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of the dissertation for the degree of
Doctor of Philosophy

RECOVERY OF DIFFUSION OPERATOR WITH SPECTRAL PARAMETER IN BOUNDARY CONDITION

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GENERAL CHARACTERISTIS OF THE WORK

The relevance and the background of the work. This dissertation is dedicated to the inverse spectral problems for a diffusion operator with a boundary condition including a linear function of spectral parameter. In spectral analysis, the inverse problem consists in recovering the linear operators from some of their given spectral data. As spectral data, we use one or more spectra, normalizing numbers, spectral functions, Weyl function, etc. The theory of inverse problems has been widely developed due to its applications in mechanics, physics, geophysics, meteorology, etc. This theory plays a special role in various fields of mathematics, including the integration of nonlinear equations.

A lot of works have been dedicated to the theory of inverse problems of spectral analysis. For the view on the current situation in this theory, we refer the readers to the works by A.M. Akhtyamov, N.Sh. Iskenderov, B.M. Levitan, V.A. Marchenko, J. Poshel and E. Trubovich, V.A. Sadovnichi, Y.T. Sultanayev and A.M. Akhtyamov, V.A. Yurko, I.M. Nabiyev, A.Kh. Khanmammadov.

The majority of results in the theory of direct and inverse spectral problems pertain to the Sturm-Liouville equation

$$ly = -y'' + q(x)y = \lambda^2 y.$$

D. Bernoulli, J. d'Alembert, L. Euler, J. Liouville and Ch.F. Sturm were first to do research in this field. They obtained some results for the equation describing the oscillations of a string. The theory of inverse problems for the operator l has been intensively developed in the 20th Century. The first result in this field was obtained in a special case of Sturm-Liouville operator with Neumann boundary conditions. The first comprehensive study of the problem of recovering the operator l from its spectral data was carried out by the Swedish mathematician G. Borg. He proved that the spectra of two Sturm-Liouville differential operators, with one of their boundary conditions being the same, determine the function $q(x)$ uniquely.

In various areas of the spectral theory of operators, transformation operators have played an important role. For the first time, these operators were applied to the inverse problems by V.A. Marchenko. He showed the uniqueness of the solution to the inverse problem for the Sturm-Liouville operator with a given spectral function (for some other inverse spectral problems too). The method of transformation operators was also used in the fundamental work by I.M. Gelfand and B.M. Levitan, where necessary and sufficient conditions and a recovery method for solving the inverse problem for the Sturm-Liouville operator with a given spectral function have been found. M.G. Krein has also done some research concerning efficient recovery of the classical Sturm-Liouville operator from two of its spectra. Borg-stated inverse problem, i.e. the problem of recovering the operator from two of its spectra in the interval, has been completely solved by M.G. Gasimov and B.M. Levitan.

In 1967, C. Gardner, J. Green, M. Kruskal and R. Miura developed the method for the integration of some nonlinear evolution equations of mathematical physics using an inverse spectral problem. Later, V.Y. Zakharov and A.B. Shabat extended this method for other equations.

Note that direct and inverse problems for differential operators with nonseparated (including periodic, antiperiodic and quasiperiodic) boundary conditions date back to 1970. These problems have been solved using various methods by I.V. Stankevich, V.A. Sadovnichiy, Ya.T. Sultanayev, A.M. Akhtyamov, V.A. Marchenko, I.V. Ostrovsky, M.G. Gasimov, H.M. Huseynov, I.M. Nabiyev, O.A. Plaksina, V.A. Yurko, Y.L. Korotyayev, A.S. Makin, etc.

Many problems of mathematical physics contain the spectral parameter both in the differential equation and in the boundary conditions. Such problems arise when applying the method of separation of variables to partial differential equations with a time derivative in the boundary conditions. For example, when a homogeneous string is fixed at one end and loaded at the other end, the equation of oscillations of a string can be reduced to the problem with a spectral parameter in the boundary condition. The history of inverse spectral problems for Sturm-Liouville equations with a spectral parameter in the boundary conditions is relatively short. Direct and inverse problems in various settings have been studied by A. Benedek and P. Pansonen, P.C. Braun and B.D. Slima, P.A. Baiding, P.C. Braun and B.A. Watson, X.R. Mammadov, N.B. Karimov, Z.S. Aliyev, V. Pivovarchik, N.C. Guliyev, etc.

First important results concerning the spectral theory of polynomial operator pencil have been obtained by M.V. Keldysh, who introduced the basic concepts of this theory, proved theorems on the completeness of systems of eigenfunctions and associated functions for important classes of linear and polynomial operator pencils, and obtained the asymptotics of the eigenvalues. Keldysh's results boosted the further development of the spectral theory of operator pencils. A.G. Kostyuchenko, A.A. Shkalikov and other mathematicians considered the direct problems of spectral analysis for differential operator pencils in the interval. Note that some partial differential equations are reduced to spectral problems for second order ordinary differential operator pencils.

M. Jolan and K. Jan, M.G. Gasimov, H.Sh. Huseynov, H.M. Huseynov, I.M. Nabiyev, R.Kh. Amirov, A.A. Nabiyev, V.A. Sadovnichiy, Y.T. Sultanayev, A.M. Akhtyamov, V.A. Yurko, S.A. Buterin, V.N. Pivovarchik, C.F. Yang, N.P. Bondarenko, X.C. Hu and other researchers treated various inverse spectral problems for the diffusion equation

$$y'' + \left[\lambda^2 - 2\lambda p(x) - q(x)\right]y = 0 \, .$$

Inverse problems generated by the above equation and nonseparated boundary conditions, with various spectral data, have been examined by H.M. Huseynov, I.M.Nabiyev and V.A.Yurko.

Most of the above works consider the problems with separated regular boundary conditions, while there are few works dedicated to the spectral problems for diffusion equation with nonseparated boundary conditions or with boundary conditions involving a spectral parameter. Spectral parameter in the boundary conditions creates additional difficulties for researchers. For example, this makes it impossible to use some well-known methods in this case, which, in turn, requires changing usual arguments. At the same time, along with creating the difficulties, spectral parameter in boundary conditions allows avoiding some complex estimation. That's why the study of direct and inverse boundary value problems generated by the diffusion equation and the nonseparated boundary conditions including a spectral parameter is of particular importance.

Thus, the study of inverse spectral problems for a quadratic pencil of Sturm-Liouville operators is relevant in case where one of the nonseparated boundary conditions includes a linear function of spectral parameter.

The object and the subject of research. The object of this research is a diffusion operator with a nonseparated boundary condition involving a linear function of spectral parameter, and the subject is the solution of direct and inverse spectral problems for the considered operator.

The purpose and the objectives of research. The main purpose of this research is to solve inverse problems for the diffusion operator in case where one of the nonseparated boundary conditions includes a linear function of spectral parameter. The main objectives are to find the asymptotics of the eigenvalues of the considered boundary value problems, to find a criterion for multiple eigenvalues, to study the scattering of the eigenvalues of two different problems with one of their boundary conditions being the same, and to investigate the inverse problem of spectral analysis with two spectra and a sequence of signs.

Research methods. We use the methods of the theory of differential and integral equations, spectral theory of differential operators, complex analysis, theory of real functions, and asymptotic methods.

Main points to be defended in this dissertation. The main points to be defended in this dissertation are:

- to find the asymptotics of the eigenvalues of diffusion operator in case where one of the nonseparated boundary conditions includes a linear function of spectral parameter;

- to find a criterion for the multiplicity of the eigenvalues for the considered boundary value problems and to prove the non-existence of associate functions for these problems;

- to prove uniqueness theorems for the solution of inverse boundary value problems with two spectra and a sequence of signs, to create solution algorithms;

- to find sufficient conditions for some sequences of real numbers to be the spectral data of boundary value problems generated by the diffusion equation and the nonseparated boundary conditions involving a spectral parameter.

Scientific novelty of research. The following main results are obtained in this dissertation:

- asymptotic formulas are found for the eigenvalues of boundary value problems generated by the diffusion equation and the nonseparated boundary conditions involving a spectral parameter;

- necessary and sufficient conditions for multiplicity of the eigenvalues of the considered boundary value problems are found, and the non-existence of associate functions for these problems is proved;

- uniqueness theorems for the solution of inverse boundary value problems with two spectra and a sequence of signs are proved, efficient solution algorithms are created;

- sufficient conditions are found for a set of some quantities to be the spectral data of a diffusion operator with one of its nonseparated boundary conditions involving a linear function of spectral parameter.

Theoretical and practical significance of research. Results obtained in this work are theoretical, and they can be successfully used in recovery problems for some differential operators. Also, these results can be useful in integration of some nonlinear evolution equations of mathematical physics and in some problems of quantum mechanics.

Approbation and applications. The results of this dissertation have been presented in the seminars held at the Department of Applied Mathematics in the Baku State University (presided by prof. H.D. Orujov), at the Department of Functional

Analysis in the Institute of Mathematics and Mechanics (presided by prof. H.I. Aslanov), and in many scientific conferences such as National Scientific Conference on Actual Problems of Theoretical and Practical Mathematics dedicated to the 100th Anniversary of Academician M.L. Rasulov (Sheki, Azerbaijan, 2016), International Conference on Actual Problems of Mathematics and Mechanics dedicated to the 55th Anniversary of the Institute of Mathematics and Mechanics (Baku, Azerbaijan, 2014), National Scientific Conference dedicated to the 85th Anniversary of prof. Y.J. Mamedov (Baku, Azerbaijan, 2015), International Conference on Mathematical Analysis, Differential Equations and their Applications" (Baku, 2015), International Conference on Mathematical Modeling of Processes and Systems (Ufa, Russia, 2017), International Conference dedicated to the 100th Anniversary of A.F. Leontyev (Ufa, Russia, 2017), International Conference on Operators, Functions, and Systems of Mathematical Physics dedicated to the 70th Anniversary of prof. H.A. Isahanly (Baku, 2018).

Author contribution statement. All the results obtained in this work belong to the applicant.

Author's publications. Author's publications include 6 articles in scientific journals recommended by the Presidential Higher Certifying Commission of Azerbaijan (three of them being cited by the Web of Science: Scopus) and 7 conference theses (6 in international and 1 in national conferences), making a total of 13 publications. The full list of author's publications is available at the end of the abstract.

The name of the organization where the dissertation work was carried out. The dissertation work was performed at the Department of Applied Mathematics of Baku State University.

The total volume of the dissertation in characters, indicating the volume of the structural sections of the dissertation separately. The dissertation work consists of a title page and table of contents (1214 characters), an introduction (42000 characters), 3 chapters of 138000 characters (Chapter I - 68000, Chapter II - 52000, Chapter III - 18000), a conclusion (879 characters) and a list of 81 references. The volume of the dissertation consists of 182093 characters.

THE CONTENT OF THE WORK

The dissertation work consists of an introduction, three chapters, a conclusion and a list of references. The introduction substantiates the relevance of the topic, indicates the purpose, objectives and scientific novelty of the research, notes its theoretical and practical significance, provides information on the approval of the work, and also comments on the brief content of the dissertation.

In the first chapter, the direct problems of spectral analysis for boundary problems arising from the non-separable boundary conditions involving the diffusion equation and the spectral parameter are studied.

Let us consider the Sobolev space $W_2^n[0,\pi]$ on the interval $[0,\pi]$ consisting of functions whose derivatives up to order $n-1$ are absolutely continuous, while the n-th order derivative is square-integrable (i.e., belongs to $L_2[0,\pi]$). Suppose that $p(x)\in W_2^1[0,\pi]$, $q(x)\in L_2[0,\pi]$ are real-valued functions.

In paragraph 1.1, we consider the diffusion equation:

$$y''+\left[\lambda^2-2\lambda p(x)-q(x)\right]y=0\,,0\le x\le \pi \tag{1}$$

with boundary conditions:

$$\begin{aligned} &y'(0)+(\alpha\lambda+\beta)y(0)+\omega y(\pi)=0\,,\\ &y'(\pi)+\gamma\, y(\pi)-\overline{\omega}y(0)=0\,, \end{aligned} \tag{2}$$

We analyze the boundary value problem satisfying these boundary conditions, where λ is the spectral parameter, α, β, γ are real numbers, and ω is a complex number. Suppose that $\alpha\omega\ne 0$, meaning that the boundary conditions (2) are non-separable, and one of these conditions includes a linear function of the spectral parameter. We will denote the boundary value problem (1)-(2) as $L(\omega,\alpha,\beta)$.

Definition: If for the value $\lambda=\lambda_0$ of the parameter λ, the boundary problem $L(\omega,\alpha,\beta)$ has a nontrivial solution $y_0(x)$, then λ_0 is called an eigenvalue of this problem, and the corresponding function $y_0(x)$ is called an eigenfunction associated with this eigenvalue. The maximum number of linearly independent eigenfunctions corresponding to the same eigenvalue λ_0 is called the multiplicity of λ_0 .

If the functions $y_1(x)$, $y_2(x)$, . . . , $y_r(x)$possess absolutely continuous derivatives and satisfy the differential equation

$$y_j''(x)+\left[\lambda_0^2-2\lambda_0 p(x)-q(x)\right]y_j(x)+[2\lambda_0-2p(x)]y_{j-1}(x)+y_{j-2}(x)=0$$

as well as the boundary conditions

$$(\alpha\lambda_0+\beta)y_j(0)+y_j'(0)+\omega y_j(\pi)+\alpha y_{j-1}(0)=0,$$
$$-\overline{\omega}y_j'(0)+\gamma y_j(\pi)+y_j'(\pi)=0,$$
$$j=1,2,3,\dots,r \quad (y_{-1}(x)\equiv 0).$$

then these functions are called functions associated with the eigenfunction $y_0(x)$.

In this section, we assume that the following condition (condition A) holds:

$$y(x)\in W_2^2[0,\pi],\ y(x)\ \not\equiv\ 0$$

For functions satisfying these conditions, the inequality:

$$\gamma|y(\pi)|^2-2\operatorname{Re}\left[\omega\overline{y(0)}y(\pi)\right]-\beta|y(0)|^2+\int_0^{\pi}\left\{|y'(x)|^2+q(x)|y(x)|^2\right\}dx>0, \tag{3}$$

is satisfied. It can be easily checked that inequality (3) holds whenever $\beta\le 0,\ \gamma\ge 0,\ |\omega|\le\sqrt{|\beta|\gamma}$, $q(x)>0$ always holds.

Let us denote by $c(x,\lambda)$ and $s(x,\lambda)$ the solutions of equation (0.1) that satisfy the following initial conditions:

$$c(0,\lambda)=s'(0,\lambda)=1,\ c'(0,\lambda)=s(0,\lambda)=0\,.$$

are used in this section to show that the eigenvalues of the boundary value problem $L(\omega,\alpha,\beta)$satisfy:

In this paragraph, using the solutions $c(x,\lambda)$ and $s(x,\lambda)$, it is shown that the eigenvalues of the boundary value problem $L(\omega,\alpha,\beta)$ coincide with the zeros of the function

$$\delta(\lambda)=2\operatorname{Re}\omega-\eta(\lambda)+|\omega|^2 s(\pi,\lambda)+(\alpha\lambda+\beta)\sigma(\lambda), \tag{4}$$

Where

$$\eta(\lambda)=c'(\pi,\lambda)+\gamma\, c(\pi,\lambda), \qquad \sigma(\lambda)=s'(\pi,\lambda)+\gamma\, s(\pi,\lambda)\,. \tag{5}$$

We will call function (4) the characteristic function of the boundary value problem $L(\omega,\alpha,\beta)$.

Theorem 1: The eigenvalues of the boundary problem $L(\omega,\alpha,\beta)$ are real and nonzero.

Theorem 2: If the function $y(x)$ is an eigenfunction of the problem $L(\omega,\alpha,\beta)$ corresponding to the eigenvalue λ, then the following relationship holds:

$$2\int_0^\pi [\lambda-p(x)]|y(x)|^2\,dx+\alpha|y(0)|^2\neq 0\,.$$

Additionally, the sign of the left-hand side of this relationship coincides with the sign of the eigenvalue λ.

Furthermore, in this section, we also investigate whether the boundary value problem $L(\omega,\alpha,\beta)$ possesses associated eigenfunctions.

Theorem 3: The boundary problem $L(\omega,\alpha,\beta)$ has no associated eigenfunctions. In this dissertation, whenever a dot appears above a function, it should be understood as differentiation with respect to the parameter λ.

Theorem 4. If $\delta(\lambda_0)=0$ and $\sigma(\lambda_0)\neq 0$, then the relation $\dot{\delta}(\lambda_0)\neq 0$ holds.

Theorem 5. The inequality $\delta(0)<0$ is valid.

In section 1.2, it is proven that the eigenvalues of the boundary value problem $L(\omega,\alpha,\beta)$ can be repeated.

Theorem 6. The eigenvalue λ_0 of the boundary value problem

$L(\omega,\alpha,\beta)$ is repeated only if ω is a real number different from zero and the following equalities hold:

$$\alpha\lambda_0+\beta+\omega c(\pi,\lambda_0)=s'(\pi,\lambda_0)+\gamma s(\pi,\lambda_0)=0 \qquad (6)$$

Furthermore, this section establishes a criterion for the characteristic function of the boundary value problem $L(\omega,\alpha,\beta)$ to have repeated zeros.

Theorem 7. For the number λ_0 to be a repeated zero of the characteristic function (4) of the boundary value problem $L(\omega,\alpha,\beta)$, a necessary and sufficient condition is that this number is a repeated eigenvalue of the boundary value problem $L(\omega,\alpha,\beta)$, and that ω is a real number different from zero, satisfying condition (6).

Additionally, in section 1.2, it is shown that the order of the zeros of the function $\delta(\lambda)$ does not exceed two.

Theorem 8. If $\delta(\lambda_0)=\dot{\delta}(\lambda_0)=0$, then $\ddot{\delta}(\lambda_0)\neq 0$, meaning that the order of the zeros of the function $\delta(\lambda)$cannot exceed two.

From this point onward, we assume that ω is a real number. In section 1.3, an asymptotic formula for the eigenvalues of the boundary value problem $L(\omega,\alpha,\beta)$is derived.

Theorem 9. For the eigenvalues $\gamma_k\ (k=\pm 0,\pm 1,\pm 2,...)$ of the boundary value problem $L(\omega,\alpha,\beta)$, the following asymptotic relation holds as $|k|\to\infty$

$$\gamma_k=k+a-\frac{1}{\pi}\operatorname{arctg}\alpha+\frac{(-1)^{k+1}b\omega-B}{k\pi}+\frac{\tau_k}{k}, \qquad (7)$$

where $a=\dfrac{1}{\pi}\displaystyle\int_0^{\pi}p(x)dx,\ B=\dfrac{\beta+\alpha p_0}{1+\alpha^2}-\gamma-\pi Q,\ p_0=p(0),$ (8)

$$b=\frac{2}{\sqrt{1+\alpha^2}},\ \ Q=\frac{1}{2\pi}\int_0^{\pi}\left[q(x)+p^2(x)\right]dx,\ \{\tau_k\}\in l_2.$$

In the last part of Chapter I, section 1.4, a proof is provided for the mutual arrangement (interlacing) of the eigenvalues of the boundary value problems. Initially, the cases $\alpha\neq 0,\ \omega\neq 0$ are considered, and as β varies, problems (1) and (2) are examined. In this case, we denote the boundary problem as $P(\beta)$.

In the following discussions, we assume that the index j takes values 1 and 2. The characteristic function of the boundary problem $P(\beta_j)$is denoted by $\delta_j(\lambda)\delta_j(\lambda)$, and the eigenvalues of this boundary problem are denoted by $\gamma_k^{(j)}$ ($k = \pm 0, \pm 1, \pm 2, \ldots$).

Based on equalities (4) and (7), the following relationships hold:

$$\delta_j(\lambda) = 2\omega - \eta(\lambda) + \omega^2 s(\pi, \lambda) + (\alpha\lambda + \beta_j)\sigma(\lambda),$$

$$\gamma_k^{(j)} = k + a - \frac{1}{\pi}\operatorname{arctg}\alpha + \frac{(-1)^{k+1} b\omega - B_j}{k\pi} + \frac{\tau_k^{(j)}}{k},$$

where $B_j = \dfrac{\beta_j + \alpha p_0}{1+\alpha^2} - \gamma - \pi Q, \quad \{\tau_k^{(j)}\} \in l_2$.

Theorem 10. If $\beta_1 < \beta_2$, then for the eigenvalues $\gamma_k^{(1)}$ and $\gamma_k^{(2)}$ of the boundary value problems $P(\beta_1)$ and $P(\beta_2)$, the following inequalities hold when $\omega < 0$ for $k = \pm 0, \pm 1, \pm 2, \ldots$:

$$0 < \gamma_{+0}^{(2)} \le \gamma_{+0}^{(1)} \le \gamma_1^{(2)} \le \gamma_1^{(1)} < \gamma_2^{(2)} \le \gamma_2^{(1)} \le \gamma_3^{(2)} \le \gamma_3^{(1)} < \ldots,$$

$$0 > \gamma_{-0}^{(2)} \ge \gamma_{-0}^{(1)} \ge \gamma_{-1}^{(2)}\gamma_{2,-1} \ge \gamma_{-1}^{(1)} > \gamma_{-2}^{(2)} \ge \gamma_{-2}^{(1)} \ge \gamma_{-3}^{(2)} \ge \gamma_{-3}^{(1)} > \ldots$$

For $\omega > 0$, the following inequalities hold:

$$0 < \gamma_{+0}^{(2)} < \gamma_{+0}^{(1)} < \gamma_1^{(2)} \le \gamma_1^{(1)} \le \gamma_2^{(2)} \le \gamma_2^{(1)} < \gamma_3^{(2)} \le \gamma_3^{(1)} \le \ldots,$$

$$0 > \gamma_{-0}^{(2)} > \gamma_{-0}^{(1)} > \gamma_{-1}^{(2)}\gamma_{2,-1} \ge \gamma_{-1}^{(1)} \ge \gamma_{-2}^{(2)} \ge \gamma_{-2}^{(1)} > \gamma_{-3}^{(2)} \ge \gamma_{-3}^{(1)} \ge \ldots$$

Thus, if $\gamma_k^{(j)} = \gamma_{k+1}^{(j)}$, then the following relation holds: $\gamma_{k-1}^{(3-j)} < \gamma_k^{(3-j)} < \gamma_{k+1}^{(3-j)}$.

In section 1.4, the cases $\beta \ne 0$, and $\omega \ne 0$ are also considered, the arrangement of the eigenvalues of problem (1), (2) as α varies is studied.. In this case, the boundary value problem is denoted as $Y(\alpha)$. The characteristic function of the boundary value problem $Y(\alpha_j)$ is denoted by $\Delta_j(\lambda)$, and the eigenvalues of this boundary problem are denoted by $\mu_k^{(j)}$ ($k = \pm 0, \pm 1, \pm 2, \ldots$).

From equation (4), we obtain:

$$\Delta_j(\lambda) = 2\omega - \eta(\lambda) + \omega^2 s(\pi,\lambda) + (\alpha_j\lambda + \beta)\sigma(\lambda).$$

On the other hand, based on relations (7) and (8), we have:

$$\mu_k^{(j)} = k + \tilde{a}_j + \frac{(-1)^{k+1} b_j \omega - A_j}{k\pi} + \frac{\eta_k^{(j)}}{k},$$

where $A_j = \dfrac{\beta + \alpha_j\, p_0}{1+\alpha_j^2} - \gamma - \pi Q,\ \tilde{a}_j = a - \dfrac{1}{\pi}\operatorname{arctg}\alpha_j,$

$$b_j = \frac{2}{\sqrt{1+\alpha_j^2}}, \quad \left\{\eta_k^{(j)}\right\} \in l_2$$

Theorem 11. For the eigenvalues $\mu_k^{(1)}$ and $\mu_k^{(2)}$ of the boundary value problems $Y(\alpha_1)$ and $Y(\alpha_2)$ where $(\alpha_1 < \alpha_2)$, the following inequalities hold when $\omega < 0$, for $k = \pm 0, \pm 1, \pm 2, \ldots$:

$$0 < \mu_{+0}^{(2)} \le \mu_{+0}^{(1)} \le \mu_1^{(2)} \le \mu_1^{(1)} < \mu_2^{(2)} \le \mu_2^{(1)} \le \mu_3^{(2)} \le \mu_3^{(1)} < \ldots,$$

$$0 > \mu_{-0}^{(2)} \ge \mu_{-0}^{(1)} \ge \mu_{-1}^{(2)} \ge \mu_{-1}^{(1)} > \mu_{-2}^{(2)} \ge \mu_{-2}^{(1)} \ge \mu_{-3}^{(2)} \ge \mu_{-3}^{(1)} > \ldots$$

For $\omega > 0$, the following inequalities hold:

$$0 < \mu_{+0}^{(2)} < \mu_{+0}^{(1)} < \mu_1^{(2)} \le \mu_1^{(1)} \le \mu_2^{(2)} \le \mu_2^{(1)} < \mu_3^{(2)} \le \mu_3^{(1)} \le \ldots,$$

$$0 > \mu_{-0}^{(2)} > \mu_{-0}^{(1)} > \mu_{-1}^{(2)} \ge \mu_{-1}^{(1)} \ge \mu_{-2}^{(2)} \ge \mu_{-2}^{(1)} > \mu_{-3}^{(2)} \ge \mu_{-3}^{(1)} \ge \ldots$$

Thus, if $\mu_k^{(j)} = \mu_{k+1}^{(j)}$, then the following relation holds: $\mu_{k-1}^{(3-j)} < \mu_k^{(3-j)} < \mu_{k+1}^{(3-j)}$.

Chapter II is dedicated to the uniqueness of the solution of the inverse problem and the construction of inverse problem solution algorithms based on the two spectra and the sequence of signs for the boundary value problems $P(\beta)$ and $Y(\alpha)$.

In section 2.1, the problem of the unique reconstruction of the characteristic function of the boundary problem $P(\beta)$ with respect to the spectrum is investigated.

Theorem 12. If $a = 0$, then the given spectrum $\{\gamma_k\}$ ($k = \pm 0, \pm 1, \pm 2, \ldots$) of the boundary value problem $P(\beta)$ uniquely determines its characteristic function $\delta(\lambda)$ via the formula:

$$\delta(\lambda)=\frac{\pi}{\cos \pi d}\left(\gamma_{-0}-\lambda\right)\left(\gamma_{+0}-\lambda\right)\prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{\gamma_k-\lambda}{k},$$

where $d=\lim_{k\to\infty}\left(\gamma_k-k\right)$.

In section 2.2, the uniqueness of the reconstruction of the boundary value problems $P(\beta)$ and $Y(\alpha)$ based on spectral data is proven. It is noted that the spectra of problems of the type $P(\beta_1), P(\beta_2)$ or $Y(\alpha_1), Y(\alpha_2)$ do not uniquely determine the coefficients $p(x)$ and $q(x)$ of equation (1) or, in general, the boundary value problems themselves. Additional spectral data are required for the unique reconstruction of these problems. Such spectral data are given by the sequence of signs $\sigma_n=\operatorname{sign} h_-(\nu_n)$, where $h_-(\lambda)=\alpha\lambda\sigma(\lambda)-\eta(\lambda)-\omega^2 s(\pi,\lambda)$.

The values $\nu_n, n=\pm1,\pm2,...$ are the zeros of the function $\sigma(\lambda)$. These zeros satisfy the following asymptotic formula as $|n|\to\infty$:

$$\nu_n=n-\frac{1}{2}\operatorname{sign} n+a+\frac{\pi Q+\gamma}{n\pi}+\frac{\xi_n}{n}, \quad \{\xi_n\}\in l_2. \qquad (9)$$

The sequences $\{\gamma_k^{(1)}\}$, $\{\gamma_k^{(2)}\}$ $(k=\pm0,\pm1,\pm2,...)$ and $\{\sigma_n\}$ $(\sigma_n=-1,0,1\,;\ n=\pm1,\pm2,...)$ are referred to as the spectral data of the boundary value problems $P(\beta_1)$ and $P(\beta_2)$. The inverse spectral problem for $P(\beta_1)$ and $P(\beta_2)$ consists of reconstructing the coefficients $p(x), q(x)$ of equation (1) and the parameters $\omega, \alpha, \beta_1, \beta_2$, γ included in the boundary conditions from the given spectral data.

Theorem 13. If $a=p_0=0$, then the boundary value problems $P(\beta_1)$ *and* $P(\beta_2)$ are uniquely reconstructed by their spectral data. Similarly, for the boundary value problems $Y(\alpha_1), Y(\alpha_2)$, the spectral data refer to the $\{\mu_k^{(1)}\}$, $\{\mu_k^{(2)}\}$ and the sequence of signs $\{\sigma_n\}$.

The inverse spectral problem for these boundary value problems consists of uniquely reconstructing the coefficients $p(x), q(x)$ of equation (1), as well as the parameters ω, β, α_1, α_2 and γ included in the boundary conditions.

Theorem 14. The boundary value problems $Y(\alpha_1)$ and $Y(\alpha_2)$ are uniquely reconstructed from their spectral data.

In section 2.3, the boundary value problem

$$-y'' + 2\lambda p(x)y + q(x)y = \lambda^2 y,\ \ 0 < x < \infty$$

$y(0) = 0$ is considered. For this boundary value problem, certain auxiliary factors related to the inverse scattering problem on the half-axis are introduced.

The functions $p(x)$, $q(x)$ satisfy the conditions:

$$q(x) = p(x) = 0, x > \pi,$$
$$q(x) \in L_2[0,\pi], p(x) \in W_2^1[0,\pi].$$

In section 2.4, the equation

$$-y'' + 2\lambda p(x)y + q(x)y = \lambda^2 y,\quad 0 \le x \le \pi \tag{10}$$

is initially considered, along with the boundary conditions

$$y(0) = y(\pi) = 0\ , \tag{11}$$
$$y(0) = y'(\pi) = 0. \tag{12}$$

Boundary value problems arising from separated boundary conditions are examined, where $q(x) \in L_2[0,\pi], p(x) \in W_2^1[0,\pi]$ are real-valued functions. The inverse problem solution algorithm for the two spectra corresponding to boundary value problems (10), (11) and (10), (12) is provided.

Algorithm 1. Suppose that the sequences $\{\lambda_n\}$ and $\{\theta_n\}$ $(n = \pm 1, \pm 2, \ldots)$, corresponding to the spectra of problems (10), (11) and (10), (12) are given.

1) Using the sequences $\{\lambda_n\}$ and $\{\theta_n\}$, we construct the functions: $s(\lambda) = \pi \prod\limits_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \dfrac{\lambda_n - \lambda}{n}$, $s_1(\lambda) = \prod\limits_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \dfrac{\theta_n - \lambda}{n - \frac{1}{2}\operatorname{sign} n}$

2) Defining $\varphi(\lambda) = e^{i\lambda\pi}[s_1(\lambda) - i\lambda s(\lambda)]$, we introduce the

function:

$$S(\lambda)=\frac{\overline{\varphi(\lambda)}}{\varphi(\lambda)},-\infty<\lambda<\infty.$$

3) The function $F(x)$is determined by the formula:

$$F(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[e^{-2i\pi a}-S(\lambda)\right]e^{i\lambda x}d\lambda$$

where $a=\lim_{n\to\infty}(\lambda_n-n)$.

4) For each fixed x, solving the equations

$$F(x+y)+\overline{K_0(x,y)}+\int_x^{\infty}K_0(x,t)F(t+y)dt=0,0\le x\le y,$$

$$iF(x+y)+\overline{K_1(x,y)}+\int_x^{\infty}K_1(x,t)F(t+y)dt=0,\ 0\le x\le y,$$

we obtain the solutions $K_0(x,y)\in L_1(x,\infty)$, $K_1(x,y)\in L_1(x,\infty)$.

5) We determine the function $\alpha(x)$ as the solution of the following Volterra Integral Equation:

$$\alpha(x)=\int_x^{\infty}V(t,\alpha(t))dt,x\ge 0$$

Where

$$V(t,z)=\left[\operatorname{Re}K_0(t,t)-\operatorname{Im}K_1(t,t)\right]\sin 2z+$$

$$+2\left[\operatorname{Re}K_1(t,t)\right]\sin^2 z-2\left[\operatorname{Im}K_0(t,t)\right]\cos^2 z.$$

6) We reconstruct the coefficients $p(x)$ and $q(x)$using the formulas:

$$p(x)=-\alpha'(x),$$

$$q(x)=-p^2(x)-2\frac{d}{dx}\left[\operatorname{Re}[K(x,y)]\cos\alpha(x)+\operatorname{Im}[K(x,y)]\sin\alpha(x)\right],$$

Where

$$K(x,y)=K_0(x,y)\cos\alpha(x)+K_1(x,y)\sin\alpha(x)$$

Using Algorithm 1, inverse problem-solving algorithms for boundary value problems of the types $P(\beta_1)$, $P(\beta_2)$ and

$Y(\alpha_1), Y(\alpha_2)$ are also provided in section 2.4.

Algorithm 2. Suppose that the sequences $\{\gamma_k^{(1)}\}$, $\{\gamma_k^{(2)}\}$, $\{\sigma_n\}$ are the spectral data of the boundary value problems $P(\beta_1)$ and $P(\beta_2)$, and let $a = p_0 = 0$.

1) We determine the parameters α and ω using the formulas:

$$\alpha = -\mathrm{tg}\,\pi d\,,\ \omega = \frac{\pi}{b}\lim_{k\to\infty} k\left(\gamma_{2k+1}^{(j)} - \gamma_{2k}^{(j)} - 1\right)$$

$$\text{where } d = \lim_{k\to\infty}\left(\gamma_k^{(j)} - k\right), \quad b = \frac{2}{\sqrt{1+\alpha^2}}.$$

2) Using the sequences $\{\gamma_k^{(j)}\}$ and the parameter α, we construct the functions $\delta_j(\lambda), j = 1,2$, in the form of an infinite product:

$$\delta_j(\lambda) = \pi\sqrt{1+\alpha^2}\left(\gamma_{-0}^{(j)} - \lambda\right)\left(\gamma_{+0}^{(j)} - \lambda\right)\prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{\gamma_k^{(j)} - \lambda}{k}\ .$$

3) We compute $\beta_2 - \beta_1$ using the equality

$$\beta_2 - \beta_1 = \pi\left(1+\alpha^2\right)\lim_{k\to\infty} k\left(\gamma_k^{(1)} - \gamma_k^{(2)}\right)$$

4) We determine the function $\sigma(\lambda)$ using the following formula:

$$\sigma(\lambda) = \frac{\delta_1(\lambda) - \delta_2(\lambda)}{\beta_1 - \beta_2}$$

5) Finding the zeros $\nu_n, n = \pm 1, \pm 2, \ldots$ of the function $\sigma(\lambda)$ and reconstructing the parameters β_j using the formula:

$$\beta_j = \left(1+\alpha^2\right)\lim_{k\to\infty}\left[b\omega - 2k\pi\left(\gamma_{2k+1}^{(j)} - \nu_{2k+1} - \frac{1}{2}\right)\right]$$

6) We construct the function $h_+(\lambda)$:

$$h_+(\lambda) = \frac{\beta_2\delta_1(\lambda) - \beta_1\delta_2(\lambda)}{\beta_2 - \beta_1}\ .$$

7) We find the values of $h_-(\lambda)$ at the zeros ν_n of $\sigma(\lambda)$ using the formula:

$$h_-(\nu_n) = (-1)^k \sigma_n \sqrt{h_+^2(\nu_n) - 4\omega^2} \ .$$

8) Using $\sigma(\lambda)$ and its zeros ν_n, we construct the function

$$g(\lambda) = \sigma(\lambda) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{g(\nu_n)}{(\lambda - \nu_n)\dot{\sigma}(\nu_n)}$$

Where

$$g(\nu_n) = h_+(\nu_n) - (-1)^n \sigma_n \sqrt{h_+^2(\nu_n) - 4\omega^2} - 2\omega^2 \frac{\sin \nu_n \pi}{\nu_n} .$$

9) We determine the function $h_-(\lambda)$ using the formula:

$$h_-(\lambda) = h_+(\lambda) - g(\lambda) - 2\omega^2 \frac{\sin \lambda\pi}{\lambda}$$

10) We reconstruct the function $s(\pi, \lambda)$ using the formula:

$$s(\pi, \lambda) = \frac{1}{2\omega^2}[h_+(\lambda) - h_-(\lambda)].$$

(This function is the characteristic function of problems (10) and (11)). We also determine its zeros λ_n, $n = \pm 1, \pm 2, ...$

11) We determine the parameter γ using the zeros λ_n of the function $s(\pi, \lambda)$ and the zeros ν_n of the function $\sigma(\lambda)$:

$$\gamma = \pi \lim_{n\to\infty} k\left(\nu_n - \lambda_n + \frac{1}{2}\right).$$

12) We construct the function $s'(\pi, \lambda)$ using the formula:

$$s'(\pi, \lambda) = \sigma(\lambda) - \gamma\, s(\pi, \lambda) \ .$$

(This function is the characteristic function of problems (10) and (12)). We find the zeros of $s'(\pi, \lambda)$, denoted by $\{\theta_n\}$.

13) Using the sequences of zeros $\{\lambda_n\}$ and $\{\theta_n\}$ of the functions $s(\pi, \lambda)$ and $s'(\pi, \lambda)$, we reconstruct the coefficients $p(x), q(x)$ of equation (1) with the help of Algorithm 1.

Algorithm 3. Suppose that the sequences $\{\mu_k^{(1)}\}$, $\{\mu_k^{(2)}\}$, $\{\sigma_n\}$ correspond to the spectral data of the boundary value problems $Y(\alpha_1)$ and $Y(\alpha_2)$.

1) We determine the quantities $\tilde{a}_j$, $j=1,2$, using the formula:

$$\tilde{a}_j = \lim_{k\to\infty}\left(\mu_k^{(j)} - k\right)$$

2) Using the sequences $\left\{\mu_k^{(j)}\right\}$ and the parameter $\tilde{a}_j$, we construct the functions $\Delta_j(\lambda)$, $j=1,2$ in the form of an infinite product:

$$\Delta_j(\lambda) = \frac{\pi}{\cos\pi\tilde{a}_j}\left(\mu_{-0}^{(j)} - \lambda\right)\left(\mu_{+0}^{(j)} - \lambda\right)\prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{\mu_k^{(j)} - \lambda}{k}$$

3) We reconstruct the quantities $a, \alpha_1, \alpha_2, \omega$ using the formulas:

$$\operatorname{tg}\pi a = \lim_{k\to\infty}\frac{\Delta_1\left(2k+\frac{1}{2}\right) + \Delta_2\left(2k-\frac{1}{2}\right)}{\Delta_1(2k) + \Delta_2(2k-1)},$$

$$\alpha_j = \frac{tg\pi a - \operatorname{tg}\pi\tilde{a}_j}{1 + tg\pi a\cdot\operatorname{tg}\pi\tilde{a}_j}$$

$$\omega = \frac{\pi\sqrt{1+\alpha_j^2}}{2}\lim_{k\to\infty}k\left(\mu_{2k+1}^{(j)} - \mu_{2k}^{(j)} - 1\right)$$

4) We determine the function $\sigma(\lambda)$ using the following formula:

$$\sigma(\lambda) = \frac{\Delta_1(\lambda) - \Delta_2(\lambda)}{\lambda(\alpha_1 - \alpha_2)}.$$

We find the zeros ν_n $(n=\pm1,\pm2,\ldots)$ of the function $\sigma(\lambda)$.

5) Using the asymptotics (9) of the zeros ν_n $(n=\pm1,\pm2,\ldots)$ of the function $\sigma(\lambda)$, we reconstruct the parameter β using the formula:

$$\beta = \frac{\alpha_1 d_2 - \alpha_2 d_1}{\alpha_1 - \alpha_2},$$

wher

$$d_j = \left(1+\alpha_j^2\right)\lim_{k\to\infty}\left[\frac{2}{\sqrt{1+\alpha_j^2}}\omega - 2k\pi\left(\mu_{2k+1}^{(j)} - \nu_{2k+1} + \frac{1}{\pi}\operatorname{arctg}\alpha_j - \frac{1}{2}\right)\right]$$

6) We construct the function $h_+(\lambda)$:

$$h_+(\lambda)=\frac{\alpha_2\Delta_1(\lambda)-\alpha_1\Delta_2(\lambda)}{\alpha_2-\alpha_1}-2\omega$$

7) We determine the values of $h_-(\lambda)$ at the zeros ν_n of the function $\sigma(\lambda)\sigma(\lambda)$ $(n=\pm1,\pm2,...)$ $(n=\pm1,\pm2)$ using the formula:

We determine the values of the function $h_-(\lambda)$ at the zeros ν_n $(n=\pm1,\pm2,...)$ of the function $\sigma(\lambda)$ using the following formula:

$$h_-(\nu_n)=(-1)^{n+1}\sigma_n\sqrt{h_+^2(\nu_n)-4\omega^2}$$

8) Using $\sigma(\lambda)$ and its zeros ν_n, we construct the function

$$g(\lambda)=\sigma(\lambda)\sum_{\substack{n=-\infty\\ n\neq0}}^{\infty}\frac{g(\nu_n)}{(\lambda-\nu_n)\dot{\sigma}(\nu_n)}.$$

Where

$$g(\nu_n)=h_+(\nu_n)-(-1)^n\operatorname{sgn}h_-(\nu_n)\sqrt{h_+^2(\nu_n)-4\omega^2}-2\omega^2\frac{\sin(\nu_n-a)\pi}{\nu_n}$$

9) We determine the function $h_-(\lambda)$ using the equality:

$$h_-(\lambda)=h_+(\lambda)-g(\lambda)-2\omega^2\frac{\sin(\lambda-a)\pi}{\lambda}$$

10) We determine the function $s(\pi,\lambda)$ using the formula:

$$s(\pi,\lambda)=\frac{1}{2\omega^2}\left[h_+(\lambda)-h_-(\lambda)\right]$$

This function is the characteristic function of problems (10) and (11). We also determine its zeros λ_n, $n=\pm1,\pm2,...$

11) Using the zeros λ_n of the function $s(\pi,\lambda)$ and the zeros ν_n of the function $\sigma(\lambda)$, we determine the parameter γ:

$$\gamma=\pi\lim_{n\to\infty}k\left(\nu_n-\lambda_n+\frac{1}{2}\right).$$

12) We construct the function $s'(\pi,\lambda)$ using the formula:

$$s'(\pi,\lambda)=\sigma(\lambda)-\gamma s(\pi,\lambda).$$

This function is the characteristic function of problems (10) and

(12). We determine the zeros of $s'(\pi,\lambda)$, denoted as $\{\theta_n\}$.

13) Using the sequences of zeros $\{\lambda_n\}$ and $\{\theta_n\}$ of the functions $s(\pi,\lambda)$ and $s'(\pi,\lambda)$, we reconstruct the coefficients $p(x), q(x)$ of equation (1) with the help of Algorithm 1.

Section III of the dissertation is devoted to finding sufficient conditions for the unique solvability of inverse problems for boundary value problems of the types $Y(\alpha_1), Y(\alpha_2)$.

In paragraph 3.1, the function $\sigma(\lambda)$, defined by equation (5), is examined.

In the second and final paragraphs of Section III, sufficient conditions are found for ensuring that the spectral data consists of the sequences $\{\mu_k^{(1)}\}$, $\{\mu_k^{(2)}\}$ $(k=\pm 0,\pm 1,\pm 2,\ldots)$, and $\{\sigma_n\}$ $(\sigma_n=-1,0,1\,;\ n=\pm 1,\pm 2,\ldots)$, forming the complete set of quantities that uniquely determine the spectral data of boundary value problems of the types $Y(\alpha_1)$ and $Y(\alpha_2)$ $(\alpha_1<\alpha_2)$.

Theorem 15. The real number sequences $\{\mu_k^{(1)}\}$, $\{\mu_k^{(2)}\}$ $(k=\pm 0,\pm 1,\pm 2,\ldots)$ and $\{\sigma_n\}$ $(\sigma_n=-1,0,1\,;\ n=\pm 1,\pm 2,\ldots)$ serve as the spectral data for the boundary value problems $Y(\alpha_1)$ and $Y(\alpha_2)$ $(\alpha_1<\alpha_2)$ if the following conditions hold:

1) When $|k|\to\infty$, the asymptotic formula

$$\mu_k^{(j)}=k+a+a_j+\frac{(-1)^{k+1}A_j-B_j}{k\pi}+\frac{\tau_k^{(j)}}{k}\ \text{ is valid,}$$

where $A_j=2\omega\cos\pi a_j$, ω, a, a_j, B_j are real numbers satisfying

$$a<0,\quad 0<a_j<\frac{1}{2},\quad a_1>a_2,\ \omega\neq 0,\ \{\tau_k^{(j)}\}\in l_2;$$

2) The numbers $\mu_k^{(1)}$ and $\mu_k^{(2)}$, $k=\pm 0,\pm 1,\pm 2,\ldots$ satisfy the inequalities:

When $\omega<0$:

$$0<\mu_{+0}^{(2)}\le\mu_{+0}^{(1)}\le\mu_1^{(2)}\le\mu_1^{(1)}<\mu_2^{(2)}\le\mu_2^{(1)}\le\mu_3^{(2)}\le\mu_3^{(1)}<\ldots,$$

$$0>\mu_{-0}^{(2)}\ge\mu_{-0}^{(1)}\ge\mu_{-1}^{(2)}\ge\mu_{-1}^{(1)}>\mu_{-2}^{(2)}\ge\mu_{-2}^{(1)}\ge\mu_{-3}^{(2)}\ge\mu_{-3}^{(1)}>\ldots$$

When $\omega > 0$:

$$0 < \mu_{+0}^{(2)} < \mu_{+0}^{(1)} < \mu_{1}^{(2)} \le \mu_{1}^{(1)} \le \mu_{2}^{(2)} \le \mu_{2}^{(1)} < \mu_{3}^{(2)} \le \mu_{3}^{(1)} \le \ldots,$$

$$0 > \mu_{-0}^{(2)} > \mu_{-0}^{(1)} > \mu_{-1}^{(2)} \ge \mu_{-1}^{(1)} \ge \mu_{-2}^{(2)} \ge \mu_{-2}^{(1)} > \mu_{-3}^{(2)} \ge \mu_{-3}^{(1)} \ge \ldots$$

These inequalities hold, and if $\mu_k^{(j)} = \mu_{k+1}^{(j)}$ then the inequalities $\mu_{k-1}^{(3-j)} < \mu_k^{(3-j)} < \mu_{k+1}^{(3-j)}$ are satisfied.

3) For every $n = \pm 1, \pm 2, \ldots$ and $j = 1, 2$ the inequality

$b_n \overset{def}{=} \left|\delta_j(\nu_n) - 2\omega\right| - 2|\omega| \ge 0$ holds,

where

$$\delta_j(\lambda) = \frac{\pi\left(\mu_{-0}^{(j)} - \lambda\right)\left(\mu_{+0}^{(j)} - \lambda\right)}{\cos \pi a_j} \prod_{\substack{k=-\infty \\ k \ne 0}}^{\infty} \frac{\mu_k^{(j)} - \lambda}{k},$$

The numbers ν_n are the zeros of the function $\delta_1(\lambda) - \delta_2(\lambda)$.

4) The quantity σ_n is equal to zero when $b_n = 0$, and when $b_n > 0$ it takes the values 1 or −1. Moreover, there exists a number $N > 0$ such that for $|n| \ge N$, we have $\sigma_n = 1$.

CONCLUSION

The dissertation is devoted to the study of direct and inverse problems of spectral analysis for the diffusion operator in the case where one of the inseparable boundary conditions includes a spectral parameter.

The following main results were obtained in the dissertation:

- asymptotic formulas for the eigenvalues of the diffusion operator in the case where one of the inseparable boundary conditions includes a linear function of the spectral parameter were obtained;

- necessary and sufficient conditions for the eigenvalues of boundary problems to be recurrent were given, and it was proved that these problems do not have adjoint functions;

- uniqueness theorems for solving inverse problems on the reconstruction of boundary problems by two spectra and a sequence of signs were proved and effective solution algorithms were given;

- sufficient conditions were found for the spectral data of the diffusion operator in which one of the inseparable boundary conditions of a set of certain quantities includes a linear function of the spectral parameter.

The main results of the dissertation are reflected in the following works:

1. Ибадзаде, Ч.Г. Кратность собственных значений оператора диффузии со спектральным параметром в граничном условии // – Баку: Journal of Qafqaz University - Mathematics and computer science, – 2015. v. 3, № 2, – p. 188-195.
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