapter 1.

In the domain  $[0, \infty) \times R_n$  we consider a Cauchy problem for the system of semilinear equations

$$\left. \begin{aligned} u_{1tt} + u_{1t} + \Delta_{I_1}^2 u_1 - \Delta_{J_1} u_1 &= f_1(u_1, u_2) \\ u_{2tt} + u_{2t} + \Delta_{I_2}^2 u_2 - \Delta_{J_2} u_2 &= f_2(u_1, u_2) \end{aligned} \right\}, \quad (1)$$

$$u_k(0, x) = \varphi_k(x), \quad u_{kt}(0, x) = \psi_k(x), \quad x \in R_N, \quad k = 1, 2 \quad (2)$$

where

$$\Delta_{I_k} = \sum_{i \in I_k} \frac{\partial^2}{\partial x_i^2}, \quad \Delta_{J_k} = \sum_{i \in J_k} \frac{\partial^2}{\partial x_j^2},$$

$$I_k \subset N_n = \{1, \dots, n\}, \quad J_k = N_n \setminus I_k, \quad k = 1, 2$$

By  $m_r = \overline{J_r}, r = 1, 2$  we denote the amount of elements by  $J_r$   $n_r = \overline{I_r} = n - m_r$  the amount of the elements  $I_r$ . For definiteness, we assume

$$m_1 \geq m_2. \quad (3)$$

The system of type (1) can be encountered when studying vibrations of deformed systems under moving loads (see for example. Kokhmanyuk S.S., Yanyutin E.T., Romanenko L.G. Vibrations of deformable systems under impulsive and moving loads. Kiev, Naukova Dumka (1980, 23 p.)

Suppose that the following conditions are fulfilled:

1.  $f_1(\cdot)$  and  $f_2(\cdot)$  are continuously differentiable functions on  $R^2$ . For any  $(u, v) \in R^2$  it is fulfilled the following estimation

$$|f_r(u, v)| \leq c |u|^{p_r} |v|^{q_r}, \quad (4)$$

where

$$p_r > 0, \quad q_r > 0, \quad p_r + q_r \geq 2, \quad , \quad (5)$$

$$\frac{n+m_1}{4} p_r + \frac{n+m_2}{4} q_r > 1 + \Psi(q_r), \quad r=1,2. \quad (6)$$

Here

$$\Psi(s) = \begin{cases} \frac{n+m_2}{4}, & s \geq 2, \\ \frac{n+m_2}{4} + \frac{(m_1-m_2)(2-s)}{8}, & 0 \leq s < 2. \end{cases}$$

$$2. \quad n+m_2 < 4 < n+m_1 \text{ if } p_r \leq \frac{n+m_1}{n+m_1-4},$$

$$n+m_2 = 4, \quad n+m_1 > 4, \quad r=1,2 \text{ if } p_r < \frac{n+m_1}{n+m_1-4}.$$

$$m_1 = m_2 = m, \quad 4 < n+m \leq 6, \quad r=1,2 \text{ if } p_r + q_r \leq \frac{n+m}{n+m-4}.$$

By  $W_{2,k}^{2s,s}$ ,  $k=1,2$  we denote functional spaces with the finite norm:

$$\|u\|_{W_{2,k}^{2s,s}}^2 = \left\{ \|u\|_{L_2(R_N)}^2 + \sum_{i \in I_k} \|D_{x_i}^{2s} u\|^2 + \sum_{j \in J_k} \|D_{x_j}^s u\|^2 \right\}^{\frac{1}{2}}.$$

Let  $U_\delta^k$  be a sphere of radius  $\delta > 0$  centered at the zero in the space  $[W_{2,k}^{2,1} \cap L_m(R_N)] \times [L_2(R_N) \cap L_m(R_N)]$  i.e.

$$U_\delta^k = \left\{ (u, v) : \|u\|_{W_{2,k}^{2,1}} + \|u\|_{L_m(R_N)} + \|v\|_{L_2(R_N)} + \|v\|_{L_m(R_N)} < \delta \right\}.$$

The main result of the given chapter is the following theorem.

**Theorem 1.1.** Let the hypotheses (3) and conditions 1-2 be fulfilled. Then there exists such  $\delta > 0$  that for any  $((\varphi_1, \psi_1), (\varphi_2, \psi_2)) \in U_\delta^1 \times U_\delta^2$  problem (1), (2) has a unique

solution

$$(u_1, u_2) \in C([0, \infty); W_{2,1}^{2,1} \times W_{2,2}^{2,1}) \cap C^1([0, \infty); L_2(R_n) \times L_2(R_n))$$

and for  $(u_1, u_2)$  the following estimations are valid:

$$\|u_\kappa(\cdot, t)\|_{L_2(R_n)} \leq C(\delta)(1+t)^{-\frac{n+m_\kappa}{8}}, \quad (7)$$

$$\sum_{i \in I_k} \|D_{x_i}^2 u_k(\cdot, t)\|_{L_2(R_n)} + \sum_{j \in J_k} \|D_{x_j} u_k(\cdot, t)\|_{L_2(R_n)} \leq C(\delta)(1+t)^{-\frac{n+m_k+4}{8}}, \quad (8)$$

$$\|u_{k_i}(t, \cdot)\|_{L_2(R_n)} \leq C(\delta)(1+t)^{-\gamma_k}, \quad (9)$$

where  $\gamma_k = \min \left\{ \frac{n+m_k+8}{8}, \frac{n+m_1}{8} p_k + \frac{n+m_1}{8} q_k \right\}, \kappa=1,2.$

In §1.2 we study local solvability of problem (1)-(2). To this end problem (1)-(2) is reduced to the Cauchy problem for a second order operator-differential equation. Using a theorem on solvability of the Cauchy problem for nonlinear differential equations in Hilbert space, we prove a theorem on the existence and uniqueness of local solutions (theorem 2.1). Applying theorem 2.1 for problem (1), (2) we get the following local solvability problem.

**Theorem 1.3.** Let conditions 1, 2 be fulfilled. Then for any  $\varphi_i \in W_{2,i}^{2,1}$ ,  $\psi_i \in L_3(R_n)$ ,  $i=1,2$  there exists such  $T' > 0$  that problem (1), (2) has a unique solution  $(u_1, u_2) \in C([0, T']; W_{2,1}^{2,1} \times W_{2,2}^{2,1}) \cap C^1([0, T']; L_2(R_n) \times L_2(R_n))$ .

If  $T_0 = \sup T'$  i.e. if  $T_0$  is the length of the maximum interval of the existence of the solution

$$(u_1, u_2) \in C([0, T_0]; W_{2,1}^{2,1} \times W_{2,2}^{2,1}) \cap C^1([0, T_0]; L_2(R_n) \times L_2(R_n))$$

then,

$$1) \text{ either } T_0 = +\infty; \quad 2) \text{ or}$$

$$\limsup_{t \rightarrow T_0 - 0} \sum_{k=1}^2 \left[ \|u_k(t, \cdot)\|_{W_{2,k}^{2,1}} + \|\dot{u}_{k_t}(t, \cdot)\|_{L_2(R_N)} \right] = +\infty .$$

From theorem 1.3 it follows that if it is fulfilled the following a priori estimation

$$\sum_{k=1}^2 \left[ \|u_k(t, \cdot)\|_{W_{2,k}^{2,1}} + \|\dot{u}_{k_t}(t, \cdot)\|_{L_2(R_N)} \right] \leq C, \quad t \in [0, T_0) \quad (10)$$

then  $T_0 = +\infty$ , i.e. in this case the global solution exists.

In §1.3 we study the asymptotics of solutions of the following problem

$$u_{tt} + u_t + \Delta_I u + \Delta_J^2 u = 0, \quad t > 0, \quad x \in R_n, \quad (11)$$

$$u(0, x) = \varphi(x), \quad u_t(0, x) = \psi(x), \quad x \in R_n, \quad (12)$$

where

$$\Delta_I = \sum_{i \in I_k} \frac{\partial^2}{\partial x_i^2}, \quad \Delta_J = \sum_{i \in J_k} \frac{\partial^2}{\partial x_j^2},$$

$$I \subset N_n = \{1, \dots, n\}, \quad J = N_n \setminus I.$$

Denote by  $m = \bar{J}$ , the amount of the elements  $J$ , by  $r = \bar{I} = n - m$  the amount of the elements  $I$ .

Let  $u_1 = F^{-1}[\hat{u}_1]$ ,  $u_2 = F^{-1}[\hat{u}_2]$ ,  $\hat{u}_1$ ,  $\hat{u}_2$  be the solutions of the following problems:

$$\begin{cases} L_\xi \hat{u}_1 = 0, \\ \hat{u}_1(0, \xi) = 0, \quad \hat{u}'_1(0, \xi) = 1 \end{cases} \quad \begin{cases} L_\xi \hat{u}_2 = 0, \\ u_2(0, \xi) = 1, \quad \hat{u}'_2(0, \xi) = 0. \end{cases}$$

where  $L_\xi \hat{u} = \hat{u}_t(t, \xi) + \left[ \sum_{k \in I} \xi_k^2 + \sum_{k \in J} \xi_k^4 \right] \hat{u}(t, \xi) + \hat{u}_t(t, \xi) = 0$ ,

$F^{-1}$  is an inverse Fourier transform.

**Theorem 1.4.** Let  $\psi \in L_2(R^n) \cap L_r(R^n)$ , then the following estimation is valid:

$$\begin{aligned} & \sum_{i \in I} \left\| D_{x_i}^2 (u_1(t, \cdot) * \psi(\cdot)) \right\|_{L_2(R_n)} + \sum_{j \in J} \left\| D_{x_j} (u_1(t, \cdot) * \psi(\cdot)) \right\|_{L_2(R_n)} \leq \\ & \leq c(1+t)^{-\left[ \frac{1}{2} + \frac{n+m}{4} \cdot \frac{2-r}{r} \right]} \left\| \psi(\cdot) \right\|_{L_r(R_n)} + ce^{-\omega t} \left\| \psi(\cdot) \right\|. \end{aligned}$$

**Theorem 1.5.** Let  $\varphi \in W_{2,I}^{2,1} \cap L_r(R^n)$ , then the following estimation is valid:

$$\begin{aligned} & \sum_{i \in I} \left\| D_{x_i}^2 (u_1(t, \cdot) * \varphi(\cdot)) \right\|_{L_2(R_n)} + \sum_{j \in J} \left\| D_{x_j} (u_1(t, \cdot) * \varphi(\cdot)) \right\|_{L_2(R_n)} \leq \\ & \leq c(1+t)^{-\left[ \frac{1}{2} + \frac{n+m}{4} \cdot \frac{2-r}{r} \right]} \left\| \varphi \right\|_{L_r(R_n)} + ce^{-\omega t} \left\| \varphi \right\| \end{aligned}$$

From the lemma (lemma 1.13-1.16 from of the dissertation) we get the following statements:

**Theorem 1.6.** Let  $\varphi, \psi \in W_{2,I}^{2,1} \cap L_r(R^n)$ , then the following estimations are valid:

$$\left\| u_1(t, \cdot) * \psi(\cdot) \right\|_{L_2(R_n)} \leq c(1+t)^{-\left[ \frac{1}{2} + \frac{n+m}{4} \cdot \frac{2-r}{r} \right]} \left\| \psi \right\|_{L_r(R_n)} + ce^{-\omega t} \left\| \psi \right\|$$

$$\|u_2(t, \cdot) * \varphi(\cdot)\|_{L_2(R_n)} \leq c(1+t)^{\left[-\frac{1}{2} + \frac{n+m}{4} - \frac{2-r}{r}\right]} \|\varphi\|_{L_r(R_n)} + ce^{-\omega t} \|\varphi\|$$

**Theorem 1.7.** Let  $\varphi \in W_{2,I}^{2,1} \cap L_1(R^n)$ ,  $\psi \in L_2(R_n) \cap L_1(R^n)$ , then the following estimations are valid:

$$\|D_t u_1(t, \cdot) * \psi(\cdot)\|_{L_2(R_n)} \leq c(1+t)^{\left[-1 + \frac{n+m}{8}\right]} \|\psi\|_{L_1(R_n)} + ce^{-\omega t} \|\psi\|_{L_2(R_n)}$$

$$\|D_t u_2(t, \cdot) * \varphi(\cdot)\|_{L_2(R_n)} \leq c(1+t)^{\left[-1 + \frac{n+m}{8}\right]} \|\varphi\|_{L_1(R_n)} + ce^{-\omega t} \|\varphi\|_{W_{2,I}^{2,1}}$$

In §1.4 using theorems 1.4-1.7 we get a priori estimation (10) for solving problem (1), (2) with initial data  $((\varphi_1, \psi_1), (\varphi_2, \psi_2)) \in U_\delta^1 \times U_\delta^2$ , where  $\delta > 0$  is a sufficiently small number.

In §1.5 we consider some general case and give some particular case of the system (1), (2).

In chapter II in the cylinder  $Q_T = [0, T] \times \Pi$  we study a mixed problem for a system of fourth order semilinear hyperbolic equations

$$\left. \begin{aligned} u_{1tt} + \Delta_{I_1}^2 u_1 - \Delta_{J_1} u_1 &= f_1(t, x, u_1, u_2) \\ u_{2tt} + \Delta_{I_2}^2 u_2 - \Delta_{J_2} u_2 &= f_2(t, x, u_1, u_2) \end{aligned} \right\} \quad (13)$$

with boundary conditions

$$u_k(t, x_1, x_2, \dots, x_i, \dots, x_n) = 0, \quad x_i = 0, x_i = 1, \quad k=1,2; \quad i=1, \dots, n; \quad (14)$$

$$\Delta_{I_k} u_k(t, x_1, x_2, \dots, x_i, \dots, x_n) = 0, \quad x_i = 0, x_i = 1, \quad i \in I_k, \quad k=1,2; \quad (15)$$

and initial conditions

$$u_k(0, x) = \varphi_k(x), \quad u_{kt}(0, x) = \psi_k(x), \quad x \in \Pi, \quad k = 1, 2, \quad (16)$$

Problem (13)-(16) was studied subject to the following conditions.

**I.** The functions  $f_1(t, x, \xi, \eta)$  and  $f_2(t, x, \xi, \eta)$  were determined for all  $t \in [0, T]$ ,  $x \in \Pi$ ,  $\xi \in R$ ,  $\eta \in R$  and are continuously differentiable with respect to  $t$ ,  $\xi$  and  $\eta$ ;

**II.** One of the following assumptions were fulfilled:  
either  $n + m_2 < 4$  or  $n + m_2 \geq 4$

**II a)** Let  $n + m_1 < 4$  and for all  $(t, x, \xi, \eta) \in [0, T] \times \Pi \times R^2$  the following estimations be fulfilled:

$$\begin{aligned} |f_k(t, x, \xi, \eta)| + |f'_{kt}(t, x, \xi, \eta)| &\leq c(\xi) [g_k(x) + |\eta|^{p_k}], \\ |f'_{k\xi}(t, x, \xi, \eta)| &\leq c(\xi) [h_k(x) + |\eta|^{q_k}], \\ |f'_{k\eta}(t, x, \xi, \eta)| &\leq c(\xi) [h_k(x) + |\eta|^{q_k}], \end{aligned}$$

where  $c(\cdot) \in C(R; R_+)$ ,  $g_k(\cdot) \in L_2(\Pi)$ ,  $p_k \in [1, \infty)$  if

$$n + m_2 = 4 \text{ and } p_k \leq \frac{n + m_2}{n + m_2 - 4} \text{ if } n + m_2 > 4.$$

**II b)** Let  $n + m_1 \geq 4$  and for all  $(t, x, \xi, \eta) \in [0, T] \times \Pi \times R^2$  the following estimations be fulfilled:

$$\begin{aligned} |f_k(t, x, \xi, \eta)| + |f'_{kt}(t, x, \xi, \eta)| &\leq c [g_k(x) + |\xi|^{p_{k1}} + |\eta|^{p_{k2}}] \\ |f'_{k\xi}(t, x, \xi, \eta)| &\leq c [h_k(x) + |\xi|^{q_{k1}} + |\eta|^{q_{k2}}]; \\ |f'_{k\eta}(t, x, \xi, \eta)| &\leq c [l_k(x) + |\xi|^{r_{k1}} + |\eta|^{r_{k2}}], \end{aligned}$$

where  $c \in R_+ = [0, \infty)$ ,  $g_k(\cdot) \in L_2(\Pi)$ ;  $h_k(\cdot), l_k(\cdot)$ ,  $p_{ki}$  and  $q_{ki}$  satisfy the following conditions:

**II b<sub>1</sub>)**  $h_k(\cdot), l_k(\cdot) \in L_p(\Pi)$ ,  $p \in (2, \infty)$ ,

$p_{ki}, q_{ki} \in (1, \infty)$ ,  $i=1,2; k=1,2$  if  $n+m_1 = n+m_2 = 4$ ;

**II b<sub>2</sub>)**  $h_k(\cdot) \in L_p(\Pi)$ ,  $p \in (1, \infty), l_k(\cdot) \in L_{\frac{n+m_2}{2}}(\Pi)$ ,  $k=1,2$ ,

$$p_{k1}, q_{k1}, r_{k1} \in (1, \infty), \quad p_{k2} \leq \frac{n+m_2}{n+m_2-4}, \quad q_{k2} < \frac{n+m_2}{n+m_2-4},$$

$$r_{k2} \leq \frac{4}{n+m_2-4}; \text{ if } n+m_1 = 4, \quad n+m_2 > 4$$

**II b<sub>3</sub>)**  $h_k(\cdot) \in L_{\frac{n+m_1}{2}}(\Pi)$ ,  $k=1,2$ ,  $l_k(\cdot) \in L_{\frac{n+m_2}{2}}(\Pi)$ ,  $k=1,2$

$$p_{k1} \leq \frac{n+m_1}{n+m_1-4}, \quad p_{k2} \leq \frac{n+m_2}{n+m_2-4}, \quad q_{k1} \leq \frac{4}{n+m_1-4},$$

$$q_{k2} \leq \frac{4(n+m_2)}{(n+m_2-4)(n+m_1)}, \quad k=1,2,$$

$$r_{k1} \leq \frac{4(n+m_1)}{(n+m_1-4)(n+m_2)}, \quad r_{k2} \leq \frac{4}{n+m_2-4}, \quad k=1,2, \quad \text{if}$$

$n+m_1 > 4$ .

Denote by

$$W_{2,k}^{2s,s} = W_2^{2s,s}(I_k, J_k) \quad k=1,2$$

functional spaces with the finite norm:

$$\|u\|_{W_{2,k}^{2s,s}(\Pi)}^2 = \left\{ \|u\|_{L_2(\Pi)}^2 + \sum_{i \in I_k} \|D_{x_i}^{2s} u\|_{L_2(\Pi)}^2 + \sum_{j \in J_k} \|D_{x_j}^s u\|_{L_2(\Pi)}^2 \right\}^{\frac{1}{2}}$$

Denote by  $\hat{W}_{2,k}^{2,1}$  the following subspace of the space  $W_{2,k}^{2,1}(\Pi)$

:

$$\hat{W}_{2,k}^{2,1} = \{u : u \in W_{2,k}^{2,1}(\Pi), u_k(t, x_1, x_2, \dots, x_i, \dots, x_n) = 0, \quad x_i = 0, x_i$$

,

and by  $\hat{W}_{2,k}^{4,2}$  denote the following subspace of the space

$$\hat{W}_{2,k}^{4,2}(\Pi):$$

$$\hat{W}_{2,k}^{4,2} = \left\{ u : u \in \hat{W}_{2,k}^{4,2}(\Pi), u_k(t, x_1, x_2, \dots, x_i, \dots, x_n) = 0, \quad x_i = 0, \quad x_i = 1, \right. \\ \left. \Delta_{I_k} u_k(t, x_1, x_2, \dots, x_j, \dots, x_n) = 0, \quad x_j = 0, \quad x_j = 1, \quad j \in I_k \right\}, \quad k = 1, 2$$

We introduce the functional space

$$H = \hat{W}_{2,1}^{2,1} \times L_2(R^n) \times \hat{W}_{2,2}^{2,1} \times L_2(R^n)$$

with a scalar product

$$\langle w^1, w^2 \rangle_H = \int_{\Pi} (\Delta_{I_1} v_1^1 \cdot \Delta_{I_1} v_1^2 + \nabla_{J_1} v_1^1 \cdot \nabla_{J_1} v_1^2) dx + \int_{\Pi} v_2^1 \cdot v_2^2 dx + \\ + \int_{\Pi} (\Delta_{I_2} v_3^1 \cdot \Delta_{I_2} v_3^2 + \nabla_{J_2} v_3^1 \cdot \nabla_{J_2} v_3^2) dx + \int_{\Pi} v_4^1 \cdot v_4^2 dx,$$

where  $w^k = (v_1^k, v_2^k, v_3^k, v_4^k) \in H$ ,  $k = 1, 2$ .

We prove the following local solvability theorem.

**Theorem 2.1.** Let conditions I-II be fulfilled. Then for any  $(\varphi_1, \psi_1, \varphi_2, \psi_2) \in H$  there exists such  $T' > 0$  that problem (13)-(16) has a unique solution

$$(u_1, u_2) \in C([0, T']; \hat{W}_{2,1}^{2,1} \times \hat{W}_{2,2}^{2,1}) \cap C^1([0, T']; L_2(R^n) \times L_2(R^n)).$$

If in addition  $(\varphi_1, \psi_1, \varphi_2, \psi_2) \in \hat{W}_{2,1}^{4,2} \times \hat{W}_{2,1}^{2,1} \times \hat{W}_{2,2}^{4,2} \times \hat{W}_{2,2}^{2,1}$  then

$$(u_1, u_2) \in C([0, T']; \hat{W}_{2,1}^{4,2} \times \hat{W}_{2,2}^{4,2}) \cap C^1([0, T']; \hat{W}_{2,1}^{2,1} \times \hat{W}_{2,2}^{2,1}) \cap C^2([0, T']; L_2(R^n) \times L_2(R^n)).$$

Moreover, if  $T_{\max}$  is the length of the maximum interval of the existence of the weak solution:

$$(u_1, u_2) \in C([0, T_{\max}); \hat{W}_{2,1}^{2,1} \times \hat{W}_{2,2}^{2,1}) \cap C^1([0, T_{\max}); L_2(R^n) \times L_2(R^n))$$

then one of the following statements is fulfilled

a) either  $T_{\max} = +\infty$  b) or  $\lim_{t \rightarrow T_{\max}^-} E(t) = +\infty$ , where

$$E(t) = \sum_{k=1}^2 \left[ \|u_k(t, \cdot)\|_{W_{2,k}^{2,1}} + \|u'_k(t, \cdot)\|_{L_2(R^n)} \right] \quad t \in [0, T_{\max}).$$

In § 2.2 we give the proof of theorem 2.1 on local solvability.

In § 2.3 in the domain  $Q_T = [0, T] \times \Pi$  we study a “global” solvability of a mixed problem for the system of fourth order semi-linear hyperbolic equations

$$\begin{cases} u_{1tt} + \Delta_{I_1}^2 u_1 - \Delta_{J_1} u_1 + |u_1|^{p_1-1} |u_2|^{p_2+1} u_1 = f_1(t, x), \\ u_{2tt} + \Delta_{I_2}^2 u_2 - \Delta_{J_2} u_2 + |u_1|^{p_1+1} |u_2|^{p_2-1} u_2 = f_2(t, x), \end{cases} \quad (17)$$

with boundary conditions (14), (15) and initial conditions (16), where

I.  $m_1 \leq m_2$ ,

II.  $f_k(t, x) \in L_2([0, T] \times \Pi)$ ,  $k = 1, 2$

III. Let the following conditions be fulfilled

1)  $p_1 > 0, p_2 > 0$ , if  $n + m_2 < 4$ ;

2)  $p_1 > 0, p_2 > 0$ , if  $n + m_2 \geq 4$  and  $n + m_1 \leq 4$

3)  $p_1 > 0, p_2 > 0$  and  $p_2 \leq \frac{4}{n + m_2 - 4}$  if  $n + m_1 < 4$  and

$n + m_2 > 4$ ;

4)  $p_1 > 0, p_2 > 0$  and  $p_1 + p_2 \leq \frac{4}{n + m - 4}$  if  $m_1 = m_2 = m$

and  $n + m > 4$ .

**Theorem 2.1.** Let conditions I-III be fulfilled. Then for any  $((\varphi_1, \psi_1), (\varphi_2, \psi_2)) \in H$  and  $T > 0$  mixed problem (14)-(17) has a unique solution  $(u_1, u_2) \in C([0, T]; W_{2,1}^{2,1} \times W_{2,2}^{2,1}) \cap C^1([0, T]; L_2(R^n) \times L_2(R^n))$ .

In § 2.4 in the domain  $Q_T = [0, T] \times \Pi$  we consider a mixed problem for a system of a fourth order semilinear hyperbolic equations

$$\begin{cases} u_{1tt} + \Delta_{I_1}^2 u_1 - \Delta_{J_1} u_1 + u_{1t} - |u_1|^{p_1-1} |u_2|^{p_2+1} u_1 = 0, \\ u_{2tt} + \Delta_{I_2}^2 u_2 - \Delta_{J_2} u_2 + u_{2t} - |u_1|^{p_1+1} |u_2|^{p_2-1} u_2 = 0, \end{cases} \quad (18)$$

with boundary conditions (14), (15) and initial conditions (16).

**Theorem 2.3.** Let conditions I-III be fulfilled. Then there exists such  $\delta > 0$  that for any  $((\varphi_1, \psi_1), (\varphi_2, \psi_2)) \in U_\delta$  mixed problem (14)-(16), (18) has a unique solution  $(u_1, u_2) \in C([0, \infty); W_{2,1}^{2,1} \times W_{2,2}^{2,1}) \cap C^1([0, \infty); L_2(R^n) \times L_2(R^n))$  and for

$$E(t) = \sum_{k=1}^2 \left[ \|u_k(t, \cdot)\|_{W_{2,k}^{2,1}} + \|u'_{kt}(t, \cdot)\|_{L_2(R^n)} \right], \quad t > 0 \quad \text{the estimation}$$

$E(t) \leq Me^{-\alpha t}$  is valid.

In chapter III we study non-existence of global solutions of the Cauchy problem and of mixed problem for fourth order semilinear equations.

In § 3.1 we study absence of global solutions of the Cauchy problem for the system of fourth order semilinear equations.

In the domain  $[0, \infty) \times R_n$  we consider the Cauchy problem

$$\left. \begin{aligned} u_{1tt} + u_{1t} + \Delta_{I_1}^2 u_1 - \Delta_{J_1} u_1 &= f_1(u_2) \\ u_{2tt} + u_{2t} + \Delta_{I_2}^2 u_2 - \Delta_{J_2} u_2 &= f_2(u_1) \end{aligned} \right\}, \quad (19)$$

with initial conditions

$$u_i(0, x) = \varphi_i(x), \quad u_{it}(0, x) = \psi_i(x), \quad x \in R^n, \quad i = 1, 2, \quad (20)$$

Here it is proved that conditions (4)-(6) are exact in some sense.

It is assumed that,

$$|f_1(u_2)| \geq c|u_2|^p, \quad |f_2(u_1)| \geq c|u_1|^q. \quad (21)$$

where

$$\max\{p, q\} \leq \frac{4}{n + m_2} \min\{p', q'\}, \quad (22)$$

$$p' = \frac{p}{p-1}, \quad q' = \frac{q}{q-1}$$

Let

$$\varphi_i(\cdot), \psi_i(\cdot) \in L_1(R^n), \quad i = 1, 2 \quad (23)$$

The pair of functions  $(u_1, u_2)$  where  $u_1 \in L_{p_2, loc}([0, \infty) \times R^n)$ ,  $u_2 \in L_{q_1, loc}([0, \infty) \times R^n)$   $f_1(u_2), f_i(u_1) \in L_{1, loc}([0, \infty) \times R^n)$  is called a weak solution of problem (19), (20) if for any

$\xi_1, \xi_2 \in C_0^{2,2}([0, \infty) \times R^n)$ ,  $\xi_i(t, x) \geq 0$ ,  $i = 1, 2$  the following identities are fulfilled,

$$\begin{aligned} & - \int_{R^n} (\varphi_1(x) + \psi_1(x)) \xi_1(x, 0) dx + \int_{R^n} \varphi_1(x) \frac{\partial \xi_1(x, 0)}{\partial t} dx + \int_0^\infty \int_{R^n} u_1 \frac{\partial^2 \xi_1}{\partial t^2} dx dt - \\ & - \int_0^\infty \int_{R^n} u_1 \frac{\partial \xi_1}{\partial t} dx dt - \int_0^\infty \int_{R^n} u_1 [\Delta_{t_1}^2 \xi_1 - \Delta \xi_1] dx dt = \int_0^\infty \int_{R^n} f_1(u_2) \xi_1 dx dt ; \end{aligned} \quad (24)$$

$$\begin{aligned}
& - \int_{R^n} (\varphi_2(x) + \psi_2(x)) \xi_2(x, 0) dx + \int_{R^n} \varphi_2(x) \frac{\partial \xi_2(x, 0)}{\partial t} dx + \int_0^\infty \int_{R^n} u_2 \frac{\partial^2 \xi_2}{\partial t^2} dx dt - \\
& - \int_0^\infty \int_{R^n} u_2 \frac{\partial \xi_2}{\partial t} dx dt - \int_0^\infty \int_{R^n} u_2 [\Delta_{I_2}^2 \xi_1 - \Delta \xi_2] dx dt = \int_0^\infty \int_{R^n} f_2(u_1) \xi_2 dx dt .
\end{aligned} \tag{25}$$

In §3.2 we prove the following theorem on the absence of global solutions.

**Theorem 3.1.** Assume that conditions (21) - (23) are fulfilled and in addition the following conditions are fulfilled as well

$$\sum_{i=1}^2 \int_{R^n} (\varphi_i(x) + \psi_i(x)) dx \geq 0, \quad i = 1, 2 \tag{26}$$

Then problem (19)-(20) does not have a weak global solution.

**Remark 3.1.** If we assume  $p \geq q$ , then  $p' \leq q'$ .

Then from (22) we get,

$$p \leq \frac{4}{n + m_2} p' = \frac{4}{n + m_2} \frac{p}{p - 1},$$

i.e.

$$q \leq p \leq 1 + \frac{4}{n + m_2} . \tag{27}$$

If we assume  $p \leq q$ , then in the similar way we get

$$q \leq p \leq 1 + \frac{4}{n + m_2} \tag{28}$$

Thus, subject to the conditions (21), (3.5), (3.8), (27) and (28) problem (19), (20) does not have a weak global solution. And this shows that conditions (4)- (6) in certain sense are important for the validity of statement of theorem 1.1.

In 3.2 we study appearance of collapse (blow-up) for a finite time.

In the domain  $[0, +\infty) \times \Pi$  we consider the following mixed problem

$$\begin{cases} u_{1tt} + \Delta_{I_1}^2 u_1 - \Delta_{J_1} u_1 + u_{1t} - |u_1|^{p_1-1} |u_2|^{p_2+1} u_1 = 0, \\ u_{2tt} + \Delta_{I_2}^2 u_2 - \Delta_{J_2} u_2 + u_{2t} - |u_1|^{p_1+1} |u_2|^{p_2-1} u_2 = 0, \end{cases} \quad (29)$$

with boundary conditions (14), (15) and initial conditions (16) and it is supposed that the following conditions are fulfilled:

- 1)  $p_1 > 0, p_2 > 0$  ; if  $n + m_2 \geq 4$  and  $n + m_1 \leq 4$
- 2)  $p_1 > 0, p_2 \leq \frac{4}{n + m_2 - 4}$  ; if  $n + m_1 < 4$  and  $n + m_2 > 4$  ;
- 3)  $m_1 = m_2 = m$  and  $n + m > 4$  if  $p_1 \leq \frac{4}{n + m - 4}$ ,  
 $p_2 \leq \frac{4}{n + m - 4}$ ; if  $n + m_1 < 4$  and  $n + m_2 > 4$ .

**Theorem 3.2.** Assume that conditions 1)-3) are fulfilled.

Let  $(\varphi_1, \psi_1, \varphi_2, \psi_2) \in \hat{W}_{2,1}^{2,1} \times L_2(R^n) \times \hat{W}_{2,2}^{2,1} \times L_2(R^n)$  and

$$\sum_{j=1}^2 (p_j + 1) \left[ |\psi_j|^2 + |\Delta_{I_j} \varphi_j|^2 + |\nabla_{J_j} \varphi_j|^2 \right] - 2G(0) < 0.$$

Then problem (14)-(16), (29) does not have a weak global solution.

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## CONCLUSIONS


1. The Cauchy problem for systems of semilinear fourth-order hyperbolic equations with an anisotropic elliptic part has unique local solution if the growth of the nonlinear part does not exceed the Sobolev exponent.
2. The mixed problem for systems of semilinear fourth-order hyperbolic equations with an anisotropic elliptic part has unique local solution if the growth of the nonlinear part does not exceed the Sobolev exponent.
3. For linear hyperbolic equations with an anisotropic elliptic part, the estimates of the solutions depend on the order of derivatives.
4. There exists a region on the plane such that, if the growth exponents of the nonlinear part belong to this region, then the corresponding Cauchy problem with sufficiently small initial data has a global solution.
5. There exists a region on the plane such that, if the growth exponents of the nonlinear part belong to this region, then the corresponding Cauchy problem, even with sufficiently small initial data, does not have a global solution.

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