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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

STUDYING SOLVABILITY AND SPECTRAL PROPERTIES OF A CLASS OF BOUNDARY VALUE PROBLEMS FOR OPERATOR-DIFFERENTIAL EQUATIONS

Specialty: 1202.01 – Analysis and functional analysis

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GENERAL CHARACTERISTICS OF THE WORK

Rationale and development degree of the topic.

Theory of operator-differential equations took its origin from the generalization of the system of finite differential equations, i.e. of matrix differential equations to the case of infinite-dimensional spaces. Famous mathematicians as E.Hille, K.Iosida, T.Kato, Z.I.Khalilov and others have studied the existence of the solution to the Cauchy problem for first order constant operator coefficient differential equations. The well-posedness of the considered Cauchy problem was determined with respect to the spectral properties of the operator contained in the equation. The works of S.Agmon and L.Nirenberg devoted to the study of the Cauchy problem for unbounded operator coefficient equation and the asymptotic properties of the solution of the problem should be especially underlined. Afterwards there appeared a number of reserarch works on studying the Cauchy problem and boundary value problems for first and second order operator-differential equations in this direction.

These results can be found in fundamental books of S.G.Krein, A.A.Dezin, V.I.Gorbachuk, M.L.Gorbachuk, Y.Y.Yakubov. At the same time, among these research we can show the works of M.G.Gasymov, M.Bayramoglu, Y.A.Dubinski, S.S.Mirzoyev, F.S.Rofe-Beketov, A.A.Shkalikov, A.R.Aliyev and ofhers.

In spite of these facts the theory of solvability of boundary value of problems for operator-differential equations has not found its solution and new researches are being created. Recently new valuable researches in this direction were carried out. The researches of S.S.Mirzoyev and his followers should be especially underlined.

As is known, discrete spectrum operators have a great role in quantum mechanics and theoretical physics. The study of eigen-values and asymptotic distribution of the given operator are the problems of great interest. First time these problems were studied by M.S.Birman, M.Z.Solomiak, A.G.Kostyuchenko, M.G.Gasymov, Y.V.Solomiak and others.

In some cases, the spectrum of the considered operator consists of negative eigen-values whose unique limit point is zero. In this case, finding of the asymptotic formula of the number of eigen-values smaller than $(\varepsilon < 0)$ is of great significance.

In this direction, we can note the works of M.Sh.Birman, V.Y.Skachek, Q.Rosenbloom, A.A.Adigezalov, M.Bayramoglu, D.R.Yafaev, A.M.Bayramov, H.I.Aslanov and N.A.Gadirli and others.

The represented dissertation work consists of the researches carried out in this direction. The works studies the well-posedness of boundary value problems stated for fourth order complete operator-differential equations on a finite segment, the discreteness of negative spectrum of second order differential operators with great importance in quantum mechanicsç estimation of distribution function of fourth order negative eigen-values and obtaining asymptotic formulas of sums consisting of fourth degree negative eigen values.

Object and subject of the research. Study of solvability condition of some boundary value problems on a finite segment for fourth order binomial operator-differential equations in Hilbert space, determination of uniqueness conditions, for the solution, determination of regular solvability conditions regular and Fredholm solvability boundary value problems of fourth order complete operator-differential equations on a finite segment. Study of negative spectrum of second order complete operator-differential equations on a semi-axis, obtaining estimations for distribution function of a negative spectrum and obtaining some asymptotic equalities for the sums consisting of fourth degree of negative eigen-values.

Goals and objectives of the research. Determine regular solvability conditions of a boundary value problem on a finite segment for binomial fourth order homogeneous operator-differential equations. To show existence and regular solvability of a unique solution of a boundary value problem for a fourth order inhomogeneous operator-differential equation. To study regular solvability of a non-zero boundary condition fourth order binomial operator-differential equation. To obtain Kolmogorov type inequalities for the norms of intermediate derivative operators in the space of smooth vector-functions on a finite segment. To prove the main theorem on the existence of regular solution to the boundary value problem with the help of Kolmogorov type inequalities. To prove Fredholm solvability of a boundary value problem for fourth order operator-differential equations. To obtain asymptotic estimations for the sums of fourth degrees of negative eigen-values of second order operator-differential equations on a semi-axis.

Research methods. The methods of theory of operator semi-groups, theory of self-adjoint operators in Hilbert space, some results obtained from Kolmogorov type inequalities, the methods of theory of differential equations were used.

The main theses to be defended.

Regular solvability of various boundary value problems for fourth order complete operator-differential equations on a finite segment was determined. These conditions are obtained by estimations obtained by means of principle part of intermediate derivative operators in Sobolev type spaces corresponding to the boundary value problem. The Fredholm solvability of equations obtained from absolutely continuous coefficient perturbation of the operator-differential equation, was determined. Negative spectrum of the second order operator-differential equation was studied, estimations for distribution function of negative eigen-values were obtained, the estimations for the sums consisting of fourth degree of negative eigen-values were obtained and asymptotic formulas for these sums was proved.

Scientific novelty of the research. The main obtained results in the work are the followings:

- Regular solvability of a boundary value problem on a finite segment was proved for binomial fourth order homogeneous operator-differential equations.
- The existence of a unique solution to a boundary value problem for an inhomogeneous operator-differential equation was proved.
- Exact estimations were obtained for norms of intermediate derivative operators corresponding to the equation in the space of smooth vector-functions.
- Sufficient conditions for regular solvability of the equation by means of inequalities were determined for the norms of intermediate

derivative operators.

- The Fredholm solvability of a boundary value problem for operator-differential equations was proved.
- Negative spectrum of second order differential equations on a semi-axis was studied, asymptotic formulas for the sum of the fourth degrees of negative eigen-values were obtained.

Theoretical and practical importance of the study. The results obtained in the dissertation work are mainly of theoretical character. The obtained results can be used when studying a class of partial differential equations. The obtained results can be applied in solving mechanical and applied problems reduced to a fourth order operator-differential equation. Taking into account that negative spectrum differential operators have significant applications in theoretical physics and quantum mechanics, it can be expected that the results obtained in this direction can be useful for the researches conducted in this field.

Approbation and application. The results of the dissertation work were reported in the seminars of the departments of “Functional analysis” and “Differential equations” of IMM, in the scientific seminars of the chairs of “Mathematical analysis and function theory” and “Differential equations and optimization” of Sumgait State University, in the İnternational Workshop dedicated to the 80 anniv. of an acad. Mirabbas Gasymov (Baku 2019), “Modern Problems of Mathematics and Mechanics” of International conference devoted to 60 years of IMM ANAS (Baku, 2019), in the Online International Symposium “Applied Mathematics and Engineering” ISAME22 (Turkey, 2022), in the II and III All-Russian scientific conference with international participation “Actual Problems of Mathematics and Informatics on Technology” (Mahachkala, 2021, 2022), “Modern Problems of Mathematics and Mechanics” Proceedings of the Inter. conf. devoted to the 110 anniv. of acad. Ibrahim Ibrahimov (Baku 2022).

Author’s personal contribution. All the results and research methods obtained in the work belong to the author.

Author’s publications. The author has published 12 papers: including 6 papers single authored, no coauthor -4, in scientific editions included in indexing systems -2.

As a result of republican and international scientific events (conferences, congresses, symposium and so on) 6 abstracts, including 3 abstracts abroad.

The name of the organization where the dissertation work was executed. The dissertation work was executed at the chair of “Mathematical analysis and function theory” of Sumgait State University.

Structure and volume of the dissertation (in signs indicating the volume of each structural subdivision separately).

Total volume of the dissertation work– 221762 signs (title page – 387 signs, contents – 2513 signs, introduction – 39186 signs, chapter I – 50000 signs, chapter II -64000 signs, chapter III -64000 signs, conclusion – 1676 signs). The dissertation work contains a list of references with 118 names.

THE MAIN CONTENT OF THE WORK

The dissertation work consists of introduction, 3 chapters and a list of references. In the introduction, the rationale of the topic of the dissertation work was justified, brief rewiev of main scientific works related to the dissertation work was given, the main results obtained in the work were commented.

Chapter I of the dissertation work was devoted to regular solvability of a boundary value problem stated for a binomial fourth order homogeneous operator-differential equation, to the existence and regular solvability of a boundary value problem stated for a fourth order inhomogeneous equation, the study of regular solvability of a fourth order homogeneous equation with nonzero boundary condition.

Assume that H is a separable Hilbert space, A is a self-adjoint positive-definite operator determined in the space H. İt is known that the domain of definition $D(A^p)$, $(x, y)_p = (A^p x, A^p y)$ of the operator $A^p (p \geq 0)$ forms the Hilbert space H_p with respect to the scalar product $x, y \in D(A^p)$. For $p = 0$ we accept $H_0 = H$.

By $L_2([0,1]; H)$ we denote the set of vector-functions whose values are contained in the Hilbert space H for any $t \in [0,1]$ and has the finite norm

$$\| f \|_{L_2([0,1];H)} = \left(\int_0^1 \| f(x) \|_H^2 \, dt \right)^{1/2}$$

In this space the scalar product of the elements f, g is determined by the equality

$$(f, g)_{L_2([0,1];H)} = \int_0^1 (f(t), g(t))_H \, dt .$$

By $W_2^4([0,1]; H)$ we denote a Hilbert space determined as follows:

$$W_2^4([0,1]; H) =$$
$$= \{ u(t); u^{(IV)}(t) \in L_2([0,1]; H), A^4 u(t) \in L_2([0,1]; H) \}.$$

In this space, the scalar product and norms of elements are determined as follows:

$$(u, v)_{W_2^4([0,1];H)} = \int_0^1 \left(u^{(IV)}(t), v^{(IV)}(t) \right)_H dt + \int_0^1 \left(A^4 u(t), A^4 v(t) \right)_H dt$$

$$(u(t))^2_{W_2^4([0,1];H)} = \left\| u^{IV}(t) \right\|^2_{L_2([0,1];H)} + \left\| A^2 u(t) \right\|^2_{L_2([0,1];H)} .$$

It is known from the trace theorem that if $u(t) \in W_2^4([0,1]; H)$ then the relations $u^{(k)}(0)$, $u^{(k)}(1) \in H_{4-k-\frac{1}{2}}$, $k = 0,1,2,3$ are valid. Here the derivatives $u^{(k)}(t)$ are understood in the sense of distributions.

In section 1.2 we consider the following boundary value problem:

$$\frac{d^4 u(t)}{dt^4} + A^4 u(t) = f(t) \tag{1}$$

$$u(0) = u'(0) = 0, \; u(1) = u'(1) = 0 \tag{2}$$

Here the functions $u(t)$, $f(t)$ are vector-functions almost every where determined on the interval $(0,1)$, A is a self-adjoint positive definition operator in space H.

Definition 1. If for the vector-function $f(t) \in L_2([0,1]; H)$ there exists such a vector-function $u(t) \in W_2^4([0,1]; H)$ that on the interval $(0,1)$ almost everywhere defines the equation (1), then $u(t)$ is called a regular solution of equation (1).

Definition 2. If for any $f(t) \in L_2([0,1]; H)$ the equation (1) has a regular solution $u(t)$ and this solution satisfies boundary conditions (2) in the sense of $\lim\limits_{t \to +0} \|u(t)\|_{H_{7/2}} = 0$, $\lim\limits_{t \to +0} \|u'(t)\|_{H_{5/2}} = 0$, $\lim\limits_{t \to 1-0} \|u(t)\|_{H_{7/2}} = 0$, $\lim\limits_{t \to 1-0} \|u'(t)\|_{H_{5/2}} = 0$, then the vector-function $u(t)$ is called a regular solution of boundary value problem (1)-(2).

Definition 3. If for any vector-function $f(t) \in L_2([0,1]; H)$ the boundary value problem (1)-(2) has a regular solution and this solution satisfies the inequality

$$\|u(t)\|_{W_2^4([0,1];H)} \le c \|f(t)\|_{L_2([0,1];H)},$$

then the boundary value problem (1)-(2) is called a regularly solvable boundary value problem. To study regular (unique) solvability of boundary value problem (1)-(2) at first we consider the following homogeneous boundary value problem:

$$\frac{d^4 u(t)}{dt^4} + A^4 u = 0 \tag{3}$$

$$u(0) = u'(0) = 0, \; u(1) = u'(1) = 0. \tag{4}$$

The following theorem is valid:

Theorem 1. If the operator A is a self-adjoint, positive-definite operator, then the problem (3)-(4) has only a zero solution.

In section 1.3, using theorem 1 in the case $f(t) \neq 0$, a theorem on the existence of a unique solution to boundary value problem (3)-(4) is proved.

Theorem 2. If the operator A is a self-adjoint, positive-definite operator in the Hilbert space H, then for any $f(t) \in L_2([0,1]; H)$ the boundary value problem (1)-(2) has a unique solution

$u(t) \in \overset{0}{W}_2[[0,1];H]$.

From the theorem we obtain that for the operator $L_0 u = \dfrac{d^4u}{dt^4} + A^4 u$ defined in the space $u(t) \in \overset{0}{W}{}_2^4((0,1);H)$ $KerL_0 = \{0\}$.

From theorem 2 we obtain that $JmL_0 = L_2([0,1];H)$, i.e. the operator L_0 one-to-one maps the space $\overset{0}{W}{}_2^4((0,1);H)$ into the space $L_2((0,1);H)$. According to the Banach theorem on the existence and boundedness of the inverse operator, we obtain the boundedness of the operator L_0^{-1} and

$$\left\| L_0^{-1} u \right\|_{\overset{0}{W}{}_2^4((0,1);H)} \leq c \| f \|_{L_2((0,1);H)}$$

or

$$\| u(t) \|_{\overset{0}{W}{}_2^4([0,1];H)} \leq c \| f \|_{L_2((0,1);H)}.$$

In section 1.4 we consider the boundary value problem

$$\frac{d^4u}{dt^4} + A^4 u = 0 \tag{5}$$

$$u(0) = \varphi(0),\ u'(0) = \varphi_1,\ u(1) = \psi_0,\ u'(1) = \psi_1 \tag{6}$$

and prove a theorem on regular solvability of this problem.

Definition 4. If for $\varphi_0, \psi_0 \in H_{7/2}$, $\varphi_1, \psi_1 \in H_{5/2}$ the equation (5) has such a regular solution that satisfies the boundary conditions (6) in the sense of

$$\lim_{t \to +0} \| u(t) - \varphi_0 \|_{H_{7/2}} = 0,\ \lim_{t \to +0} \| u'(t) - \varphi_1 \|_{H_{5/2}} = 0, \tag{7}$$

$$\lim_{t \to 1-0} \| u(t) - \psi_0 \|_{H_{7/2}} = 0,\ \lim_{t \to 1-0} \| u'(t) - \psi_1 \|_{H_{5/2}} = 0 \tag{8}$$

then $u(t)$ is said to be a regular solution of the problem (5)-(6).

Definition 5. If for any $\varphi_0, \psi_0 \in H_{7/2}$, $\varphi_1, \psi_1 \in H_{5/2}$ the problem (5)-(6) has a regular solution and the estimation

$$\|u(t)\|_{\overset{0}{W}{}_2^4((0,1);H)}\le c\left(\|\varphi_0\|_{H_{7/2}}+\|\varphi_1\|_{H_{5/2}}+\|\psi_0\|_{H_{7/2}}+\|\psi_1\|_{H_{5/2}}\right) \qquad (9)$$

is satisfied for this solution, then the problem (5)-(6) is called a regularly solvable problem.

The following theorem is valid:

Theorem 3. If the operator A is self-adjoint, positive-definite operator in H, then the boundary value problem (5)-(6) is regularly solvable.

Chapter II of the dissertation work was devoted to the study of regular and Fredholm solvability of a class of boundary value problems for fourth order complete operator-differential equations on a finite segment.

In separable Hilbert space H we consider the following boundary value problem:

$$Lu=\frac{d^4u(t)}{dt^4}+A^4u(t)+\sum_{j=0}^{4}A_{4-j}u^{(j)}(t)=f(t),\ \ t\in[0,1] \qquad (10)$$

$$u(0)=u'(0)=0,\ \ u(1)=u'(1)=0 \qquad (11)$$

Here the functions $f(t)$ and $u(t)$ are vector functions whose values determined in the interval $(0,1)$ are included in space H.

It is assumed that the coefficients of the equation (10) satisfy the following conditions:

1) A is a self-adjoint positive-definite operator.

2) The operators $B_j=A_jA^{-j}\left(j=\overline{0,4}\right)$ are bounded operators in the space H.

Definition 6. If for the given $f(t)\in L_2((0,1);H)$ thre is a function $u(t)\in W_2^4((0,1);H)$ satisfying almost everywhere on the interval $(0,1)$ the equation (10), it is said a regular solution of the equation (10).

Definition 7. If for any $f(t)\in L_2((0,1);H)$ equation (10) has a such a regular solution that satisfies the boundary conditions (11) on the sense of

$$\lim_{t\to+0}\|u(t)\|_{H_{7/2}}=0,\quad \lim_{t\to+0}\|u'(t)\|_{H_{5/2}}=0,\qquad \lim_{t\to1-0}\|u(t)\|_{H_{7/2}}=0,$$

$\lim\limits_{t\to 1-0}\|u'(t)\|_{H_{5/2}} = 0$ then the vector-function $u(t)$ is said to be a regular solution of the boundary value problem (10)-(11).

Definition 8. If for any $f(t)\in L_2((0,1);H)$ the problem (10)-(11) has a regular solution and this solution satisfies the estimation

$$\|u(t)\|_{W_2^4((0,1);H)} \le c\|f\|_{L_2((0,1);H)},$$

then the problem (10)-(11) is called a regularly solvable problem.

To determine regular solvability of the boundary value problem (10)-(11) we use exact estimations of the norms of the operators called intermediate derivative operators.

In section 2.2. the following theorem on the estimation of the norms of intermediate derivative operator was is proved.

Theorem 4. Assume that the operator A is self-adjoint positive-definite operator. Then for all the functions $u(t)\in \overset{0}{W}{}_2^4((0,1);H)$ the following inequalities are valid:

$$\left\|A^4u(t)\right\|_{L_2((0,1);H)} \le c_0\|L_0u(t)\|_{L_2((0,1);H)} \tag{12}$$

$$\left\|A^3u'(t)\right\|_{L_2((0,1);H)} \le c_1\|L_0u(t)\|_{L_2((0,1);H)} \tag{13}$$

$$\left\|A^2u''(t)\right\|_{L_2((0,1);H)} \le c_2\|L_0u(t)\|_{L_2((0,1);H)} \tag{14}$$

$$\|Au'''(t)\|_{L_2((0,1);H)} \le c_3\|L_0u(t)\|_{L_2((0,1);H)} \tag{15}$$

$$\left\|u^{(IV)}(t)\right\|_{L_2((0,1);H)} \le c_4\|L_0u(t)\|_{L_2((0,1);H)} \tag{16}$$

here $c_0 = c_4 = 1,\ c_2 = \frac{1}{2},\ c_1 = \frac{1}{\sqrt{2}},\ c_3 = \sqrt{3}+1+\frac{1}{2}\sqrt{1(2+\sqrt{3})}.$

In section 2.3 using the estimations obtained in theorem 4 we prove the following theorem on regular solvability of the boundary value problem (10)-(11).

Theorem 5. Assume that the operator A is a self-adjoint operator in the Hilbert space H and the operators $B_j = A_jA^{-j}\left(j=\overline{0,4}\right)$ are bounded operators in the space H. In addition, assume that the algebraic condition

$$h=\sum_{j=1}^{4} c_j \left\| B_{4-j} \right\| < 1$$

is satisfied. Then the problem (10)-(11) is regularly solvable. Here the numbers c_0, c_1, c_2, c_3, c_4 are the numbers obtained from theorem 4.

In section 2.4 we study regular solvability of a homogeneous equation with non-zero boundary condition. Let us consider the following problem:

$$\frac{d^4 u(t)}{dt^4} + A^4 u(t) + \sum_{j=0}^{4} A_{4-j} u^{(j)}(t) = 0 \qquad (17)$$

$$u(0)=\varphi_0,\ u'(0)=\varphi_1,\ u(1)=\psi_0,\ u'(1)=\psi_1 . \qquad (18)$$

Definition 9. If the equation (17) has a regular solution, and this solution satisfies the boundary conditions (18) in the sense of

$$\lim_{t\to+0}\left\|u(t)-\varphi_0\right\|_{H_{7/2}}=0,\ \lim_{t\to+0}\left\|u'(t)-\varphi_1\right\|_{H_{5/2}}=0,$$

$$\lim_{t\to1-0}\left\|u(t)-\psi_0\right\|_{H_{7/2}}=0,\ \lim_{t\to1-0}\left\|u'(t)-\psi_1\right\|_{H_{5/2}}=0,$$

then $u(t)$ is called a regular solution of boundary value problem (17)-(18).

Definition 10. If for any $\varphi_0 \in H_{7/2}$, $\varphi_1 \in H_{5/2}$, $\psi_0 \in H_{7/2}$, $\psi_1 \in H_{5/2}$ the problem (17)-(18) has a regular solution and the estimation

$$\left\|u(t)\right\|_{\overset{0}{W}{}_2^4((0,1);H)} \le$$

$$\le const\left(\left\|\varphi_0\right\|_{H_{7/2}} + \left\|\varphi_1\right\|_{H_{5/2}} + \left\|\psi_0\right\|_{H_{7/2}} + \left\|\psi_1\right\|_{H_{5/2}}\right)$$

is satisfied for this solution, then the problem (17)-(18) is said to be a regularly solvable boundary value problem.

The following theorem was proved.

Theorem 6. If the operator A is a self-adjoint, positive-definite operator in the space H, the operators $B_j = A_j A^{-j}\left(j=\overline{0,4}\right)$ are bounded and the condition

$$h = \sum_{j=1}^{4} c_j \left\| B_{4-j} \right\| < 1$$

is satisfied, then the boundary value problem (17)-(18) is regularly solvable. Here the constants $c_j \left(j = \overline{0,4} \right)$ are the numbers found by means of theorem 4.

Section 2.5 studies regular solvability of an inhomogeneous equation with a non-zero boundary condition. Let us consider the following problem:

$$\frac{d^4 u(t)}{dt^4} + A^4 u(t) + \sum_{j=0}^{4} A_{4-j} u^{(j)}(t) = f(t), \ t \in [0,1) \tag{19}$$

$$u(0) = \varphi_0, \ u'(0) = \varphi_1, \ u(1) = \psi_0, \ u'(1) = \psi_1 \tag{20}$$

Definition 11. If for any $f(t) \in L_2((0,1); H)$ and for any $\varphi_0 \in H_{7/2}, \varphi_1 \in H_{5/2}, \psi_0 \in H_{7/2}, \psi_1 \in H_{5/2}$ there exists such a vector-function $u(t) \in W_2^4((0,1); H)$ that almost everywhere satisfies the equation (19) that satisfies the boundary conditions (20) in the sense of

$$\lim_{t \to +0} \left\| u(t) - \varphi_0 \right\|_{H_{7/2}} = 0, \ \lim_{t \to +0} \left\| u'(t) - \varphi_1 \right\|_{H_{5/2}} = 0,$$

$$\lim_{t \to 1-0} \left\| u(t) - \psi_0 \right\|_{H_{7/2}} = 0, \ \lim_{t \to 1-0} \left\| u'(t) - \psi_1 \right\|_{H_{5/2}} = 0$$

and the estimation

$$\left\| u(t) \right\|_{W_2^4((0,1);H)} \le const \Big(\| f \|_{L_2((0,1);H)} +$$

$$+ \left\| \varphi_0 \right\|_{H_{7/2}} + \left\| \varphi_1 \right\|_{H_{5/2}} + \left\| \psi_0 \right\|_{H_{7/2}} + \| \psi_1 \|_{H_{5/2}} \Big)$$

is valid for it, then the boundary problem (19)-(20) is regularly solvable.

The following theorem is valid:

Theorem 7. Assume that the coefficients of the equation (19) satisfy the following conditions:

1) The operator A is a self-adjoint positive-definite operator in the space H;

2) The operators $B_j = A_j A^{-j} \left(j = \overline{0,4} \right)$ are bounded in space H;

3) The algebraic condition $h=\sum_{j=1}^{4} c_j \left\| B_{4-j} \right\| < 1$ is satisfied.

Then the boundary value problem (19)-(20) is regularly solvable.

In section 2.6 a theorem on Fredholm solvability of boundary value problems under certain conditions was proved. Let us consider the following boundary value problem:

$$\frac{d^4 u(t)}{dt^4} + A^4 u(t) + \sum_{j=0}^{3} A_{3-j} u^{(j)}(t) + \sum_{i=0}^{3} T_{3-i} u^{(i)}(t) = f(t) \qquad (21)$$

$$u(0)=0,\ u'(0)=0,\ u(1)=0,\ u'(1)=0 \qquad (22)$$

Let us define the following operators in space $\overset{0}{W}{}_2^4((0,1);H)$:

$$Lu = \frac{d^4 u(t)}{dt^4} + A^4 u(t) + \sum_{j=0}^{3} A_{3-j} u^{(j)}(t)$$

$$Ku = \sum_{j=0}^{3} K_{3-j} u^{(j)}(t)$$

$$Qu = Lu + Ku\ .$$

The following theorem is valid:

Theorem 8. Assume that the coefficients of the equation (21) satisfy the following conditions:

1) the operator A is a self-adjoint positive-definite operator in space H and its inverse operator A^{-1} is continuous;

2) the operators $B_j = A_j A^{-j} \left(j=\overline{0,3}\right)$ are bounded operators in space H;

3) the operators $K_i = T_i A^{-i} \left(i=\overline{0,3}\right)$ are completely continuous operators;

4) the algebraic condition $\tilde{h} = \sum_{j=0}^{3} c_j \left\| B_{3-j} \right\| < 1$ is satisfied;

here the numbers $c_j \left(j=\overline{0,3}\right)$ are the numbers found by means of theorem 4.

In this case, the operator Q is a Fredholm type operator acting from the space $\overset{0}{W}{}_2^4((0,1);H)$ to the space $L_2((0,1);H)$.

In chapter III the negative spectrum of the Sturm-Liouville operator given on the semi-axis was studied, the discretenness of the negative spectrum was proved, the estimation of the distribution function of negative eigen-value were obtained, asymptotic formulas of the sums consisting of the degrees of negative eigen-values, were proved.

In section 3.1 some inequalities related to eigen-values of the Sturm-Liouville operator were proved.

In the space $L_2[0,\infty)$ by L we denote the operator determined by the differential expression

$$l(y) = -y'' - q(x)y \tag{23}$$

and boundary condition

$$y'(0) = 0 \tag{24}$$

Assume that the function $q(x)$ satisfies the following conditions:

1) the function $q(x)$ is a continuous, monotonically decreasing positive function in the interval $[0,\infty)$.

2) $\lim\limits_{x\to+\infty} q(x) = 0$.

The operator L is a self-adjoint, lower bounded operator and negative part of its spectrum is discrete. We denote the negative eigen-values of the operator L by $\lambda_1 < \lambda_2 < \lambda \cdots < \lambda_n < \cdots$. Let us denote the inverse of the function $q(x)$ by $p(x)$. Let us take the number $\varepsilon \in (0, q(0))$ and define the following operators:

1) By L' we define an operator determined by the differential expression $l(y) = -y'' - q(x)y$ and the boundary condition $y'(p(\varepsilon)) = 0$ in the space $L_2[p(\varepsilon),\infty)$.

2) By L_0 and L_1 we denote the operator determined by the differential expression $l(y) = -y'' - q(x)y$ and by the conditions $y(0) = (p(\varepsilon)) = 0$ and $y'(0) = y'(p(\varepsilon)) = 0$ in the space $L_2[0, p(\varepsilon)]$.

3) Let us take the part $0 = x_0 < x_1 < x_2 < \cdots < x_m = \rho(\varepsilon)$ of the

segment $[0, p(\varepsilon)]$. By L_{0i} and L_{1i} we denote the operators determined by the differential expression (23) and the boundary conditions $y(x_{i-1}) = y(x_i) = 0$ and $y'(x_{i-1}) = y'(x_i) = 0$ in the space $L_2[x_{i-1}, x_i]$.

4) Let us denote by $\overline{L}_{0i}$ the operator determined by the differential expression $l(y) = -y'' - q(x_i)y$ and boundary conditions $y(x_{i-1}) = y(x_i) = 0$ in the space $L_2[x_{i-1}, x_i]$. By $\overline{L}_{1i}$ the operator determined by the differential expression $l(y) = -y'' - q(x_{i-1})y$ and the boundary condition $y'(x_{i-1}) = y'(x_i) = 0$ in the space $L_2[x_{i-1}, x_i]$.

The following theorem is valid:

Theorem 9. If the function $q(x)$ satisfies the condition 1) then for any $y \in D(L')$ the inequality

$$(L'y, y) \geq -\varepsilon(y, y) \tag{25}$$

is valid.

By $N(\alpha)$, $N_0(\alpha)$ and $N_1(\alpha)$ we denote the amount of negative eigen-values of the operators L, L_0 and L' less than $(-\alpha)-$ and $(\alpha \in (0, \infty))$. We denote orthonormal eigen functions of the operator L corresponding to the eigen-values $\lambda_1, \lambda_2, \cdots \lambda_n \cdots$ by $u_1, u_2, \cdots, u_n, \cdots$.

Let $T = L + \alpha E$, $T_0 = L_0 + \alpha E$, $T_1 = L_1 + \alpha E$. (E-is a unit operator).

Theorem 10. If the function $q(x)$ satisfies the condition 1), 2) in this case for any $\alpha \in (0, \infty)$ the inequality

$$N(\alpha) \geq N_0(\alpha) \tag{26}$$

is valid.

Theorem 11. If the function $q(x)$ satisfies the conditions 1), 2) in this case, for any $\alpha \in [\varepsilon, \infty)$ the inequality

$$N(\alpha) \leq N_1(\alpha) \tag{27}$$

is valid.

Theorem 12. If the function $q(x)$ satisfies the conditions 1) in this case the relation $L_{0i} < \overline{L}_{0i}$ və $L_{1i} < \overline{L}_{1i}$ is valid.

In section 3.2 some inequalities for the sum of fourth degree of negative eigen-values of the operator L were proved.

By $n_{0i}(\alpha)$ and $\overline{n}_{0i}(\alpha)$ we denote the numbers of eigen-values of the operators L_{0i} and $\overline{L}_{0i}$ less than $(-\alpha)$ $n_{0i}(\varepsilon) = n_{0i}$, $\overline{n}_{0i}(\varepsilon) = \overline{n}_{0i}$

Let us enumerate the eigen-values of the operator $\overline{L}_{0i}$ as $\mu_{i(1)} \le \mu_{i(2)} \le \cdots$. Using the R.Courant variation principle, we obtain the inequality

$$N(\alpha) \ge \sum_{i=1}^{m} n_{0i}(\alpha) \tag{28}$$

and the relation

$$\sum_{i=1}^{N(\varepsilon)} \lambda_j^4 \ge \sum_{i=1}^{M} \sum_{m=1}^{\overline{n}_{0i}} \mu_{im}^4 \tag{29}$$

The following important theorem is valid:

Theorem 13. For the sum of fourth degree of eigen values of the operator $\overline{L}_{0i}$ less than $(-\varepsilon)$ the following inequality is valid:

$$\sum_{m=1}^{\overline{n}_{0i}} \mu_{i(m)}^4 > \frac{\delta}{3/5\pi} \sqrt{q(x_i) - \varepsilon} \{128\, q^4(x_i) + 64 q^3(x_i)\varepsilon +$$
$$+ 48 q^2(x_i)\varepsilon^2 + 40 q(x_i)\varepsilon^3 + 45\varepsilon^4\} - 2q^4(x_i), \ (\delta = x_i - x_{i-1})$$

Now, we show theorems consisting of inequalities that shows the estimation of the sum of fourth degrees of eigen values of the operator L:

Theorem 14. If the function $q(x)$ satisfies the conditions 1), 2) then for the small positive values of ε the following inequality is valid:

$$\sum_{j=1}^{N(\varepsilon)} \lambda_j^4 > \frac{1}{3/5\pi} \int_0^{p(\varepsilon)} f(x,\varepsilon)dx - cp^k(\varepsilon) \tag{30}$$

here $f(x,\varepsilon) = \sqrt{q(x) - \varepsilon}\,[128 q^4(x) + 64 q^3(x)\varepsilon +$
$$+ + 48 q^2(x)\varepsilon^2 + 40 q(x)\varepsilon^3 + 45\varepsilon^4]$$

and $c > 0$ is a positive constant.

Theorem 15. If the function $q(x)$ satisfies the conditions 1), 2) then for the eigen-values of the operator $\bar{L}_{1i}$ less than $(-\varepsilon)$ the following inequality is valid:

$$\sum_{m=1}^{\bar{n}_{1i}} \bar{\gamma}_{im}^{4} < \frac{\delta}{3/5\pi} f(x_{i-1}, \varepsilon) + q^4(x_{i-1}) \qquad (31)$$

Theorem 16. If the function $q(x)$ satisfies the conditions 1), 2) then for the rather small positive values of the number ε the following inequality is valid:

$$\sum_{j=1}^{N(\varepsilon)} \lambda_j^4 < \frac{1}{3/5\pi} \int_0^{p(\varepsilon)} f(x, \varepsilon)dx - c_1 \int_0^{\delta} f(x, \varepsilon)dx + c_2 p^k(\varepsilon), \qquad (32)$$

c_1 and c_2 are positive constants.

In section 3.3 imposing some additional conditions on the function $q(x)$, we prove asymptotic formulas for the sums $\varepsilon \to 0$ as $\sum_{\lambda_j < -\eta} \lambda_j^4$.

Assume that the function $q(x)$ in addition to the above conditions 1), 2) satisfies the following condition as well:

3) For the constant $k_0 \in \left(0, \frac{2}{9}\right)$ and the number $\eta > 0$

$$\lim_{x \to \infty} q(x) x^{k_0 - \eta} = \lim_{x \to \infty} \left[q(x) x^{k_0 + \eta} \right] = 0.$$

In this case the following theorem was proved:

Theorem 17. If the function $q(x)$ satisfies the conditions 1)-3) then as $\varepsilon \to 0$ the following asymptotic formula is valid:

$$\sum_{\lambda_j < -\varepsilon} \lambda_j^4 = \frac{1}{3/5\pi} \left[1 + O\left(\varepsilon^{-t_0}\right) \right] \int_{q(x) > \varepsilon} \sqrt{q(x) - \varepsilon} \times \qquad (33)$$

$$\times \left[128 q^4(x) + 64 q^3(x)\varepsilon + 48 q^2(x)\varepsilon^2 + 40 q(x)\varepsilon^3 + 35\varepsilon^4 \right] dx,$$

here t_0 is a positive number.

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CONCLUSION

The dissertation work was devoted to the regular and Fredholm solvability of some boundary value problems on finite segment for fourth order complete operator-differential equations in Hilbert space, to the study of negative spectrum of second order differential operators being one of the important applications of theoretical physics and quantum mechanics, and obtaining some asymptotic equalities for the sums of negative eigen values consisting of fourth degrees.

The following scientific results were obtained:

- Regular solvability of a boundary value problem on a finite segment was obtained for binomial fourth order homogeneous operator-differential equations.
- The existence of a unique solution to a boundary value problem for an inhomogeneous operator-differential equation was proved.
- Exact estimations for the norms of intermediate derivative operators corresponding to the equation in the space of smooth vector-functions, were obtained.
- Using the estimations for the norms of intermediate derivative operators sufficient conditions for regular solvability of the equation were determined.
- The Fredholm solvability of a boundary value problem for a class of operator-differential equations was proved.
- Negative spectrum of second order differential equations on a semi-axis was studied, asymptotic equalities for the sum of the fourth degrees of negative eigen-values on a semi-axis were proved.

The basic results of the dissertation work are in the following works:

1. Ейвазлы, Г.М. Об однозначной разрешимости краевой задачи для операторно-дифференциального уравнения четвертого порядка в гильбертовом пространстве //-Baku: Journal of Baku Engineering University, Mathematics and computer science, -2018. v.2, №2, -p.59-66.
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