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ABSTRACT

of the dissertation for the degree of Doctor of Science

DEVELOPMENT AND APPLICATION OF DECISION-MAKING METHODS BASED ON FUZZY LOGIC AND ITS EXTENSIONS

Specialty: 3338.01 – System Analysis, Management and Information Processing (control and decision making)

Field of science: Technical sciences

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GENERAL DESCRIPTION OF THE DISSERTATION

The actuality of the topic. Although existing methods are widely used in the decision-making process, there is a great need to create new decision-making approaches based on fuzzy logic and its extensions. An analysis of solutions to the multi-criteria decision-making problem showed that as new methods emerged and old methods improved, the role of decision-making tools in various applications increased significantly. This dissertation analyzes different multi-criteria decision-making methods, evaluates the relative advantages, and disadvantages of these methods and determines their application to different situations, the use of these methods and the evolution of their use over time. The development of technology in recent decades has allowed the formation of more sophisticated decision-making methods. The experience of dealing with extended multi-criteria decision-making methods provided a completely new approach to multi-stage decision analysis. The main relevance of this dissertation is that it forms an appropriate advantage that more adequately describes the decision-maker's preference under uncertainty. Such an advantage is based on data generalization, which involves higher-level uncertainty. This dissertation is devoted to the study of the advantages of fuzzy logic and its extensions (as type-2 fuzzy sets and Z-numbers) which allow to more adequately describe uncertainty of real decision problems and to develop and apply new decision-making methods to a wide spectrum of problems. From this point of view, the activity of the research conducted in the dissertation is evident.

Goal of the dissertation. The purpose of the dissertation is to develop and apply decision-making approaches based on fuzzy logic and its extensions in various fields. Taking this into account, the dissertation covers: comparative analysis of existing decision-making methods, analysis of general classification of decision-relevant information, analysis of fuzzy sets and their extensions, solving of a problem of multi-stage decision-making based on fuzzy dynamic programming, and application of Z-decision-making tools.

Main highlights, brought forward for dissertation defense.

The following provisions are submitted for defense in the dissertation:

- A comparative analysis of various multi-criteria decision-making tools was conducted, and the relative advantages and disadvantages of these methods were assessed;
- A method of analysis of preference by using consistency index in the framework of Z-extension of fuzzy logic has been proposed;
-The classical Delphi method has been extended to a fuzzy environment;
-The method of aggregation of information, widely used in decision analysis, was developed for Z-environment;
- Exact and approximate methods of solving linear equations (which are widely used in decision-making problems under uncertainty) with parameters and variables described by Z-numbers have been proposed;
- The problem of construction of a Z-regression model for identification of control and decision-making objects characterized by bimodal information has been solved;
- It is proposed to describe the preference knowledge based on type-2 fuzzy sets in multi-criteria decision-making;
- The solution method and algorithm of the multi-stage decision-making problem are developed on the basis of fuzzy dynamic programming;
- Type-1 and type-2 fuzzy and Z-decision-making approaches have been applied to various problems.

Research methods. Research methods in the dissertation include fuzzy logic, Z-extension of fuzzy logic, computation with fuzzy and Z-numbers, eigenvalue and eigenvector analysis, fuzzy relation equations, distance and similarity measures, fuzzy set theory methods, operations on type-2 fuzzy sets, determination of probability distributions of Z-numbers.

Scientific value of the thesis.

- The formulation of the preference knowledge of decision-maker on the basis of type-2 fuzzy sets expands the decisions analysis from the scientific point of view;

- Z-preference analysis is a new approach to decision making;
- Multi-stage decision-making approach based on fuzzy dynamic programming, development and application of solution algorithm expands the analysis of classical decisions for a dynamic environment;
-The proposed method of aggregation of Z-valued information is new;
- The fuzzy Delphi method is new and is widely used in the scientific literature;
- Solution of the system of linear equations (widely used in multicriteria decision-making, planning and forecasting) extended for bimodal environment by using the Z-number theory is new;
- The proposed Z-regression model for modeling decision-making and control objects characterized by bimodal information extends regression analysis.

Practical value of the thesis. The proposed multi-criteria decision-making tools and methods can be applied to various fields of the economy, production, business. Different methods and approaches suggested in the dissertation have been applied.

Approbation of dissertation. Main results of dissertation were presented in the international conferences:

- ***ICAFS-2010*** 9th International Conference on Application of Fuzzy Systems and Soft Computing, Prague, Czech Republic;
- ***ICSCCW-2011*** 6th International Conference on Soft Computing and Computing with Words and Perceptions, Antalya, Turkey;
- ***ICAFS-2012*** 10th International Conference on Application of Fuzzy Systems and Soft Computing, Lisbon, Portugal;
- ***WCIS-2012*** 7th World Conference on Intelligence Systems for Industrial Automation, Tashkent, Uzbekistan;
- ***ICSCCW-2015*** 8th International Conference on Soft Computing and Computing with Words and Perceptions, Antalya, Turkey;
- ***ICAFS-2016*** 12th International Conference on Application of Fuzzy Systems and Soft Computing, ICAFS 2016, 29-30 August 2016, Vienna, Austria;
- ***ICSCCW-2017*** 9th International Conference on Theory and

Application of Soft Computing, Computing with Words and Perceptions – ICSCCW-2017, 22-23 August 2017, Budapest, Hungary;

- ***ICAFS-2018*** 13th International Conference on Theory and Application of Fuzzy Systems and Soft Computing - ICAFS-2018, 26-27 August 2018, Warsaw, Poland;
- ***ICAFS-2020*** 14th International Conference on Theory and Application of Fuzzy Systems and Soft Computing – ICAFS-2020, 27-28 August 2020, Budva, Montenegro;
- ***WCIS-2020*** 11th World Conference on Intelligent systems for industrial automation – WCİS-2020, 26-28 November, Tashkent, Uzbekistan;
- ***ICSCCW-2021*** 11th International Conference on Theory and Application of Soft Computing, Computing with Words and Perception – Antalya, Turkey, on August 23–24, 202.

Organization where dissertation was realized: Azerbaijan State Oil and Industry University, department of "Instrument Engineering".

Structure of dissertation. Manuscript of dissertation includes introduction, 6 chapters, conclusion, and references.

Publications. In total, 61 scientific works were published. Of the 30 scientific works published on the study, 25 are articles (11 are from the Web of Science, 16 are from SCOPUS, included in the databases) and 5 are conference materials.

MAIN CONTENT OF THE WORK

The introduction highlights the relevance of the topic, the goals and objectives of the research, the main provisions to be defended, research methods, the theoretical and practical significance of the research. The analysis of solutions to the problem of multicriteria decision-making showed that as new methods emerged and old methods improved, the role of these decision-making tools in various fields of application increased significantly. This dissertation analyzes different multi-criteria decision-making methods and evaluates the relative advantages and disadvantages of these methods and determines their application to different situations, the use of these methods and the evolution of their use over time. The development of technology in recent decades has allowed formation of more sophisticated decision-making methods. Research shows that while some well-known multicriterial decision-making methods are more appropriate for certain types of problems, a series of real world problems cannot be solved by these methods. Development of new methods would eliminate the shortcomings that may appear in existing methods.

Chapter 1 deals with comparative analysis of existing decision-making methods - AHP, TOPSIS and other multi-criteria decision-making methods. The main advantages and disadvantages of these methods are discussed. A lot of real problems are characterized by multiple criteria. Decision-making is considered as a specific process of human activity aimed at selecting the most appropriate option among available alternatives. Recently, there have been new trends in the development of decision-making technologies in a multi-criteria environment. Nowadays, various methodologies are used to solve problems by selection of the most suitable option among alternatives through pairwise comparisons. One of these important approaches is the Analytic Hierarchy Process (AHP) methodology. This method was first proposed for the analysis of the political situation when it was not possible to use a mathematical (numerical) description of the problem in the decision-making process. Structural description of the problem includes hierarchical description of criteria and alternatives, pairwise comparison of decision elements, calculation of weights of criteria, construction of the decision matrix, evaluation and ranking of alternatives. The advantages of the Analytic Hierarchy Process are universality, reduction of subjectivity due to the consideration of the human factor, verification of data inconsistency; the disadvantages - high labor input, large amount of initial data, limited nature of the assessment scale. Disadvantages of this method: AHP is limited to using only a 9-point scale which sometimes makes it difficult for the decision maker to compare criteria; as the number of pairwise comparisons increases, it takes more time to complete the calculations;

AHP has a limited number of elements; aggregation of individual judgments leads to loss of information. It is difficult to judgements to conduct pairwise comparisons in the AHP, Fuzzy AHP methods. The main problems arise from the interdependence between criteria and alternatives, and the mismatch between judgments and criteria weights. In this case, the TOPSIS method, a widely used multi-criteria decision-making tool, can be applied. The TOPSIS method allows to remove the limit on the number of alternatives and criteria. Based on this method, multi-criteria optimization problems are performed by using fuzzy numbers (triangular or trapezoidal) for linguistic qualitative variables that reflect the criteria values. The optimal solution is found based on the distance between the best solution and the worst solution. The main idea of the TOPSIS method is to identify positive ideal and negative ideal alternatives. A positive ideal alternative is one that maximizes benefit criteria and minimizes cost criteria. The negative ideal one maximizes cost criteria and minimizes benefit criteria. The main difference between TOPSIS method and other decision-making methods is ease of application, universality, determination of distances to an ideal solution. Some of the advantages of TOPSIS methods are simplicity, rationality, comprehensibility, good computational efficiency, and ability to measure the relative performance for each alternative in a simple mathematical form. These advantages make TOPSIS a key multi-criteria decision-making technique compared to other methods. In fact, TOPSIS is a sound method that directly compares alternatives based on data in evaluation matrices and weights. The disadvantage of this method is that when calculating the Euclidean distance, the relationship between the criteria, the correlation is not considered, it is difficult to determine the weight of the criteria, and it is impossible to determine the consistency of the decision matrix. In this case, the use of the AHP-TOPSIS hybrid method is more appropriate.

Although there are many methods and tools for multi-criteria decision making, none of them are perfect. Most of them are based on assumptions applied to real life issues. Therefore, often using an extension of these methods, a synergy of several methods, leads to more realistic results. Classical multi-criteria decision-making methods can be classified into different groups according to their similar characteristics. The technologies used in decision-making depend not only on the nature of the decision-making problem, but also on the character, experience, style and environment in which the decision-maker is located. When considering the methodologies used to solve decision-making problems, it is observed that various multi-criteria decision-making methods can be applied, which can be grouped into several categories. The methods used in this category revolve around various tools to assess the relative importance of multiple criteria and alternatives. Most of the methods in this category are based on weight determination. Examples of methods in this category include the fuzzy simplified weight method, the fuzzy analytical hierarchy process method, the fuzzy conjunctival-disjunctive methods, and the max-min methods. Goal, target, and reference models are goal programming and TOPSIS are the most basic methods in this group. This category includes several ways to determine the ranking of fuzzy multimedia decision-making methods. These include optimality, Hamming distance, comparison function, fuzzy numerical mean and deviation, ideal, left and right estimation ratios, center size, area size, and linguistic sorting methods. Thus, multi-criteria decision-making methods are effective tools for solving complex decision problems, including alternatives with many qualitative and quantitative criteria. Qualitative criteria are often vague and difficult to evaluate precisely. Therefore, it is difficult to obtain overall evaluation results according to all criteria. Multi-criteria decision-making methods allow to define these criteria, calculate the total evaluation based on each decision-making expert opinion, and help decision-makers find the optimal alternative. The evaluation of alternatives in different types of problems is based on several criteria. In addition to the AHP and TOPSIS methods, the ANP method, which is designed to model more complex and dynamic environments under the influence of ever-changing external factors, allows qualitative values to be quantified, the PROMETHEE method, which is used to rank and select between conflicting criteria, the VIKOR method, which is designed for multi-criteria optimization of complex systems, and the simplified SAW method can be used. Utilization of any decision-making technique involving numerical analysis of alternatives requires steps such as identifying appropriate criteria and alternatives, determining the relative importance of the criteria and criteria evaluation of alternatives, and determining overall numerical values for ranking of the alternatives. Multiple-criteria decision-making is a sub-discipline of operations research that explicitly evaluates multiple conflicting criteria in decision making- both in daily life and in settings such as business, government and medicine. Conflicting criteria are typical in evaluating options: cost or price is usually one of the main criteria, and some measure of quality is typically another criterion, easily in conflict with the cost. In purchasing a car, cost, comfort, safety, and fuel economy may be some of the main criteria we consider – it is unusual that the cheapest car is the most comfortable and the safest one. In portfolio management, managers are interested in getting high returns while simultaneously reducing risks; however, the stocks that have the potential of bringing high returns typically carry high risk of losing money. In a service industry, customer satisfaction and the cost of providing service are fundamental conflicting criteria. Multi-criteria decision-making approaches are now very important and widely used to make practical and optimal decisions. All multicriteria decision-making methods require alternative selection based on different criteria by incorporating decision maker judgments and differ in the way decision-relevant information is combined and processed. In general, multi-criteria decision-making methods provide relative weights for different criteria. One of the key issues is to determine the most appropriate option from the multi-criteria decision-making methods, to define the alternatives and to evaluate them. Multi-criteria decision-making is based on modeling the decision process according to criteria and determination of alternative that maximizes the decision-maker's utility at the end of the process.

In this chapter, by conducting a comparative analysis of a series of MCDM methods, a possibility of applying these methods in different settings is determined. The analysis shows that it is needed to describe consistent preferences to more adequately model a decision maker's preferences under uncertainty. Such preferences rely on information that expresses a higher level of uncertainty. The use of preferences described by fuzzy logic and its extensions as type-2 fuzzy logic and Z-extension and development of related decision making methods are scientific problems to be solved. For example, for Z-valued preferenses and even for fuzzy preferences, consistency of preference knowledge is not suffisiently studied. In general, works devoted to eigensolutions of Z-valued preference matrices are scarce. The research devoted to solvingdynamic multistage fuzzy decision making and control problems is also at its initial stage. There is a large need to account for a high-level uncertainty to adequately describe preferences of decision maker. In view of this, analysis of preferences by using type-2 fuzzy sets is also problem to be solved. Despite that Delphi method is widely used for decision making, a method that allows to account for uncertainty of expert knowledge is not developped. The main features of Delphi method are anonimity of experts, repeated iterations of experts interviewing, statistical approach to analysis of structured feedback provided by experts etc. An analysis of evaluations-related information by classical approaches does not allow to account for uncertainty. As a result, researchers find it difficult to use classical Delphi method. To cope with these problems, it is needed to develop fuzzy Delphi method. Another problem to be considered is solving Z-linear equations that may be widely used in analysis of eigensolutions of Z-matrices describing preferences under bimodal infromation and the use of Leontief input-output model in economic problems. Z-number is a pair of fuzzy numbers. Its first component is fuzzy restriction on the value of random variable, and the second component is a measure of a related reliability described by a fuzzy number. Most existing methods for handling decision relevant information are developed for crisp settings. Some methods are proposed for fuzzy settings and its Z-valued extensions. Such methods allow to cope with important formal problems where objects of computations are Z-numbers. However, these methods are to be revisited.

Chapter 2 deals with information on fuzzy sets, classical fuzzy sets, type-2 fuzzy sets, Z-sets and Z-numbers. The main areas of research in fuzzy logic are artificial intelligence, intelligent systems, decision support systems, possibility theory, probability models and optimization methods, approximate reasoning, fuzzy forecasting, fuzzy management, evolutionary modeling etc.

In this chapter, the problem of aggregation of information, which is widely used in decision-making theories, is considered for the Z-environment. The solution of such problems requires the aggregation of fuzzy information collected from various sources. Formally, aggregation is the combination of numbers, linguistic terms, and so on, and common result is the process of 'merging'. Aggregation problems have been studied in many scientific fields, such as decision-making under uncertainty, multi-criteria decision-making, data collection. The nature of information is one of the important issues. In the case of linguistic information, the problem of aggregation is based on fuzzy sets. The aggregation process in various applications is complicated by fuzzy information. On the other hand, the linguistic description of information is partially reliable. This partial reliability is also fuzzy because constraints on possible probabilities are fuzzy. Z-numbers are widely used to describe this type of uncertainty. It is proposed to solve the problem of aggregation of discrete Z-numbers based on t-norm and t-conorm operators. T-norm and T-conorm operators are used successfully to process uncertainty in system analysis, decision analysis, control, modeling, and forecasting. Assume that Z_1 and Z_2 are Z -numbers that represent random and fuzzy information of variables. When building an algorithm for calculating the T-norm operator for Z -numbers, the first step is to calculate the T-norm of fuzzy numbers. After that, the probability distribution is determined using linear programming:

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \to b_j \tag{1}$$

$$\left.\begin{array}{c} v_1 + v_2 + \cdots + v_n = 1 \\ v_1, v_2, \cdots, v_n \geq 0 \end{array}\right\} \quad (2)$$

where , $c_k = \mu_{A_j}(x_{jk})$ and $v_k = p_j(x_{jk}), k = 1, \cdots, n_j$.

In the next step, construct the $p_{12} = p_1 {}^\circ p_2$ convolution for the random variable $X_{12} = T(X_1, X_2)$ as a result of the T-norm operator:

$$p_{12s}(x_{12}) = \sum_{x_{12}=T(x_1,x_2)} p_1(x_1) p_2(x_2), \forall x_{12} \in X_{12}; x_1 \in X_1, x_2 \in X_2 \quad (3)$$

The fuzzy sets of p_{12} convolutions are constructed in relation to the fuzzy sets of probability distributions p_j :

$$\mu_{p_{12}}(p_{12}) = \max_{p_{12}=p_1 \circ p_2} \min\left\{\mu_{p_1}(p_1), \mu_{p2}(p_1)\right\}$$

s.t.

$$\mu_{p_j}(p_j) = \mu_{B_j}\left(\sum_{k=1}^{n_j} \mu_{A_j}(x_{jk}) p_j(x_{jk})\right), j = 1, 2 \quad (4)$$

Then, a discrete B_{12} number is constructed for p_{12s}, which is related to the fuzzy value of $P(A_{12}) = \sum_{k=1}^{n} \mu_{A_{12}}(x_{12k}) p_{12}(x_{12k})$:

$$\mu_{B_{12}}(b_{12}) = \max(\mu_{p_{12}}(p_{12}))$$

s.t.

$$b_{12} = \sum_{k=1}^{n} \mu_{A_{12}}(x_{12k}) p_{12}(x_{12k})$$

As a result, $Z_{12} = T(Z_1, Z_2)$ is represented as $Z_{12} = (A_{12}, B_{12})$.

Let's look at the application of Z-T-norm and Z-T-conorm aggregation of expert opinions. Suppose three experts intend to make a common commercial decision. Due to the uncertainty and ambiguity, each expert expresses his opinion Q_i with Z-numbers:

$$A_{Q_1} = {0}/{1} + {0.3}/{1} + {0.4}/{2} + {0.7}/{3} + {1}/{4} + {0.8}/{5} + {0.6}/{6} + {0}/{7}$$

$$B_{Q_1} = {0}/{0.4} + {0.01}/{0.5} + {0.14}/{0.6} + {0.6}/{0.7} + {1}/{0.8} + {0.6}/{0.9}$$

$$A_{Q_2} = {0.2}/{0} + {0.4}/{1} + {1}/{2} + {0.4}/{3} + {0.2}/{4} + {0}/{5}$$

$$B_{Q_2} = {0}/{0.4} + {0.01}/{0.5} + {0.14}/{0.6} + {0.6}/{0.7} + {1}/{0.8} + {0.6}/{0/9}$$

$$A_{Q_3} = {0}/{1} + {0.5}/{2} + {0.6}/{3} + {0.7}/{4} + {1}/{5} + {0.7}/{6} + {0}/{7}$$

$$B_{Q_3} = {0}/{0.4} + {0.01}/{0.5} + {0.14}/{0.6} + {0.6}/{0.7} + {1}/{0.8} + {0.6}/{0.9}$$

In addition, the level of knowledge of experts is different. The level of knowledge of experts is also expressed in discrete $Z_{w_i} = (A_{w_1}, B_{w_1})$ numbers:

$$A_{w_1} = {0}/{0} + {0.6}/{1} + {0.8}/{2} + {1}/{3} + {0.7}/{4} + {0}/{5}$$

$$B_{w_1} = {0}/{0.4} + {0.01}/{0.5} + {0.14}/{0.6} + {0.6}/{0.7} + {1}/{0.8} + {0.6}/{0.9}$$

$$A_{w_2} = {0}/{1} + {0.4}/{2} + {0.6}/{3} + {1}/{4} + {0.8}/{5} + {0}/{6}$$

$$B_{w_2} = {0}/{0.4} + {0.01}/{0.5} + {0.14}/{0.6} + {0.6}/{0.7} + {1}/{0.8} + {0.6}/{0.9}$$

$$A_{w_3} = {0}/{2} + {0.4}/{3} + {0.6}/{4} + {1}/{5} + {0.8}/{6} + {0}/{7}$$

$$B_{w_3} = {0}/{0.4} + {0.01}/{0.5} + {0.14}/{0.6} + {0.6}/{0.7} + {1}/{0.8} + {0.6}/{0.9}$$

The problem is to determine the opinion of the expert group by using the Z-T-norm and Z-T-conorm operators-based aggregation:

$$Aggreg(Z_1, Z_2, Z_3) = Z$$

The problem is solved as follows:
In first step, the experts weighted opinions $Z_{Qw_i} = (A_{Qw_i}, B_{Qw_i})$ are determined by the T-norm operator The minimum function is used as the T-norm $T(Z_Q, Z_w) = \min(Z_Q, Z_w)$. The result is as follows:

$$A_{Qw_1} = 0.2/0 + 0.4/1 + 1/2 + 0.4/3 + 0.2/4 + 0/5$$
$$B_{Qw_1} = 0/0.80 + 0.01/0.82 + 0.14/0.85 + 0.6/0.87 + 1/0.89 + 0.6/0.9$$
$$A_{Qw_2} = 0/0 + 0.6/1 + 0.8/2 + 1/3 + 0.7/4 + 0/5$$
$$B_{Qw_2} = 0/0.64 + 0.01/0.68 + 0.14/0.73 + 0.6/0.77 + 1/0.82 + 0.6/0.90$$
$$A_{Qw_3} = 0/1 + 0.5/2 + 0.6/3 + 0.7/4 + 1/5 + 0.7/6 + 0/5$$
$$B_{Qw_3} = 0/0.77 + 0.01/0.79 + 0.14/0.81 + 0.6/0.84 + 1/0.86 + 0.6/0.9$$

In second step, aggregation of weighted opinions based on T-conorm is performed[1]. Initially, the T-conorm is calculated for $Z_{Qw_1} = (A_{Qw_1}, B_{Qw_1})$ and $Z_{Qw_2} = (A_{Qw_2}, B_{Qw_2})$ (belonging to the first and second experts). The maximum $S(Z_{Qw_1}, Z_{Qw_2}) = \max(Z_{Qw_1}, Z_{Qw_2})$ is used as the T-conorm. The result is as follows:

$$A_{12} = 0/1 + 0.5/2 + 0.6/3 + 0.7/4 + 1/5 + 0.7/6 + 0/5$$
$$B_{12} = 0/0.46 + 0.01/0.55 + 0.14/0.63 + 0.61/0.72 + 1/0.81 + 0.6/0.9$$

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[1] R.R. Aliev, O.H. Huseynov, K.R. Aliyeva Z-valued T-norm and T-conorm operators-based aggregation of partially reliable information, Procedia computer science,102, pp. 12-17,2016.

Finally, T-conorm $S(Z_{12}, Z_{Qw_3}) = (A, B)$ is calculated to obtain the final group estimate:

$$A = {}^{0}\!/_{0} + {}^{0.5}\!/_{2} + {}^{0.6}\!/_{3} + {}^{0.7}\!/_{4} + {}^{1}\!/_{5} + {}^{0.7}\!/_{6} + {}^{0}\!/_{7}$$
$$B = {}^{0}\!/_{0.73} + {}^{0.01}\!/_{0.76} + {}^{0.14}\!/_{0.78} + {}^{0.61}\!/_{0.82} + {}^{1}\!/_{0.85} + {}^{0.6}\!/_{0.91}$$

Thus, the aggregation of individual expert opinions based on T-norm and T-conorm operations results in the evaluation of the final expert group.

In this chapter, the classical Delphi method, which is widely used in decision-making and forecasting, has been extended to the fuzzy environment. The application of the fuzzy Delphi approach plays an important role in solving different complex decision problems. The computations in the Delphi method are based on the opinions of experts. For this reason, any errors, or inconsistencies in the assessment of expert opinions affect the results of the decision. The main goal of this method is to create and maintain a structured group opinion, so that the opinions of experts with complex problems, and uncertain results can be combined. The Delphi method is essentially a technology of structured aggregation of individual judgments. Although classical Delphi approaches use the mental abilities of experts for comparison, sometimes the numerical expression of knowledge does not fully reflect the human way of thinking. The main features of the Delphi method are the anonymity of experts, repeated surveys, structured feedback that is statistically analyzed for further evaluation, and so on. Analysis of the assessed data by statistical means does not allow to consider the problems of uncertainty. The classical Delphi method is based on numerical information, does not consider uncertainty, and therefore creates difficulties for researchers. To overcome these difficulties, the fuzzy Delphi method (FDM) has been proposed in this chapter. The fuzzy Delphi method consists of the following steps:

Step 1. Qualitative data are assigned, information is collected to prepare initial questionnaires.
Step 2. Triangular membership functions are defined for each parameter. For example, if the model includes various factors (political, economic, etc.), the appropriate triangular membership function must be defined for each factor. Parameters are defined by three values: minimum (minJ), maximum (maxJ) and, geometrical mean- $\sqrt[n]{\prod mean_j}$ where j is the number of samples, n is the number of experts.
Step 3. The values of the parameters found in step 2 is added to the next questionnaire and sent back to the experts.
Step 4. Statistical tests are performed to determine whether the membership functions are compatible with each other.
Step 5. If the membership functions match, a fuzzy number is defined to describe each linguistic term.

Chapter 3 deals with the application of the type-2 fuzzy sets-based method in multi-criteria decision-making and the concept of ideal solutions for type 2-fuzzy sets. Type-1 fuzzy sets have been used successfully to solve many problems under uncertainty. However, the use of this method is characterized by limitations in modeling an uncertainty environment and in considering the effects of uncertainty. The main reason for this is that the degrees of membership of the elements in fuzzy sets can be uncertain. To solve these problems, the concept of type-2 fuzzy set is used. The type-2 fuzzy sets allow for better modeling of uncertainties of the human opinion in decision making. Fuzzy type-1 multicriteria decision-making problems have been extended to fuzzy type-2 multicriteria decision-making problems. Type-2 fuzzy sets and systems generalize standard type-1 fuzzy sets and systems so that higher level of uncertainty can be considered. Thus, to solve the problem of uncertainty about the degree of membership function, Zadeh proposed more complex type-2 fuzzy sets, where the degree of membership of each element itself is fuzzy.
Type-2 fuzzy sets and systems are generalization so that standard type-1 fuzzy sets can be processed in a higher level of fuzzy environment.

Since the beginning of the use of fuzzy sets, the fact that the membership function of fuzzy sets is fuzzy has made it difficult to make better decisions under uncertainty. The answer to the question of what to do when the degree of membership is fuzzy was given in 1975 by the author of fuzzy sets, Lotfi A. Zadeh, when proposing more complex fuzzy sets, the first of which he called type-2 fuzzy sets. In 1975, Sambuc[2] introduced interval type fuzzy sets to solve problem when it was not possible to determine the exact values of the elements of the set. In fuzzy sets expressed in intervals, an interval is used to indicate the values of the degree to which the elements belong. This chapter deals with the decision-making process, which is based on the concept of type-2 fuzzy sets of decision-maker's preferences (preference knowledge) and is based on an ideal solution approach. Assume that $A=\{A_1,A_2,\cdots,A_n\}$ is the set of alternatives under consideration and $C=\{C_1,C_2,\cdots,C_m\}$ is the set of criteria by which they are characterized. Since the decision-maker's preference knowledge is described by type 2 fuzzy information, the C_j value of criteria for each A_i alternative is considered as follows:

$$A_{ij}=\{\tilde{a}_{i1},\tilde{a}_{i2},\cdots,\tilde{a}_{im}\},\quad A_{ij}=\{\tilde{a}_{ij}\} \tag{5}$$

where $\tilde{a}_{ij}$ is type-2 fuzzy number.

In many cases, the importance weights of criteria in decision-making for real-life problems are assigned by the decision-maker or experts. In this case, the weights of the criteria are more adequately expressed by type-2 fuzzy numbers:

$$W_j=\{\tilde{w}_j\},\quad j=1,..m \tag{6}$$

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[2] Sambuc, R. (1975). Fonctions ϕ-floues, application a l’aide au diagnostic en pathologie thyroidienne. These de Doctorat en Medicine: University of Marseille, France.

where $\tilde{w}_j$ is the value (weight) of the j-th criterion.

It is known that the decision-maker's preference is symbolically expressed in the form of a decision matrix:

$$D= \begin{array}{c|cccc} & C_1 & C_2 & \cdots & C_m \\ \hline A_1 & \tilde{a}_{11} & \tilde{a}_{12} & \cdots & \tilde{a}_{1m} \\ A_2 & \tilde{a}_{21} & \tilde{a}_{22} & \cdots & \tilde{a}_{2m} \\ & \cdots & \cdots & \cdots & \cdots \\ \cdots & & & & \\ A_n & \tilde{a}_{n1} & \tilde{a}_{n2} & \cdots & \tilde{a}_{nm} \end{array} \tag{7}$$

We will use the concept of ideal positive and ideal negative solutions to select the optimal alternative. The ideal positive solution is described as follows:

$$A_{positive}^{ideal} = \left\{ \tilde{a}_{positive_1}, \tilde{a}_{positive_2}, \cdots, \tilde{a}_{positive_m} \right\} \tag{8}$$

The negative ideal solution is described as follows:

$$A_{negative}^{ideal} = \left\{ \tilde{a}_{\mathrm{negative}_1}, \tilde{a}_{\mathrm{negative}_2}, \cdots, \tilde{a}_{\mathrm{negative}_m} \right\} \tag{9}$$

Choosing the optimal solution involves the following steps:
1. Construct a weighted matrix using (6) and (7).
2. Calculate the normalized Euclidean distance between (8) and (9) and each alternative using type-2 fuzzy arithmetic operations.
3. Calculate the degree of relative similarity of each alternative to the ideal solution.
4. Selection of the optimal alternative as one with the highest degree of similarity.

Chapter 4 deals with the problem of multi-stage decision-making based on fuzzy dynamic programming, the development of a solution method and a solution algorithm. Dynamic programming is one of the most powerful optimization methods. Rational decision-making, selection of the best options, optimal control tasks are solved by specialists of different profiles. Dynamic programming is extremely important due to the simplicity and clarity of its main principle - the principle of optimality. Application of the principle of optimality in multi-stage, discrete processes leads to recursive-functional equations in relation to the optimal value of the quality criterion. The obtained equations allow to write the optimal controls for the initial problem in sequence. The advantage here is that the task of calculating control for the whole process is divided into different simple tasks of calculating control for different stages of the process. Assume that $X=\{X_1, X_2, \cdots, X_n\}$ and $U=\{U_1, U_2, \cdots, U_m\}$ are the vectors of the disturbance and control parameters, and Y is the fuzzy output parameter of the object. N fuzzy data of disturbance, control, and output variables because of object observation can be represented as,

t	X_1	X_2	$\cdots$	X_n	U_1	U_2	$\cdots$	U_m	Y
0	x_{10}	x_{20}	$\cdots$	x_{n0}	u_{10}	u_{20}	$\cdots$	u_{m0}	Y_0
1	x_{11}	x_{21}	$\cdots$	x_{n1}	u_{11}	u_{21}	$\cdots$	u_{m1}	Y_1
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
N	x_{1_N}	x_{2_N}	$\cdots$	x_{n_N}	u_{1_N}	u_{2_N}	$\cdots$	u_{m_N}	Y_N

First, let's look at fuzzy identification. Suppose an object is expressed by a fuzzy difference equation:

$$
\begin{aligned}
Y_{k+1} &= A_0 \oplus A_1 \otimes x_{1k} \oplus A_2 \otimes x_{2k} \oplus \cdots \oplus A_n \otimes x_{nk} \oplus A_{n+1} \otimes y_k \oplus \\
&\oplus B_1 \otimes u_{1k} \oplus B_2 \otimes u_{2k} \oplus \cdots \oplus B_m \otimes u_{mk}
\end{aligned}
\tag{10}
$$

$A=(A_o, A_1, \ldots, A_{n+1})$ and $B=(B_o, B_1, \ldots, B_m)$ are vectors of fuzzy coefficients. The problem of identifying the object expressed by equation (10) is aimed at estimating the coefficients $A_i\,(i=\overline{0,n+1})$ and $B=(B_o, B_1, \ldots, B_m)$. To evaluate the quality of the model, we will compare the current value of the output quantity Y with the value obtained from equation (10). The result is the following system of fuzzy equations:

$$\begin{cases} A_0 \oplus A_1 \otimes x_{10} \oplus \cdots \oplus A_n \otimes x_{n0} \oplus A_{n+1} \otimes y_0 \oplus B_1 \otimes u_{10} \oplus \cdots \oplus B_m \otimes u_{m0} = \tilde{Y}_1 \\ A_0 \oplus A_1 \otimes x_{11} \oplus \cdots \oplus A_n \otimes x_{n1} \oplus A_{n+1} \otimes y_1 \oplus B_1 \otimes u_{11} \oplus \cdots \oplus B_m \otimes u_{m1} = \tilde{Y}_2 \\ A_0 \oplus A_1 \otimes x_{1,N-1} \oplus \cdots \oplus A_n \otimes x_{n,N-1} \oplus A_{n+1} \otimes y_{N-1} \oplus B_1 \otimes u_{1,N-1} \oplus \cdots \oplus B_m \otimes u_{m,N-1} = \tilde{Y}_N \end{cases} \quad (11)$$

The main task is to determine the values of the coefficients $A_i\,(i=\overline{0,n+1})$ and $B_j\,(j=\overline{1,m})$ in such a way that equality (11) is satsified with a certain accuracy. In practice, it is necessary to determine the values of $A_i\,(i=\overline{0,n+1})$ and $B_j\,(j=\overline{1,m})$ in such a way that the difference between the left and right sides of equation (10) is minimal.

$$\begin{aligned} J = \{ & (A_0 \oplus A_1 \otimes x_{10} \oplus \cdots \oplus A_n \otimes x_{n0} \oplus A_{n+1} \otimes y_0 \oplus B_1 \otimes u_{10} \oplus \cdots \oplus B_m \otimes u_{m0} - Y)^2 \oplus \\ & \oplus (A_0 \oplus A_1 \otimes x_{11} \oplus \cdots \oplus A_n \otimes x_{n1} \oplus A_{n+1} \otimes y_1 \oplus B_1 \otimes u_{11} \oplus \cdots \oplus B_m \otimes u_{m1} - Y_2)^2 \oplus \\ & \oplus (A_0 \oplus A_1 \otimes x_{1,N-1} \oplus \cdots \oplus A_n \otimes x_{n,N-1} \oplus A_{n+1} \otimes y_{N-1} \oplus B_1 \otimes u_{1,N-1} \oplus \cdots \oplus B_m \otimes u_{m,N-1} - Y_N)^2 \} / N \end{aligned} \quad (12)$$

It is necessary to note two cases that arise in the calculation of (12). In the first case, the coefficients $A_i\,(i=\overline{0,n+1})$ and $B_j\,(j=\overline{1,m})$ are fuzzy numbers. In this case, addition, multiplication, subtraction operations are carried out according to the following formulas:

$$\mu_{A*X}\left(A*X\right)=\mu_1/A_1*\mu_1/x_1+\mu_2/A_2*\mu_2/x_2+\cdots+\mu_p/A_p*\mu_p x_p,$$
$$\mu_A(A)=\mu_1/A_1+\mu_2/A_2+\cdots+\mu_p/A_p$$
$$\mu_X(X)=\mu_1/X_1+\mu_2/X_2+\cdots+\mu_p/X_p$$

Otherwise, the coefficients $A_i(i=\overline{0,n+1})$ and $B_j(j=\overline{1,m})$ are ordinary numbers, and algebraic operations on ordinary and fuzzy numbers are performed as follows:

$$\mu_{A*X}\left(A*X\right)=\mu_1/A_1*x_1+\mu_2/A_2*x_2+\cdots+\mu_p/A_p*x_p$$

In the second case, the function (12) is a function of ordinary fuzzy variables. To minimize this function, it is decomposed into α-cut, and the optimization method is applied to each level of α. The main mathematical technic used to solve multi-stage decision-making problems is the dynamic programming method. In dynamic systems, the situation, solutions, transition functions, transformation, control, and result can be fuzzy. In other words, the system may exist in a completely or partially fuzzy environment. In such cases, the application of the dynamic programming method for the optimal control of such systems was proposed by Bellman and Zadeh[3]:

$$\begin{cases}\mu_{G^{N-1}}(X_{N-\upsilon})=\max\limits_{u_{N-\upsilon}}\min((\mu_{u_{N-\upsilon}}(u_{N-\upsilon}),\mu_{G^{N-\upsilon+1}}(X_{N-\upsilon+1})) \\ X_{N-\upsilon+1}=f\left(X_{N-\upsilon},u_{N-\upsilon}\right),\quad \upsilon=\overline{1,N}\end{cases}\tag{13}$$

The recurrent equation for solution (13) is expressed as follows: if γ is a constant and the function g is any function of u_{N-1}, then the following identity is true:

$$\max_{u_{N-1}}\left(\gamma\wedge g\left(u_{N-1}\right)\right)=\gamma\wedge\max_{u_{N-1}}g\left(u_{N-1}\right)$$

Given this feature, (13) can be expressed as follows:

[3] Bellman R. E and Zadeh L. A 1970 Decision-Making in a Fuzzy Environment Management, pp.141-164.

$$\mu_D(U_0^*,\cdots,U_{N-1}^*) = \max_{u_0,\cdots,u_{N-2}} \min\left\{\mu_{u_0}(u_0),\cdots,\mu_{u_{N-2}}(u_{N-2}),\mu_{G^{N-1}}(X_{N-1})\right\}$$

where

$$\mu_{G^{N-1}}(X_{N-1}) = \left\{\max_{u_{N-1}} \min \mu_{u_{N-1}}(u_{N-1}),\mu_{G^N}(f(X_{N-1},u_{N-1})\right\}$$

$\mu_{G^{N-1}}\left(x_{N-1}\right)$ is taken as a membership function obtained in the intermediate stage $N-1$ with the help of fuzzy goal G^N. By repeating the iteration in the opposite direction (similar to the method of dynamic programming for classical systems), we can write the following system of recurrent equations, which is a fuzzy analogue of the recurrent equations of dynamic programming:

$$\begin{cases} \mu_{G^{N-1}}(X_{N-\upsilon}) = \left\{\max_{u_{N-\upsilon}} \min \mu_{u_{N-\upsilon}}(u_{N-\upsilon}),\mu_{G^{N-\upsilon+1}}(X_{N-\upsilon+1})\right\} \\ X_{N-\upsilon+1} = f\left(X_{N-\upsilon},u_{N-\upsilon}\right), \quad \upsilon = \overline{1,N} \end{cases} \tag{14}$$

The solution of the system of equations (14) is carried out in the opposite direction.

Chapter 5 deals with some key issues in Z-information based decision-making. One of the typical problems in mathematics is the solution of linear equations. Linear equations have a wide range of applications in ecology, economics, production, and everyday life. This chapter deals with exact and approximate solutions of linear equations whose parameters and variables are expressed in Z-numbers. Consider two types of equations with Z-numbers:

$$Z_1 + Z_X = Z_2 \tag{15}$$

$$Z_1 \cdot Z_X = Z_2 \tag{16}$$

Unlike real-valued equations, there is not always a solution to the Z-equations, and if there is a Hukuhara difference, there is a solution to such equations:

$$Z_{21} = Z_2 -_h Z_1 = (A_{21}, B_{21})$$

$$Z_X = Z_{21}$$

Hukuhara difference is based on certain conditions. However, these conditions are very restrictive and the calculation of the Hukuhara difference is itself complicated. Thus, in many cases, the approximate $Z'_X = (A'_{21}, B'_{21})$ solution of this equation can be found as a result of the standard difference:

$$Z'_X = Z_2 - Z_1.$$

where $A_{21} \subseteq A'_{21},\ B_{21} \subseteq B'_{21}$.

In other words, the approximate solution Z'_X contains the exact solution Z_X, but there is uncertainty in the former. Similarly, in the case of the Hukuhara division, there is a solution to the second equation:

$$Z_{21} = Z_2 /_h Z_1 = (A_{21}, B_{21})$$

$$Z_X = Z_{21}.$$

In equation (16), similarly, we can use the standard division operation as an approximate solution:

$$Z'_X = Z_2 / Z_1 = (A'_{21}, B'_{21}),$$

where,

$$A_{21} \subseteq A'_{21},\ B_{21} \subseteq B'_{21}$$

Linear equations (15) and (16) are special cases of the following equation:

$$Z_1 Z_X + Z_2 = Z_3. \qquad (17)$$

The exact solution of this equation is:

$$Z_X = \left(Z_3 -_h Z_2\right) /_h Z_1.$$

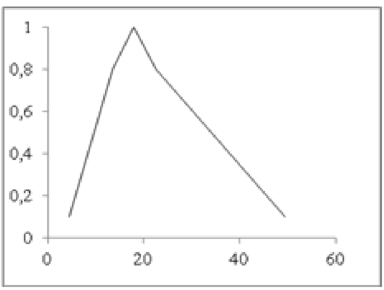
As an approximate solution, the following can be used:

$$Z_X = \left(Z_3 - Z_2\right) / Z_1.$$

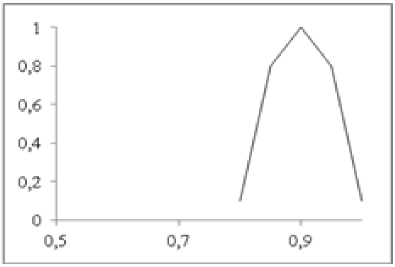
Assume that the numbers $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ in linear equation (15) are expressed as follows. (Figures 1, 2):

$A_1 = 0/4.5 + 0.8/13.5 + 1/18 + 0.8/22.5 + 0.1/49.5$,
$B_1 = 0.1/0.8 + 0.8/0.85 + 1/0.9 + 0.8/0.9 + 0.8/0.95$;
$A_2 = 0.1/9 + 0.1/18 + 0.1/22.5 + 0.1/27 + 0.1/31.5$
$+ 0.8/40.5 + 0.8/45 + 0.8/49.5 + 1/54 + 0.8/58.5 +$
$0.8/63 + 0.1/67.5 + 0.1/72 + 0.1/76.5 + 0.1/85.5 +$
$+ 0.1/90 + 0.1/99$,

$B_2 = 0.1/0.589 + 0.1/0.594 + 0.1/0.616 + 0.1/0.623 +$
$0.8/0.664 + 0.8/0.686 + 0.8/0.703 + 0.8/0.706 +$
$0.8/0.734 + 0.8/0.783 + 0.8/0.784 + 1/0.795 +$
$+ 0.8/0.799 + 0.8/0.800 + 0.8/0.835 + 0.8/0.850 +$
$0.8/0.862 + 0.8/0.875 + 0.1/0.900 + 0.1/0.950.$



a) A_1



b) B_1

Figure 1. Z-number $Z_1 = (A_1, B_1)$.

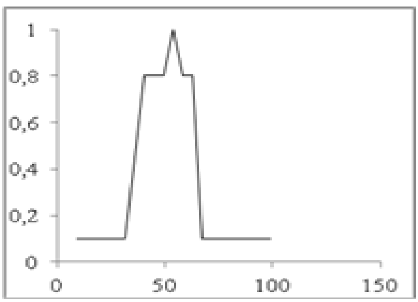
Checking for the solution of equation (15) means checking whether there is a difference in Hukuhara $Z_{21} = Z_2 -_h Z_1$. For this purpose, we use the following approach.

So, we must first check whether $A_{21} = A_2 -_h A_1$ exists:

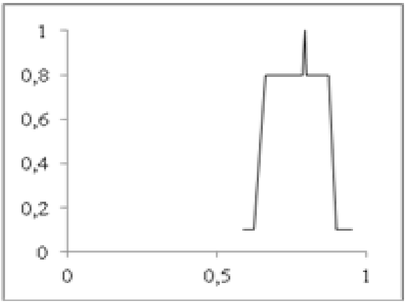
$$A_1^{\alpha=0.1} + \{4.5, 27, 36, 40.5, 49.5\} = A_2^{\alpha=0.1},$$
$$A_1^{\alpha=0.8} + \{27, 36, 40.5\} = A_2^{\alpha=0.8},$$
$$A_1^{\alpha=1} + \{36\} = A_2^{\alpha=1}.$$

In this case, the Hukuhara difference $A_{21} = A_2 -_h A_1$ is as follows:

$$A_{21} = A_1 -_h A_2 = {}^{0.1}\!/_{4.5} + {}^{0.8}\!/_{27} + {}^{1}\!/_{36} + {}^{0.8}\!/_{40.5} + {}^{0.1}\!/_{49.5}$$



a) A_{12}



b) B_{12}

Figure 2. Z-number - $Z_2 = (A_2, B_2)$

Now we need to check if there exists such an B_{12}:

$$Z_1 = Z_2 + Z_{21} = (A_2, B_2) + (A_{21}, B_{21}) = (A_1, B_1) ,$$

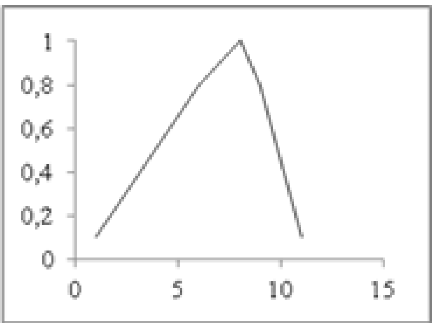
So,

$$Z_{21} = Z_2 -_h Z_1 = (A_2, B_2) -_h (A_1, B_1) = (A_{21}, B_{21}) .$$

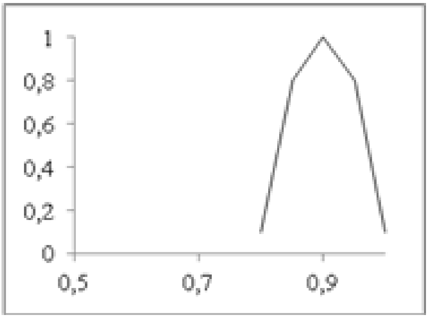
Using this approach, we see that such an B_{21} is as follows:

$$B_{21} = 0.1/0.6 + 0.8/0.7 + 1/0.8 + 0.8/0.9 + 0.1/1.$$

Thus, there is a solution Z_X of equation (15), and the resulting Hukuhara difference is $Z_X = (A_{12}, B_{12})$, and is expressed as in Figure 3:



a) A_{12}



b) B_{12}

Figure 3. Z-number - $\boldsymbol{Z_x = \left(A_{12}, B_{12}\right)}$

A method of constructing a regression model with variables and coefficients expressed as Z-numbers has been proposed to model decision-making and control objects characterized by fuzzy and probabilistic uncertainties. The advantage of the method is the reduction of complexity in terms of calculation. Consider a multi-input and single-output process characterized by Z-number information. The expression of variables $X_1, X_2, \cdots, X_N$ and the output parameter Y is in the form Z-numbers $Z_{X_i,k}, i=1,\cdots,N$ and $Z_{Y,k}, k=1,\cdots,K$ (K- is the number of observations). For simplicity, consider a Z-linear regression model of the process under study. The linear regression function of the variable Z is constructed as follows:

$$Z_{Y^M}(Z_{X_1}, Z_{X_2}, ..., Z_{X_N}) = Z_{C_0} + \sum_{i=1}^{N} Z_{C_i} Z_{X_i} \tag{18}$$

Using the bandwith method:

$$Z_{C_i}\times Z_{X_i}=((A_{C_i}^{Bandwith}\times A_{X_i}^{Bandwith}),(B_{C_i}\times B_{X_i})) \tag{19}$$

$$Z_{Y^M}=Z_{C_0}+Z_{C_1}\times Z_{X_1}+\ldots+Z_{C_N}\times Z_{X_N}=$$
$$(A_{C_0}^{Bandwith}+(A_{C_1}^{Bandwith}\times A_{X_1}^{Bandwith})+\ldots+(A_{C_N}^{Bandwith}\times A_{X_N}^{Bandwith}),$$
$$B_{C_0}\times(B_{C_1}\times B_{X_1})\times\ldots\times(B_{C_N}\times B_{X_N}))$$

The construction of the regression model (18) involves the calculation of a large number of Z-numbers. In this case, the issue of computational complexity becomes very relevant. We propose to use the bandwith method for equation (18) to obtain a trade-off between adequacy and computational complexity. The construction of Z_{Y^M} requires to determine the Z-valued coefficients Z_{C_i} on the basis of data on Z-inputs $Z_{X_i,k}, i=1,\cdots,N$ and Z-outputs Z_Y in such a way as to minimize the error:

$$\sum_{k=1}^{K}\left|Z_{Y,k}^{M}-Z_{Y,k}\right|\to\min \tag{20}$$

The "-" sign represents the standard output operation of the Z-numbers, the $|\;|$-sign represents the absolute value of the Z-number, and $\sum$ represents the sum of the Z-numbers. The regression model (18) describes the general case when all variables and coefficients are expressed in Z-numbers. In this case, the use of classical techniques, such as gradient-based methods, is not suitable for constructing a regression model due to the complexity of the description of the derivative of the Z-number function. In this chapter, a problem of computation of criteria weights is solved by using consistency index of Z-matrix describing decision maker’s preferences and further determination of eigenvector and eigenvalue of this matrix. A continuous Z-number is a pair $Z=\left(A,B\right)$, where A is a continuous fuzzy number, and B is also a continuous fuzzy number with membership functions $\mu_B:\left[0,1\right]\to\left[0,1\right]$. Decision problems in economics, management, business, and other fields are described as linear algebra problems. In particular, MCDM problems, one needs to account for consistency of pairwise comparisons of altarnatives. The classical methods of linear algebra are well developed, but sometimes it is necessary to deal with uncertainty inherent in information related to real-world problems[4]. A number of problems are based on the handling various types of uncertainties in linear algebraic models. Studies on probability uncertainty in such models are based on the theory of random matrices[5]. A random matrix consists of elements with random variables, and many studies have been devoted to the calculation of eigenvalues and eigenvectors of such matrices. Currently, random matrices are used in machine learning, signal processing, computer graphics, econometrics and other fields. A number of works are devoted to fuzzy linear algebra. The problems in the real world are characterized by a combination of fuzzy and probabilistic uncertainties. One of the methodologies developed to handle such uncertainties is dealing with Z-numbers[6]. There are a number of studies on the theory and application of Z numbers[7].

Let's look at the consistency of the Z-matrix. Suppose that the preference of the decision-maker is formed as the following matrix:

$$Z = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & \cdots & Z_{1n} \\ Z_{21} & Z_{22} & Z_{23} & \cdots & Z_{21} \\ Z_{11} & Z_{11} & Z_{11} & \cdots & Z_{11} \\ \vdots & \vdots & \vdots & & \vdots \\ Z_{n1} & Z_{n2} & Z_{n3} & \cdots & Z_{nn} \end{bmatrix} \tag{21}$$

Suppose that this matrix does not meet the conditions of consistency. An updated, consistent Z' matrix needs to be defined.

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$$Z^{'}=\begin{bmatrix} Z^{'}_{11} & Z^{'}_{12} & Z^{'}_{13} & \cdots & Z^{'}_{1n} \\ Z^{'}_{21} & Z^{'}_{22} & Z^{'}_{23} & \cdots & Z^{'}_{2n} \\ Z^{'}_{n1} & Z^{'}_{n2} & Z^{'}_{n3} & \cdots & Z^{'}_{nn} \end{bmatrix} \tag{22}$$

The following optimization problem is used to solve this problem:

$$J=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|Z_{ij}-Z_{ij}{'}\right| \to \min \tag{23}$$

$$Z_{ij}{'} \otimes Z_{ji}{'} \approx Z_{(1)} \quad i=\overline{1,n}\ ,\ j=\overline{1,n} \tag{24}$$

$$Z_{ij}{'} \otimes Z_{jk}{'} \approx Z_{ik}{'} \text{ for all i, j, k} \tag{25}$$

$$Z_{ij}{'} \geq Z_{ij(0)} \tag{26}$$

Constraint (24) describes reciprocity conditions. Constraint (21) describes transitivity conditions. For solving problem (23)-(26), one may use Euclid distance or another one. Consider computation of distance between $Z_{ij}=\left(A_{ij},B_{ij}\right)$ və $Z^{'}_{ij}=\left(A^{'}_{ij},B^{'}_{ij}\right)$. The distance between A_{ij} and $A^{'}_{ij}$ is computed as:

$$D_{A}^{\alpha}(A_{ij},A^{'}_{ij})=\left|\frac{A_{ijL}(\alpha)+A_{ijR}(\alpha)}{2}-\frac{A^{'}_{ijL}(\alpha)+A^{'}_{ijR}(\alpha)}{2}\right| \tag{27}$$

The distance between B_{ij} and $B^{'}_{ij}$ is based on the formulas:

$$D_{B}^{1\alpha}(B_{ij},B^{'}_{ij})=\left|\frac{B_{ijL}(\alpha)+B_{ijR}(\alpha)}{2}-\frac{B^{'}_{ijL}(\alpha)+B^{'}_{ijR}(\alpha)}{2}\right| \tag{28}$$

$$D_{B}^{2\alpha}(B_{ij},B^{'}_{ij})=\left|\left(B_{ijR}(\alpha)+B_{ijL}(\alpha)\right)-\left(B^{'}_{ijR}(\alpha)+B^{'}_{ijL}(\alpha)\right)\right| \tag{29}$$

Finally, distance between B_{ij} and $B^{'}_{ij}$ is computed as

$$D^{total}(Z_{ij},Z^{'}_{ij})=\left[\beta D_{A}^{\alpha}\left(A_{ij},A^{'}_{ij}\right)\right]+(1-\beta)\times D_{P}\left(P_{Z_{ij}},P^{'}_{Z_{ij}}\right) \tag{30}$$

The solution to problem (23)-(26) provides new consistent preferences.

Let us have 3×3 Z-matrix with Z-numbers described by triangular fuzzy numbers-based components:

$$(Z_{ij})=\begin{pmatrix} ((0.93,0.95,1),(0.88,0.94,1)) & ((2,2.5,3),(0.56,0.78,1)) & ((1.5,2,2.25),(0.7,0.8,0.9)) \\ ((0.33,0.4,0.5),(0.67,0.78,0.97)) & ((0.93,0.95,1),(0.88,0.94,0.97) & ((1,1.5,2),(0.65,0.75,0.85)) \\ ((0.4,0.5,0.67),(0.65,0.75,0.85)) & ((0.5,0.67,1),(0.7,0.72,0.75)) & ((0.93,0.95,1),(0.88,0.94,0.97) \end{pmatrix}$$

If Z-eigenvectors $(Z_{Ys})=((A_{Ys1},p_{Ys1}),\ldots,(A_{Ys3},p_{Ys3}))$ of the matrix are described as

$$(Z_{Y1})=\begin{pmatrix} ((0.84,0.844,0.844),(0.7,\ 0.5)) \\ ((0.41,0.416,0.4253),(0.3,\ 0.3)) \\ ((0.345,0.349,0.36),(0.28,\ 0.2)) \end{pmatrix},$$

$$(Z_{Y2})=\begin{pmatrix} ((0.83,0.833,0.853),(0.51,\ 0.6)) \\ ((-0.27,-0.22,-0.17),(-0.13+0.23\mathrm{i},\ 0.32)) \\ ((-0.22,-0.16,-0.1),(-0.11-0.19\mathrm{i},\ 0.26)) \end{pmatrix},$$

$$(Z_{Y3})=\begin{pmatrix} ((0.83,0.833,0.853),(0.51,\ 0.6)) \\ ((-0.27,-0.22,-0.17),(-0.13-0.23\mathrm{i},\ 0.3)) \\ ((-0.22,-0.16,-0.1),(-0.11+0.19\mathrm{i},\ 0.3)) \end{pmatrix}$$

then, matrix can be considered as consistent, i.e. (23)-(26) conditions are satisftied.

Chapter 6 deals with the application of fuzzy and Z decision-making tools to various sectors of the economy and business.

Application of the fuzzy AHP-TOPSIS hybrid method to the advertising media selection. Advertising media selection is the process of choosing the most efficient media for an advertising campaign. The choice of media advertising for the company requires an in-depth analysis of the existing media. There are four companies (A, B, C, D), three types of advertising media (TV advertising, Radio advertising, Internet advertising) and nine sub-criteria (design work, visual benefits, simplicity (for television), distinction, diction, general appearance (for radio), size, shape, location (for internet)) are evaluated and prioritized. First, a decision matrix of triangular fuzzy numbers is constructed. Pairwise comparison of decision criteria are created in pairs and supplemented with the relative scores. The expression of linguistic terms in fuzzy numbers is given in table 1**.**

Table 1. Expression of linguistic terms in fuzzy numbers

Importance degree	Triangular fuzzy scale
Very good	(3, 5, 7)
Good	(1, 3, 5)
Moderate	(1, 1, 1)
Poor	(1/5, 1/3, 1)

First, the pairwise comparison matrix of the criteria is constructed as in table 2.

Table 2. Pairwise comparison matrix

	Television advertising	Radio advertising	Internet advertising
Television advertising	(1, 1, 1)	(1, 3, 5)	(3, 5, 7)
Radio advertising	(1/5,1/3,1)	(1, 1, 1)	(1, 3, 5)
Internet advertising	(1/7,1/5,1/3)	(1/5,1/3,1)	(1, 1, 1)

In this case, the weight of the criteria is calculated as follows:

$$w = \frac{G_i}{G_T} = \frac{(l_i, m_i, u_i)}{(\sum_{i=1}^{k} l_i, \sum_{i=1}^{k} m_i, \sum_{i=1}^{k} u_i)} = \left(\frac{l_i}{\sum_{i=1}^{k} u_i}, \frac{m_i}{\sum_{i=1}^{k} m_i}, \frac{u_i}{\sum_{i=1}^{k} l_i} \right) \tag{31}$$

where

$$\tilde{G}_i = (l_i, m_i, u_i)$$

$$l_i = (l_{i1} \otimes l_{i2} \otimes \ldots \otimes l_{ik})^{\frac{1}{k}}, \quad m_i = (m_{i1} \otimes m_{i2} \otimes \ldots \otimes m_{ik})^{\frac{1}{k}},$$

$$u_i = (u_{i1} \otimes u_{i2} \otimes \ldots \otimes u_{ik})^{\frac{1}{k}}$$

$$G_T = \sum_{i=1}^{k} l_i, \sum_{i=1}^{k} m_i, \sum_{i=1}^{k} u_i \qquad (32)$$

$$l_{i1} = (1 \otimes 0.25 \otimes 0.14)^{1/3} = 0.17; \quad l_{i2} = (1 \otimes 1 \otimes 0.2)^{1/3} = 0.58; \quad l_{i3} = (3 \otimes 1 \otimes 1)^{1/3} = 1.43$$

$$\sum_{i=1}^{k} l_i = 2.18$$

$$m_{i1} = (1 \otimes 0.25 \otimes 0.14)^{1/3} = 0.58; \quad m_{i2} = (3 \otimes 1 \otimes 0.33)^{1/3} = 0.96; \quad m_{i3} = (5 \otimes 1 \otimes 1)^{1/3} = 1.7$$

$$\sum_{i=1}^{k} m_i = 3.24$$

$$u_{i1} = (3 \otimes 1 \otimes 1)^{1/3} = 1.43; \quad u_{i2} = (5 \otimes 3 \otimes 1)^{1/3} = 2.4; \quad u_{i3} = (7 \otimes 5 \otimes 1)^{1/3} = 3.23$$

$$\sum_{i=1}^{k} u_i = 7.06$$

$$w_{televis.} = \left[\frac{l_{iT}}{\sum_{i=1}^{3} u_i} = \frac{0.17}{7.06} = 0.02, \frac{m_{iT}}{\sum_{i=1}^{3} m_i} = \frac{0.58}{3.24} = 0.18, \frac{u_{iT}}{\sum_{i=1}^{3} l_i} = \frac{1.43}{2.18} = 0.65 \right]$$

$$w_{televis.} = [0.02, 0.18, 0.65], \quad w_{radio.} = [0.08, 0.29, 1.1], \quad w_{internet} = [0.2, 0.52, 1.5]$$

A pairwise comparison matrix is also constructed for each sub-criterion for each advertising factor (table 3) and fuzzy weights are determined.

Table 3. Pairwise comparison matrix and fuzzy weights for the television advertising criterion related sub-factors

Television advertising	Design	Visuality	Simplicity
Design	(1, 1, 1)	(3, 5, 7)	(1, 3, 5)

Visuality	(1/7,1/5,1/3)	(1, 1, 1)	(3, 5, 7)
Simplicity	(1/5,1/3,1)	(1/7,1/5,1/3)	(1, 1, 1)

The fuzzy weights of the sub-criteria for a television commercial are calculated as follows:

$$w_{design} = [1.66, 3, 4.33]$$

$$w_{visual} = [1.38, 2.06, 2.77]$$

$$w_{simplicity} = [0.44, 0.51, 0.77]$$

In next step we determine pairwise comparison matrix and fuzzy weights for different advertising factors related sub-factors.

Individual criterion	Fuzzy sub-criterion	Global fuzzy criterion
Television advertising [0.02,0.18,0.65]	[1.66,3,4.33] [1.38,2.06,2.77] [0.44,0.51,0.77]	(0.033,0.54,2.81) (0.027,0.37,1.80) (0.009,0.09,0.5)
Radio advertising [0.08,0.29,1.1]	[1.66,3,4.33] [1.4,2.1,1,3] [0.42,0.46,0.55]	(0.13,0.99,4.76) (0.11,0.61,3.3) (0.03,0.13,0.6)
Internet advertising [0.2,0.52,1.5]	[2.33,3.66,5] [0.71,1.4,2.11] [0.22,0.51,0.77]	(0.46,1.9,7.5) (0.14,0.73,3.16) (0.04,0.26,1.15)

Next, we identify global fuzzy priorities for different alternatives. Global fuzzy priorities for alternatives are listed below:

	Global fuzzy value	Triangular fuzzy value	Weight

A	[1.66,3,4.33]	(1, 3, 5)	(1.66,9,21.65)
	[1.38,2.06,2.77]	(3, 5, 7)	(4.14,10.3,19.39)
	[0.44,0.51,0.77]	(1/5, 1/3, 1)	(0.09,0.19,0.77)
	[1.66,3,4.33]	(1/2, 1, 3/2)	(0.63,3,6.5)
	[1.4,2.1,1.3]	(1, 3, 5)	(1.4,6.3,6.5)
	[0.42,0.46,0.55]	(1, 3, 5)	(0.42,1.38,2.75)
	[2.33,3.66,5]	(1/2, 1, 3/2)	(1.16,3.66,7.5)
	[0.71,1.4,2.11]	(1/7, 1/5, 1/3)	(0.1,0.28,0.69)
	[0.22,0.51,0.77]	(1/2, 1, 3/2)	(0.11,0.51,0.76)
B	[1.66,3,4.33]	(1/2, 1, 3/2)	(0.83,3,6.5)
	[1.38,2.06,2.77]	(1/2, 1, 3/2)	(0.69,2.06,4.15)
	[0.44,0.51,0.77]	(1, 3, 5)	(0.44,1.53,3.85)
	[1.66,3,4.33]	(1/5, 1/3, 1)	(0.33,1,4.33)
	[1.4,2.1,1.3]	(1, 3, 5)	(1.4,6.3,6.5)
	[0.42,0.46,0.55]	(3, 5, 7)	(0.17,2.3,3.85)
	[2.33,3.66,5]	(1/7, 1/5, 1/3)	(0.33,0.73,1.65)
	[0.71,1.4,2.11]	(1/2, 1, 3/2)	(0.35,1.4,3.16)
	[0.22,0.51,0.77]	(1, 3, 5)	(0.22,1.53,3.85)
	[1.66,3,4.33]	(1, 3, 5)	(1.66,9,21.65)
C	[1.38,2.06,2.77]	(1/2, 1, 3/2)	(0.69,2.06,4,15)
	[0.44,0.51,0.77]	(1/5, 1/3, 1)	(0.09,0.16,0.77)
	[1.66,3,4.33]	(1/5, 1/3, 1)	(0.33,1,4.33)
	[1.4,2.1,1.3]	(1/7, 1/5, 1/3)	(0.2,0.42,0.44)
	[0.42,0.46,0.55]	(3, 5, 7)	(0.17,2.3,3.85)
	[2.33,3.66,5]	(1/5, 1/3, 1)	(0.46,1,2,5)
	[0.71,1.4,2.11]	(1, 3, 5)	(0.71,4.26,10.5)
	[0.22,0.51,0.77]	(1, 3, 5)	(0.22,1.53,3.85)

D	[1.66,3,4.33]	(1, 3, 5)	(1.66,9,21.65)
	[1.38,2.06,2.77]	(3, 5, 7)	(4.14,10.3,19.39)
	[0.44,0.51,0.77]	(1/5, 1/3, 1)	(0.09,0.16,0.77)
	[1.66,3,4.33]	(1/2, 1, 3/2)	(0.83,3.6.5)
	[1.4,2.1,1,3]	(1, 3, 5)	(1.4,6.3,6.5)
	[0.42,0.46,0.55]	(1/2, 1, 3/2)	(0.21,0.46,0.82)
	[2.33,3.66,5]	(1/7, 1/5, 1/3)	(0.33,0.73,1.65)
	[0.71,1.4,2.11]	(1, 3, 5)	(0.71,4.26,10.5)
	[0.22,0.51,0.77]	(3, 5, 7)	(0.66,2.55,5.39)

Next, we rank the alternatives**.** The best alternative for advertising can be determined by using preference rank order of C_{i^*}. The best company is the one that has the smallest distance to the ideal solution. The relationship of alternatives points out that any alternative which has the smallest distance to the ideal solution is guaranteed to have the longest distance to the negative-ideal solution. The relative closeness of an alternative A_i with respect to the ideal solution A^* is defined as follows:

$$C_{i^*} = \frac{S_{i^-}}{S_{i^*} + S_{i^-}}, \quad 0 \le C_{i^*} \le 1 \tag{33}$$

C_A=[1.66/(21.65+1.66)+4.14/(19.39+4.14)+0.09/(0.77+0.09)+0.63/ /(0.63+6.5)+1.4/(1.4+6.5)+0.42/(2.75+0.42)+1.16/(7.5+1.16)+0.1/ /(0.67+0.1)+0.11/ (0.76+0.11)]/9=0.125

C_B=[0.83/(6.5+0.83)+0.69/(4.15+0.69)+0.44/(3.85+0.44)+0.33/ /(4.33+0.33)+1.4/(6.5+1.4)+0.17/(3.85+0.17)+0.33/(1.65+0.33)+0.35 /(3.16+0.35)+0.22/(3.85+0.22)]/9=0.101

C_C=[1.66/(21.65+1.66)+0.69/(4.15+0.69)+0.09/(0.77+0.09)+0.33/ /(4.33+0.33)+0.2/(0.44+0.2)+0.17/(3.85+0.17)+0.46/(2.5+0.46)+0.71 /(10.5+0.71)+0.22/(3.85+0.22)]/9=0.11

C_D=[1.66/(21.65+1.66)+4.14/(19.39+4.14)+0.09/(0.77+0.09)+0.83/ /(6.5+0.83)+1.4/(6.5+1.4)+0.21/(0.82+0.21)+0.33/(1.65+0.33)+0.71/ /(10.5+0.71)+0.66/(5.39+0.66)]/9=0.108

Ranking the preference order

	C_A	C_B	C_C	C_D
$C_{i*} =$	0.12	0.101	0.11	0.108

Results determined from the relative closeness to the ideal solutions were used to rank the preference order in the selection of company for advertising: $C_A > C_C > C_D > C_B$. The results show that alternative A is the best company for advertising.

Determination of fuzzy eigenvalues and fuzzy eigenvectors to personnel selection. Different methods have been developed that use a pairwise comparison of alternatives and criteria to solve a problem in multi-criteria decision making. These methods allow to replace choices based on experience or intuition, to make decisions based on scientifically based arguments, to model the system of preferences in the mind of the decision maker. Consistency analysis of the fuzzy AHP determines how consistent the established decision matrix is. The main purpose of this method is to determine whether the priority values of the criteria in the decision matrix formed because of comparing different pairs. For this, two main conditions must be satisfied - the reciprocity and transitivity. Since consistency analysis contains different and contradictory criteria, let's apply the verification of the consistency of the decision matrix to the problem of personnel selection. Let’s assume that the problem of multi-criteria decision-making in personnel selection includes four criteria C_1 - relevant experience, C_2 - education, C_3 - technical skills, C_4 - communication and four alternatives. Pairwise comparisons of the criteria are given in in next form:

$$
D = \begin{array}{c|cccc}
 & C_1 & C_2 & C_3 & C_4 \\
C_1 & (1;1;1) & (6;7,8) & (2;3;4) & (4;5;6) \\
C_2 & (1/8;1/7;1/6) & (1;1;1) & (1/6;1/5;1/4) & (1/4;1/3;1/2) \\
C_3 & (1/4;1/3;1/2) & (4;5;6) & (1;1;1) & (2;3;4) \\
C_4 & (1/6;1/5;1/4) & (2;3;4) & (1/4;1/3;1/2) & (1;1;1)
\end{array}
$$

D is expressed by the fuzzy λ, which is the fuzzy eigenvale of the square fuzzy matrix. The solution of the fuzzy characteristic equation $D \otimes u = \lambda \otimes u$ is a fuzzy vector with fuzzy numbers, not equal to 0. u is a fuzzy eigenvector. According to fuzzy calculations, the above fuzzy formula is expressed as follows:

$$\left(d_i^1 \otimes u_1\right) \oplus \ldots \oplus \left(d_i^p \otimes u_p\right) = \lambda \otimes u_i \qquad (34)$$

or

$$\left(\bigcup_{\alpha\in[0,1]} \alpha D^\alpha\right) \otimes \left(\bigcup_{\alpha\in[0,1]} \alpha u^\alpha\right) = \left(\bigcup_{\alpha\in[0,1]} \alpha \lambda^\alpha\right) \otimes \left(\bigcup_{a[0,1]} \alpha u^a\right) \qquad (35)$$

We have the following interval equation through the α-cut of the fuzzy equation:

$$(D \otimes u)^\alpha = (\lambda \otimes u)^\alpha \Leftrightarrow D^\alpha \otimes u^\alpha = \lambda^\alpha \otimes u^\alpha, \alpha \in [0,1] \qquad (36)$$

For a fuzzy matrix $D = \left(d_i^j\right)$, each α-cut of the fuzzy matrix $D^\alpha = \left(d_i^j\right)^\alpha$ is an interval matrix, the elements are ordinary closed intervals:

$$D^a = (d_i^j)^a = [(d_i^j)_l^a, \left(d_i^j\right)_r^a] = \left[D_l^a, D_r^a\right] \qquad (37)$$

$D=\left(d_i^j\right)$ and $D^{\alpha}=(d_i^j)^{\alpha}$ matrices are the right and left boundaries of the α -cut, respectively. Then,

$$\lambda^a=[\lambda_l^a,\lambda_r^a], \quad u^a=\left[u_l^a,u_r^a\right]=\left[(u_i)_l^a,(u_i)_r^a\right] \tag{38}$$

For some values of $\alpha\in[0,1]$, we have the following:

For the left side

$$D_l^{\alpha}=\begin{bmatrix} 1 & 6+\alpha & 2+\alpha & 4+\alpha \\ 0.015\alpha+0.125 & 1 & 0.17+0.03\alpha & 0.25+0.08\alpha \\ 0.25+0.08\alpha & 4+\alpha & 1 & 2+\alpha \\ 0.17+0.03\alpha & 2+\alpha & 0.25+0.08\alpha & 1 \end{bmatrix}$$

If $\alpha=0$, then

$$D_l^{0}=\begin{bmatrix} 1 & 6 & 2 & 4 \\ 0.125 & 1 & 0.17 & 0.25 \\ 0.25 & 4 & 1 & 2 \\ 0.17 & 2 & 0.25 & 1 \end{bmatrix}$$

If $\alpha=1$, then

$$D_l^{1}=\begin{bmatrix} 1 & 7 & 3 & 5 \\ 0.14 & 1 & 0.2 & 0.33 \\ 0.33 & 5 & 1 & 3 \\ 0.2 & 3 & 0.33 & 1 \end{bmatrix}$$

For the right side

$$D_r^a = \begin{bmatrix} 1 & 8-a & 4-a & 6-a \\ 0.17-0.03a & 1 & 0.25-0.05a & 0.5-0.17a \\ 0.5-0.17a & 6-a & 1 & 4-a \\ 0.25-0.05a & 4-a & 0.5-0.17a & 1 \end{bmatrix}$$

If $\alpha = 0$, then

$$D_r^0 = \begin{bmatrix} 1 & 8 & 4 & 6 \\ 0.17 & 1 & 0.25 & 0.5 \\ 0.5 & 6 & 1 & 4 \\ 0.25 & 4 & 0.5 & 1 \end{bmatrix}$$

If $\alpha = 1$, then

$$D_r^1 = \begin{bmatrix} 1 & 7 & 3 & 5 \\ 0.14 & 1 & 0.2 & 0.33 \\ 0.33 & 5 & 1 & 3 \\ 0.2 & 3 & 0.33 & 1 \end{bmatrix}$$

Using the MATLAB environment, we obtain the fuzzy eigenvalues of the matrix D:

For the left side of a triangular fuzzy number:

$$(\lambda_1)_l^0 = 3.389$$
$$(\lambda_2)_l^0 = 0.157$$
$$(\lambda_3)_l^0 = 0.1576$$
$$(\lambda_4)_l^0 = 0.2957$$

$$(u_1)_l^0 = \begin{bmatrix} 0.889 \\ 0.095 \\ 0.407 \\ 0.185 \end{bmatrix}$$

$$(\lambda_1)_l^1 = 4.1042 \qquad (\lambda_2)_l^1 = 0.0027 \qquad (\lambda_3)_l^1 = 0.0027 \qquad (\lambda_4)_l^1 = 0.0987$$

$$(u_1)_l^1 = \begin{bmatrix} 0.889 \\ 0.086 \\ 0.411 \\ 0.184 \end{bmatrix}$$

For the right side:

$$(\lambda_1)_r^0 = 5.0535 \qquad (\lambda_2)_r^0 = 0.2176 \qquad (\lambda_3)_r^0 = 0.2176 \qquad (\lambda_4)_r^0 = 0.6183$$

$$(u_1)_r^0 = \begin{bmatrix} 0.879 \\ 0.087 \\ 0.427 \\ 0.193 \end{bmatrix}$$

$$(\lambda_1)_r^1 = 4.1042 \qquad (\lambda_2)_r^1 = 0.0027 \qquad (\lambda_3)_r^1 = 0.0027 \qquad (\lambda_4)_r^1 = 0.0987$$

$$(u_1)_r^1 = \begin{bmatrix} 0.889 \\ 0.086 \\ 0.411 \\ 0.184 \end{bmatrix}$$

Application of fuzzy type-2 decision-making method in project selection. In many cases, the decision-making process in the project system is based on deep uncertainties and imprecise information. In such cases, classical decision-making methods, including type-1 fuzzy methods, are not useful for solving these problems, and the use of type-2 fuzzy sets to solve the problem is considered acceptable in decision making. This section discusses the application of the type-2 fuzzy decision-making method in the selection of projects with different criteria.Four projects are offered for selection: A_1, A_2, A_3, A_4. These projects are characterized by criteria such as C_1 - technical characteristics, C_2 - market potential, C_3 - originality and C_4 - investment risk. On the basis of alternative projects, selection is performed by means of fuzzy type-2 sets using the decision-making method. The linguistic terms to be used and their fuzzy type-2 numerical values are listed below:

Extremely Poor	(0,0,0;0)	(0,0,0;0)
Very Poor	(0.1,0.2,0.3;0.3)	(0.1,0.2,0.3;0.3)
Poor	(0.2,0.3,0.4;0.4)	(0.2,0.3,0.4;0.4)
Slightly Poor	(0.25,35,0.45;0.45)	(0.25,35,0.45;0.45)
Fair	(0.3,0.4,0.5;0.5)	(0.3,0.4,0.5;0)
Slightly Good	(0.4,0.5,0.6;0.6)	(0.45,0.55,0.65;0.65)
Good	(0.5,0.6,0.7;0.7)	(0.55,0.65,0.75;0.75)
Very Good	(0.6,0.7,0.8;0.8)	(0.65,0.75,0.85;0.85)
Extremely Good	(0.7,0.8,0.9;0.9)	(0.7,0.8,0.9;0.9)

The scores of alternatives concerning criteria are shown below:

	C_1	C_2	C_3	C_4
A_1	(0.2,0.3,0.4,0.4; 0.25,35,0.45;0.45)	(0.4,0.5,0.6,0.6; 0.55,0.65,0.75;0.75)	(0.3,0.4,0.5,0.5; 0.55,0.65,0.75;0.75)	(0.5,0.6,0.7,0.7; 0.55,0.65,0.75,0.75)
A_2	(0.6,0.7,0.8,0.8; 0.7,0.8,0.9,0.9)	(0.2,0.3,0.4,0.4; 0.25,35,0.45,0.45)	(0.5,0.6,0.7,0.7; 0.55,0.65,0.75, 0.75)	(0.3,0.4,0.5,0.5; 0.55,0.65,0.75;0.75)
A_3	(0.3,0.4,0.5,0.5; 0.55,0.65,0.75;0.75)	(0.4,0.5,0.6,0.6; 0.55,0.65,0.75;0.75)	(0.6,0.7,0.8,0.8; 0.7,0.8,0.9,0.9)	(0.5,0.6,0.7,0.7; 0.55,0.65,0.75,0.75)
A_4	(0.5,0.6,0.7,0.7; 0.55,0.65,0.75,0.75)	(0.6,0.7,0.8,0.8; 0.7,0.8,0.9,0.9)	(0.3,0.4,0.5,0.5; 0.55,0.65,0.75;0.75)	(0.4,0.5,0.6,0.6; 0.55,0.65,0.75;0.75)
w_j	(0.2,0.3,0.4,0.4; 0.25,35,0.45;0.45)	(0.4,0.5,0.6,0.6; 0.55,0.65,0.75;0.75)	(0.3,0.4,0.5,0.5; 0.55,0.65,0.75;0.75)	(0.5,0.6,0.7,0.7; 0.55,0.65,0.75,0.75)

By using equation (39), we get the ranking weight value w_j for criterion $\tilde{C}_j$ $(j=1,2,3,4)$:

$$R\left(\tilde{A}_i\right)=\frac{1}{2}\left\{\frac{1}{3}\frac{c_i^3\left(b_i-a_i\right)+c_i\left(b_i^3-a_i^3\right)-a_ib_i\left(b_i^2-a_i^2\right)}{c_i^2\left(b_i-a_i\right)+c_i\left(b_i^2-a_i^2\right)-a_ib_i\left(b_i-a_i\right)}+\frac{b_i\left(a_i^{'}+2b_i^{'}+c_i^{'}\right)}{2}\right\} \quad (39)$$

The normalized weights are obtained by using the following formula:

$$w_j=\frac{R\left(w_j\right)}{\sum_{j=1}^{n}R\left(w_j\right)} \quad (40)$$

$$R\left(\tilde{A}_1\right)=\frac{1}{2}\left(\frac{1}{3}\frac{0.3^3\left(0.4-0.2\right)+0.3\left(0.4^3-0.2^3\right)-0.2\times0.4\left(0.4^2-0.2^2\right)}{0.3^3\left(0.4-0.2\right)+0.3\left(0.4^2-0.2^2\right)-0.2\times0.4\left(0.4-0.2\right)}+\frac{0.4\left(0.25+2\times0.45+0.35\right)}{2}\right)=0.20$$

$$R\left(\tilde{A}_2\right)=0.50\ , \qquad R\left(\tilde{A}_3\right)=0.40\,, \quad R\left(\tilde{A}_4\right)=0.54$$

$$\sum R(w_j)=0.20+0.50+0.40+0.54=1.64$$

$$w_1=\frac{0.20}{1.64}=0.12\,,\ w_2=\frac{0.50}{1.64}=0.30\,,\ w_3=\frac{0.40}{1.64}=0.25\,,$$
$$w_4=\frac{0.54}{1.64}=0.33\,.$$

We calculate the weighted ranking value for alternative $\tilde{A}_i\ \left(i=1,2,3,4\right)$ by equation (41):

$$R_{w_j}\left(\tilde{A}_i\right)=\sum_{j=1}^{n}w_jR\left(S_{ij}\right) \quad (41)$$

$$R_w\left(\tilde{A}_{11}^L\right)=0.11,\quad R_w\left(\tilde{A}_{11}^U\right)=0.126$$

$$R_w\left(\tilde{A}_{12}^L\right)=0.45,\quad R_w\left(\tilde{A}_{12}^U\right)=0.59$$

$$R_w\left(\tilde{A}_{13}^L\right)=0.3,\quad R_w\left(\tilde{A}_{13}^U\right)=0.49$$
$$R_w\left(\tilde{A}_{14}^L\right)=0.59,\quad R_w\left(\tilde{A}_{14}^U\right)=0.64$$
$$Rank\left(A_1^L\right)=\frac{1}{n(n-1)}\cdot\left(\sum P\left(A_1^L\right)+\frac{n}{2}-1\right)=\frac{1}{12}(1.45+1)=0.21$$
$$Rank\left(A_1^U\right)=\frac{1}{n(n-1)}\cdot\left(\sum P\left(A_1^U\right)+\frac{n}{2}-1\right)=0.24$$
$$Rank(A_1)=\frac{Rank(A_1^L)+Rank(A_1^U)}{2}=\frac{0.21+0.24}{2}=0.225$$

We apply all these calculations for other alternatives:

$$Rank(A_2)=\frac{Rank(A_2^L)+Rank(A_2^U)}{2}=\frac{0.20+0.23}{2}=0.215$$
$$Rank(A_3)=\frac{Rank(A_3^L)+Rank(A_3^U)}{2}=\frac{0.23+0.26}{2}=0.245$$
$$Rank(A_4)=\frac{Rank(A_4^L)+Rank(A_4^U)}{2}=\frac{0.22+0.26}{2}=0.24$$

In the last step, we rank the alternatives and select the best one in accordance with weighted ranking values $R_{w_j}\left(\tilde{A}_i\right)$.

$$Rank(A_1)=0.225\ ;\quad Rank(A_2)=0.215$$
$$Rank(A_3)=0.245\ ;\quad Rank(A_4)=0.24$$
$$A_3 > A_4 > A_1 > A_2\ .$$

This ranking reveal that alternative A_3 is the best, and alternative A_2 is the worst alternative in project selection.

Problem of supplier selection with the application of the multi-criteria fuzzy TOPSIS method and the Hamming distance. The supply chain is an important part of business operations, and if any part of this chain fails, it can face problems that negatively affect the reputation of the enterprise. The supplier selection process depends on various factors such as quality, reliability, flexibility, price. The exact criterion-measurement procedure is based on the specific company's strategy and priorities. Supplier selection is the process by which a company identifies, evaluates and contracts with suppliers. The main goal of this process is to reduce the risk of purchase, maximize the total cost to the buyer and develop close and long-term relationships between buyers and suppliers. Although seemingly simple, choosing the right supplier is a quite complex process. Various methods have been proposed to address this issue. Let's look at the application of the fuzzy TOPSIS method and the Hamming distance to this process. We choose the best supplier from four alternatives - A_1, A_2, A_3, A_4. The main criteria for these alternatives are C_1- quality, C_2- agility, C_3- reliability and C_4- price[8]. The relative weights of the criteria are estimated as $w_1 = 0.4$, $w_2 = 0.3$, $w_3 = 0.20$, $w_4 = 0.1$. The linguistic expressions of the criteria with fuzzy numbers is shown below:

Medium	High	Very high
(1,3,5)	(3,5,7)	(5,7,9)

The linguistic expression of the alternatives according to the criteria is shown below:

	C_1 - quality	C_2 - flexibility	C_3 - reliability	C_4 - cost
A_1	High	High	High	Medium
A_2	Medium	Very high	High	High
A_3	High	Very high	Medium	Very high
A_4	High	Medium	High	Very high

[8] T.C. Chu, Selecting plant location via a fuzzy TOPSIS approach, The International Journal of Advanced Manufacturing Technology 20 (2002) 859–864.

The fuzzy scores of the alternatives concerning criteria and the weights of the criteria are shown below:

	C_1 - quality	C_2 - flexibility	C_3 -reliability	C_4 - cost
Weight	0.4	0.3	0.20	0.1
A_1	(3,5,7)	(3,5,7)	(3,5,7)	(1,3,5)
A_2	(1,3,5)	(5,7,9)	(3,5,7)	(3,5,7)
A_3	(3,5,7)	(5,7,9)	(1,3,5)	(5,7,9)
A_4	(3,5,7)	(1,3,5)	(3,5,7)	(5,7,9)

In next step each element of the fuzzy decision matrix is normalized. The normalized decision matrix is shown below:

	A_1	A_2	A_3	A_4
C_1	(1.2,2,2.8)	(0.3,0.9,1.5)	(0.6,1,1.4)	(0.3,0.5,0.7)
C_2	(1.2,2,2.8)	(1.5,2.1,2.7)	(1,1.4,1.8)	(0.1,0.3,0.5)
C_3	(1.2,2,2.8)	(0.9,1.5,2.1)	(0.2,0.6,1)	(0.3,0.5,0.7)
C_4	(0.4,1.2,2)	(0.9,1.5,2.1)	(1,1.4,1.8)	(0.5,0.7,0.9)

Using the fuzzy normalized decision matrix, formulas (42) and (43), where a^*- fuzzy positive ideal solution and a^- - fuzzy negative ideal solution, are calculated.

$$a^* = \left\{ \left(\max_i v_{ij} \,\middle|\, j \in J), (\min_i v_{ij} \,\middle|\, j \in J^{'} \right) \middle| i = 1, 2, 3, ..., M \right| \right\} = \left\{ v_{1^*}, v_{2^*}, ..., v_{N^*} \right\}$$

(42)

$$a^- = \left\{ \left(\min_i v_{ij} \,\middle|\, j \in J), (\max_i v_{ij} \,\middle|\, j \in J^{'} \right) \middle| i = 1, 2, 3, ..., M \right| \right\} = \left\{ v_{1^-}, v_{2^-}, ..., v_{N^-} \right\} \quad (43)$$

The fuzzy positive ideal solution and the fuzzy negative ideal solution for the decision matrix are shown below:

	C_1	C_2	C_3	C_4
a^*	(1.2,2,2.8)	(0.3,0.9,1.5)	(1,1.4,1.8)	(0.5,0.7,0.9)
a^-	(0.4,1.2,2)	(0.9,1.5,2.1)	(0.2,0.6,1)	(0.1,0.3,0.5)

The distances from the fuzzy positive ideal solution and the fuzzy negative ideal solution are determined for each alternative using the Hamming distance values are shown below:

	A_1^*	A_2^*	A_3^*	A_4^*
C_1	0	1.05	0.6	0.3
C_2	0	0.9	0	0.6
C_3	0	0	1.2	0.3
C_4	1.2	0	0	0

	A_1^-	A_2^-	A_3^-	A_4^-
C_1	1.2	0	0.6	0.3
C_2	1.2	1.8	1.2	0
C_3	1.2	0.9	0	0.3
C_4	0	0.9	1.2	0.6

Using the ratios closest to the fuzzy positive and negative ideal solutions, we select the best supplier by determining the order of the alternatives. Separation distances are shown below:

C_1				C_2			
A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4

a^*	0	1.05	0.6	0.3	0	0.9	0	0.6
a^-	1.2	0	0.6	0.3	1.2	1.8	1.2	0

	C_3				C_4			
	A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4
a^*	0.9	0	1.2	0.6	1.2	0	0	0
a^-	1.2	0.9	0	0.3	0	0.9	1.2	0.6

The relative similarity of all alternatives to the fuzzy positive ideal solution is calculated by the following formula:

$$C_1^* = S_1^- \big/ (S_1^* + S_1^-) \qquad (44)$$

C_1* =[1.2/(1.2+0)+0/(0+1.05)+0.6/(0.6+0.6)+0.3/(0.3+0.3)]/4=0.5,

C_2* =[1.2/(1.2+0)+1.8/(1.8+0.9)+1.2/(1.2+0)+0/(0+0.6]/4=0.67,

C_3*=[1.2/1.2+0.9/0.9+0/1.2+0.3/0.6]/4=0.62

C_4* =[0/1.2+0.9/0.9+1.2/1.2+0.6/0.6]/4=0.75,

As a result, we can determine the ranking of all alternatives and choose the best alternative:

$A_4 > A_2 > A_3 > A_1$.

The choice of supplier determines that the best alternative is A_4 and the most inappropriate alternative is A_1.

House selection by using fuzzy multi-criteria decision-making method based on the utility function. One of the most pressing issues is the application of fuzzy multi-criteria decision-making in the process of purchasing houses. Some important criteria, such as the condition of the house, price, size, neighborhood, distance to work, etc. basically the decision to buy the best house should be made. There are several solutions to the problem of choosing a home. For some reason, there are difficulties in making the right choice in the process of buying and selling houses. Therefore, researchers need a more reliable method of choosing a house based on certain criteria. In this section, we will use a type 2 fuzzy multi-criteria decision-making method based on the utility function. We consider house selection problem which consists of 3 houses A_1, A_2, A_3 examined by five criteria C_1- Condition; C_2 - Cost; C_3 - Size; C_4 - Neighborhood; C_5 - Distance to work. Preference of DM concerning importance of criteria C_1, C_2, C_3 , C_4 and C_5 is given as weight vector W= {0.25, 0.2, 0.2, 0.2, 0.15}. To take the decision on these houses, DM gives the linguistic performance ratings in terms of fuzzy type-2 numbers:

	A_1	A_2	A_3
C_1	Good	Very good	Average
C_2	155	160	130
C_3	160	150	170
C_4	Good	Average	Very good
C_5	10	12	$\tilde{6}$

The description of linguistic terms with fuzzy type-2 numbers is as follows:

$Good = \ < \ 0.7, \ 0.9, \ 1, \ 1, \ 1; \ 0.7, \ 0.9, \ 1, \ 1, \ 0.8 \ >$

$Very\ good \ = \ < \ 0.8, \ 0.9, \ 1, \ 1, \ 1; \ 0.8, \ 0.9, \ 1, \ 1, \ 0.9 \ >$

$Average \ = \ < \ 0.4, \ 0.5, \ 0.6, \ 0.7, \ 1; \ 0.4, \ 0.5, \ 0.6, \ 0.7 \ , \ 0.6 \ >$

$\tilde{6} = \ < \ 5, \ 6, \ 6, \ 7, \ 1; \ 5, \ 6, \ 6, \ 7, \ 0.6 \ >$

$\tilde{10} \ = \ <8, \ 10, \ 12, \ 12, \ 1; \ 8, \ 10, \ 10, \ 12, \ 0.9 \ >$

$\tilde{12} = \ < \ 10, \ 12, \ 12, \ 14, \ 1; \ 10, \ 12, \ 12, \ 14, \ 0.85 \ >$

$\tilde{130} = \ < \ 120, \ 130, \ 130, \ 140, \ 1; \ 120, \ 130, \ 130, \ 140, \ 0.7 \ >$

$\tilde{150} = \ < \ 135, \ 150, \ 150, \ 165, \ 1; \ 135, \ 150, \ 150, \ 165, \ 0.8 \ >$

$\tilde{155} = \ < \ 140, \ 155, \ 155, \ 160, \ 1, \ 140, \ 155, \ 155, \ 160, \ 0.8 \ >$

$\tilde{160} = \ < \ 150, \ 160, \ 160, \ 170, \ 1 \ ; \ 150, \ 160, \ 160, \ 170, \ 0.85 \ >$

$\tilde{170} = \ < \ 150, \ 170, \ 170, \ 180, \ 1; \ 150, \ 170, \ 170, \ 180, \ 0.9 \ >$

We will use the U utility function[9] to make a decision. The solution for the problem in question will be expressed by the maximum value of A_i. The normalized decision matrix will be as follows:

$$D^N = \begin{bmatrix} d_{11}^N & d_{12}^N & d_{13}^N & d_{14}^N & d_{15}^N \\ d_{21}^N & d_{22}^N & d_{23}^N & d_{24}^N & d_{25}^N \\ d_{31}^N & d_{32}^N & d_{33}^N & d_{34}^N & d_{35}^N \end{bmatrix} \tag{45}$$

where

[9] Keeney R. L. Multiplicative utility functions, Operations Research 22, 1974, 22-34.

$d_{11} = d_{14} = < 0.5, 0.9, 1, 1, 1; 0.7, 0.9, 1, 1, 0.8 >$
$d_{21} = d_{34} = < 0.4, 0.7, 0.7, 0.8, 1; 0.4, 0.7, 0.7, 0.8, 0.8 >$
$d_{13} = d_{22} = < 0.6, 0.8, 0.8, 1, 1; 0.6, 0.8, 0.8, 1, 0.85 >$
$d_{15} = < 0.33, 0.55, 0.55, 0.77, 1; 0.33, 0.55, 0.55, 0.77, 0.9 >$
$d_{12} = < 0.8, 0.9, 1, 1, 1; 0.8, 0.8, 0.9, 1, 0.9 >$
$d_{23} = < 0, 0.33, 0.33, 0.66, 1; 0, 0.33, 0.33, 0.66, 0.8 >$
$d_{24} = d_{31} = < 0, 0.16, 0.33, 0.5, 1; 0, 0.16, 0.33, 0.5, 0.6 >$
$d_{25} = < 0.55, 0.77, 0.77, 1, 1; 0.55, 0.77, 0.77, 1, 0.85 >$
$d_{32} = < 0, 0.2, 0.2, 0.4, 1; 0, 0.2, 0.2, 0.4, 0.7 >$
$d_{33} = < 0.33, 0.77, 0.77, 1, 1; 0.33, 0.77, 0.77, 180, 0.9 >$
$d_{35} = < 0, 0.11, 0.11, 0.22, 1; 0, 0.11, 0.11, 0.22, 0.6 >$

In the next step, we calculate the utility function values for each of the 3 alternatives :

$$U_{A_1} = d_{11} C_1 + d_{12} C_2 + d_{13} C_3 + d_{14} C_4 + d_{15} C_5$$
$$U_{A_2} = d_{21} C_1 + d_{22} C_2 + d_{23} C_3 + d_{24} C_4 + d_{25} C_5 \qquad (46)$$
$$U_{A_3} = d_{31} C_1 + d_{32} C_2 + d_{33} C_3 + d_{34} C_4 + d_{35} C_5$$

As a result, we get the following :

$U_{A_1} = < 0.49, 0.83, 0.88, 0.97, 1; 0.49, 0.83, 0.88, 0.97, 0.803 >$
$U_{A_2} = < 0.4025, 0.626, 0.69, 0.832, 1; 0.4025, 0.626, 0.69, 0.832, 0.851 >$
$U_{A_3} = < 0.226, 0.464, 0.476, 0.559, 1; 0.226, 0.464, 0.476, 0.559, 0.803 >$

In the last step, we rank the utility functions U_{A_1} , U_{A_2} ,U_{A_3} using the expected values of the alternatives[10] to select the best alternative. Assume that the decision is optimistic and $\mu = 0.7$. Expected values for alternatives are as follows :

$$EV_{U_{A_1}} = 0.76, \quad EV_{U_{A_2}} = 0.63, \quad EV_{U_{A_3}} = 0.42$$

As a result, we define that the best house for selection is the first alternative. In the process of purchasing houses, decision-making is carried out under conditions of high uncertainty and taking into account a number of criteria. In this case, the theory of type-2 fuzzy sets is used to solve the problem of decision-making, and the most suitable house is selected. It was determined that the obtained results coincided with the expert opinion.

Type-2 fuzzy group decision making. Supplier selection is an important process that sets a foundation for a long-term business-to-business partnership with suppliers that can greatly contribute to the success or failure of a business. Suppliers are the lifeblood of any supply chain. They can determine a company’s profitability and its ability to deliver quality products and services. That’s why all businesses should have a comprehensive supplier selection process in place. Going beyond just choosing based on price, it considers every aspect of a vendor and their offerings. Hence, you’ll have a higher chance of making the right decision every time. In many cases, the decision-making process in supplier selection is based on high uncertainties and imprecise information. In such cases, classical decision-making methods, including type-1 fuzzy methods, are not useful for solving these problems, and the use of type-2 fuzzy sets to solve the problem is considered acceptable in decision making. This section discusses the application of the type 2 fuzzy decision-making

[10] Wang, X.F., Wang, J.Q. and Yang, W.E. (2014), “multi-criteria group decision making method based on intuitionistic linguistic aggregation operators”, Journal of Intelligent & Fuzzy Systems, Vol. 26 No. 1, pp. 115-125.

method in the selection of projects with different criteria. Assume that you need to choose a supplier among 3 alternatives on 5 criteria. The criteria are quality, reliability, experience, flexibility, communication. The linguistic scores of the alternatives concerning criteria and the weights of the criteria are expressed in fuzzy type-2 numbers. Fuzzy linguistic expression of the weights of the criteria are shown below:

Criteria	Decision makers		
	D_1	D_2	D_3
Quality	High	Very high	Average
Reliability	Very high	Very high	Very high
Experience	Very high	High	High
Flexibility	Very high	Very high	Very high
Communication	Average	Average	Average

Linguistic values of alternatives concerning criteria are shown below:

Criteria	Alternatives	Linguistic expressions of decision makers		
Quality	x_1	Medium	Good	Medium
	x_2	Good	Good	Medium
	x_3	Very good	Good	Poor
Reliability	x_1	Good	Medium	Poor
	x_2	Very good	Very good	Very good
	x_3	Medium	Good	Very good

Experience	x_1	Good	Good	Good
	x_2	Very good	Very good	Good
	x_3	Good	Medium	Very good
Flexibility	x_1	Very good	Good	Very good
	x_2	Very good	Very good	Very good
	x_3	Good	Very good	Medium
Communication	x_1	Poor	Poor	Poor
	x_2	Very good	Medium	Good
	x_3	Good	Good	Medium

Expression of linguistic scores are shown below:

Linguistic expression	Fuzzy type-2 sets
Very good	((9,10,10,10;1,1), (9,10,10,10;1,1))
Good	((7,9,9,10;1,1), (7,9,9,10;1,1))
Medium	((5,7,7,9;1,1), (5,7,7,9;1,1))
Poor	((3,5,5,7;1,1), (3,5,5,7;1,1))

Expression of criteria weights with type-2 fuzzy numbers are shown below:

Very high- (0.9,1,1,1;1,1), (0.9,1,1,1;1,1))

High- ((0.7,0.9,0.9,0.1;1,1), (0.7,0.9,0.9,0.1;1,1))

Medium- ((0.5,0.7,0.7,0.9;1,1), (0.5,0.7,0.7,0.9;1,1))

Initially, the average decision matrix is constructed:

$$\bar{Y} = \begin{array}{c} Quality \\ \mathrm{Re}\,liability \\ Experience \\ Flexibility \\ Communication \end{array} \begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & \tilde{a}_{13} \\ \tilde{a}_{21} & \tilde{a}_{22} & \tilde{a}_{23} \\ \tilde{a}_{31} & \tilde{a}_{32} & \tilde{a}_{33} \\ \tilde{a}_{41} & \tilde{a}_{42} & \tilde{a}_{43} \\ \tilde{a}_{51} & \tilde{a}_{52} & \tilde{a}_{53} \end{pmatrix} \qquad (47)$$

Once the mean decision matrix is formed, the average weight vector $\bar{W}$ is determined:

$$\bar{W} = \left[\tilde{w}_1 \quad \tilde{w}_2 \quad \tilde{w}_3 \quad \tilde{w}_4 \quad \tilde{w}_5\right] \qquad (48)$$

where

$\tilde{w}_1$ = ((0.7, 0.85, 0.85, 0.95; 1, 1), (0.7, 0.85, 0.85, 0.95; 1, 1))

$\tilde{w}_2$ = ((0.9, 1, 1, 1, ; 1, 1), (0.9, 1, 1, 1, ; 1, 1))

$\tilde{w}_3$ = ((0.75, 0.95, 0.95, 1; 1, 1), (0.75, 0.95, 0.95, 1; 1, 1))

$\tilde{w}_4$ = ((0.9, 1, 1, 1; 1, 1), (0.9, 1, 1, 1; 1, 1))

$\tilde{w}_5$ = ((0.45, 0.65, 0.65, 0.85; 1, 1), (0.45, 0.65, 0.65, 0.85; 1, 1))

Then the weighted decision matrix is constructed:

$\tilde{v}_{21}$ = ((0.45, 0.7, 0.7, 0.85; 1, 1), (0.45, 0.7, 0.7, 0.85; 1, 1))

$\tilde{v}_{22}$ = ((0.80, 1, 1, 1; 1, 1), (0.80, 1, 1, 1; 1, 1))

$\tilde{v}_{23}$ = ((0.65, 0.85, 0.85, 0.95; 1, 1), (0.65, 0.85, 0.85, 0.95; 1, 1))

$\tilde{v}_{31}$ = ((0.45, 0.70, 0.70, 0.9; 1, 1), (0.45, 0.70, 0.70, 0.9; 1, 1))

$\tilde{v}_{32}$ = ((0.65, 0.9, 0.9, 1; 1, 1), (0.65, 0.9, 0.9, 1; 1, 1))

$\tilde{v}_{33}$ = ((0.55, 0.80, 0.80, 0.95; 1, 1), (0.55, 0.80, 0.80, 0.95; 1, 1))

$\tilde{v}_{41}$ = ((0.75, 0.95, 0.95, 1; 1, 1), (0.75, 0.95, 0.95, 1; 1, 1))

$\tilde{v}_{42}$ = ((0.80, 1, 1, 1; 1, 1), (0.80, 1, 1, 1; 1, 1))

$\tilde{v}_{43}$ = ((0.65, 0.85, 0.85, 0.95; 1, 1), (0.65, 0.85, 0.85, 0.95; 1, 1))

$\tilde{v}_{51}$ = ((0.15, 0.30, 0.30, 0.60; 1, 1), (0.15, 0.30, 0.30, 0.60; 1, 1))

$\tilde{v}_{52}$ = ((0.3, 0.55, 0.55, 0.81; 1, 1), (0.3, 0.55, 0.55, 0.81; 1, 1))

$\tilde{v}_{53}$ = ((0.25, 0.55, 0.55, 0.80; 1, 1), (0.25, 0.55, 0.55, 0.80; 1, 1))

For each alternative, normalized Euclidean distances $d^{U+}(x_j)$, $d^{L+}(x_j)$ and $d^{U-}(x_j)$, $d^{L-}(x_j), 1 \le j \le 3$ to the fuzzy positive ideal solution and the fuzzy negative ideal solutions[12] are determined.[11]

j	$d^{U+}(x_j)$	$d^{L+}(x_j)$	$d^{U-}(x_j)$	$d^{L-}(x_j)$
1	1.9	1.9	3.4	3.4
2	1.2	1.2	4.1	4.1
3	1.55	1.55	3.75	3.75

$C^*(x_j)$, $(1 \le j \le 3)$, relative proximity degrees are calculated, and the results are shown below:

j	$C_1(x_j)$	$C_2(x_j)$	$C^*(x_j)$
1	0.65	0.65	0.65
2	0.75	0.75	0.75
3	0.70	0.70	0.70

Finally, comparison is performed in terms of relative closeness. Since $C^*(x_1) < C^*(x_3) < C^*(x_2)$, we can state that $x_1 < x_3 < x_2$ and the second supplier is considered to be the most suitable.

Port selection by fuzzy type-2 decision-making. The decision of selection a port is proposed by shipping companies, which have a global network to carry out international cargo transportation. In this case, the issue of port selection is solved by type-2 fuzzy decision-making method. Three alternatives - A_1, A_2, A_3 and four main criteria

[11] Ghaemi Nasab, F., & Rostamy-Malkhalifeh, M. (2010). Extension of TOPSIS for group decision-making based on the Type-2 fuzzy positive and negative ideal Solutions. Int. J. Industrial Mathematics Vol. 2, No. 3 (2010) 199-213.

- C_1 - port performance, C_2 - port suitability, C_3 - port traditions and C_4 - government support were used to select the most suitable port. The relative weights of the criteria were determined as w_1 = 0.35, w_2 = 0.3, w_3 = 0.20, w_4 = 0.15.

Assume that the linguistic performance ratings are shown below:

	C_1 - performance	C_2 - suitability	C_3 - traditions	C_4 - gov. support
A_1	High	High	High	Average
A_2	Average	Average	High	High
A_3	Average	Very high	Average	Very high

Very high = < 0.8, 0.9, 1, 1, 1; 0.8, 0.9, 1, 1, 0.9 >
High = < 0.7, 0.9, 1, 1, 1; 0.7, 0.9, 1, 1, 0.8 >
Average = < 0.4, 0.5, 0.6, 0.7, 1; 0.4, 0.5, 0.6, 0.7, 0.6 >

To select the most suitable port, an alternative with the maximum value of the U utility function is defined:

$$\tilde{U}\left(A^*\right) = \max \tilde{U}\left(A_i\right); \quad A_i \in \{A_1, A_2, A_3\} \tag{49}$$

In the first step, the following decision matrix is established for port selection:

$$\tilde{d}_{11} = \tilde{d}_{12} = \tilde{d}_{13} = \tilde{d}_{23} = \tilde{d}_{24} = < 0.7, 0.9, 1, 1, 1; 0.7, 0.9, 1, 1, 0.8 >$$
$$\tilde{d}_{22} = \tilde{d}_{32} = \tilde{d}_{34} = < 0.8, 0.9, 1, 1, 1; 0.8, 0.9, 1, 1, 0.9 >$$
$$\tilde{d}_{14} = \tilde{d}_{21} = \tilde{d}_{31} = \tilde{d}_{33} = < 0.4, 0.5, 0.6, 0.7, 1; 0.4, 0.5, 0.6, 0.7, 0.6 >$$

Then the decision matrix is normalized:

$\tilde{d}_{11}=\tilde{d}_{12}=\tilde{d}_{13}=\tilde{d}_{23}=\tilde{d}_{24}=$ < 0.4, 0.7, 0.7, 0.8, 1; 0.4, 0.7, 0.7, 0.8, 0.8 >

$\tilde{d}_{22}=\tilde{d}_{32}=\tilde{d}_{34}=$ < 0.7, 0.9, 1, 1, 1; 0.7, 0.9, 1, 1, 0.9 >

$\tilde{d}_{14}=\tilde{d}_{21}=\tilde{d}_{31}=\tilde{d}_{34}=$ < 0, 0.16, 0.33, 0.5, 1; 0, 0.16, 0.33, 0.5, 0.6 >

The utility function for all three ports is calculated as follows:

$\tilde{U}_{A_1} = \tilde{d}_{11}C_1 + \tilde{d}_{12}C_2 + \tilde{d}_{13}C_3 + \tilde{d}_{14}C_4$

$\tilde{U}_{A_2} = \tilde{d}_{21}C_1 + \tilde{d}_{22}C_2 + \tilde{d}_{23}C_3 + \tilde{d}_{24}C_4$

$\tilde{U}_{A_3} = \tilde{d}_{31}C_1 + \tilde{d}_{32}C_2 + \tilde{d}_{33}C_3 + \tilde{d}_{34}C_4$

$\tilde{U}_{A_1}$ = < 0.47, 0.81, 0.86, 0.95, 1; 0.47, 0.81, 0.86, 0.95, 0.801 >

$\tilde{U}_{A_2}$ = < 0.38, 0.60, 0.67, 0.81, 1; 0.38, 0.60, 0.67, 0.81, 0.851 >

$\tilde{U}_{A_3}$ = < 0.21, 0.44, 0.45, 0.53, 1; 0.21, 0.44, 0.45, 0.53, 0.803 >

To determine the most suitable port, utility functions $\tilde{U}_{A_1}, \tilde{U}_{A_2}, \tilde{U}_{A_3}$ are ranked. Assume that the decision is close to optimistic and is λ =0.7. The expected value for different alternatives is calculated as follows.

$$E_\lambda(A) = \frac{E_\lambda\left(A^L\right) + E_\lambda\left(A^U\right)}{2} \tag{50}$$

$E_\lambda(A_1^U)$=1[(1-0.7) (0.47+0.81) +0.7(0.86+0.95)]/2=0.82

$E_\lambda(A_1^L)$= 0.8[(1-0.7) (0.47+0.81) +0.7(0.86+0.95)]/2=0.66

$EV_{U_{A_1}}$ = 0.74

$E_\lambda(A_2^U)$=1[(1-0.7)(0.38+0.60)+0.7(0.67+0.81)]/2=0.66

$E_\lambda\left(A_2^L\right)$=0.85[(1-0.7)(0.38+0.60)+0.7(0.67+0.81)]/2=0.56

$EV_{U_{A_2}}$ = 0.61

$E_\lambda(A_3^U)$=1[(1-0.7) (0.21+0.44) +0.7(0.45+0.53)]/2=0.44

$E_\lambda\left(A_3^L\right)$=0.8[(1-0.7) (0.21+0.44) +0.7(0.45+0.53)]/2=0.35

$EV_{U_{A_3}}$ = 0.40

$EV_{U_{A_1}} = 0.74,\quad EV_{U_{A_2}} = 0.61,\quad EV_{U_{A_3}} = 0.40$

After calculation and ranking, it is determined that the most suitable port is A_1.

Selection of equipment on base of multi-criteria decision-making method. The equipment selection process is considered in the first stage of the design process, and the equipment selection process takes into account the quality, cost and reliability that are important for customer satisfaction. There are different technical criteria to consider when selecting the appropriate equipment:

-Production capacity: the required amount of equipment products needed, innovation and quality are determined by marketing research.

-Product type: Product type is an important factor in equipment selection.

-Product diversity: When product diversity is required, more or less appropriate equipment should be selected.

-Equipment automation level: Equipment automation level depends on the number, type and size of equipment, production speed and convenience, etc. is an important factor influencing the processes.

In selection of equipment, we should also consider the cost of purchasing equipment and various costs such as transportation costs, installation costs, operating costs, repair costs and the cost of spare parts. The first step in assessing equipment's environmental impact is to identify a number of alternative technologies that are claimed to be the best. The criteria are the degree of environmental pollution determined for each alternative, the level of negative impact of the products on the environment. The main goal is to select the optimal technological alternative that best ensures the comprehensive prevention or minimization of adverse effects on the environment. Each criterion is of relative importance in terms of its weight, its impact on the environment. Decision-making in the equipment selection process is carried out under conditions of high uncertainty and taking into account different criteria. In this case, the method of interval approximation of fuzzy numbers is used to solve the decision problem. The Hamming distance between the intervals and the estimated size of numbers are determined, and a model representing the approximate interval for L-R fuzzy numbers is determined. Assume that on the basis of C_1 - technology, C_2 - economical, C_3 - ecology and C_4 - social criteria, it is necessary to choose between three A_1, A_2, A_3 equipments. Let's apply the method of interval approximation of fuzzy numbers to decide on the choice of equipment.

Criteria	A_1	A_2	A_3
C_1 technology	Very good	Good	Average
C_2 economical	Good	Good	Very good
C_3 - ecology	Average	Very good	Good
C_4 -social	Good	Average	Good

Weights of criteria are represented as, W={0.3, 0.2, 0.2, 0.1, 0.2}. The expression of linguistic terms in fuzzy numbers is as follows:
Very good = < 0.8, 0.9, 1, 1>
Good = < 0.6, 0.7, 0.8, 0.9>
Average = < 0.4, 0.5, 0.6, 0.7>

Finding the expected interval of a fuzzy number is based on the Hamming distance and is shown in Figure 4.

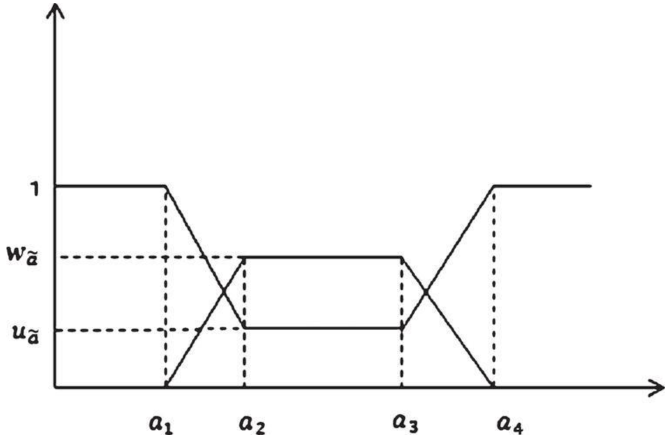

Figure 4. Hamming distance between interval and a -fuzzy number.

It contains the following rules for applying this methodology.
Rule 1. If the number a is a fuzzy number, the following formulas are used to determine the right and left limits of L-R:

$$W_A = \overline{a} - \underline{a} + \alpha_A * \lambda + \beta_A * \rho \quad (51)$$

$$L\left(\frac{\underline{a}-x}{\alpha}\right) = 1 - \left(\frac{\underline{a}-x}{\alpha}\right);\ R\left(\frac{x-\overline{a}}{\beta}\right) = 1 - \left(\frac{x-\overline{a}}{\beta}\right) \quad (52)$$

Rule 2 . If $\lambda = \rho$, then

$$z_0 = \underline{a} - \alpha_A \lambda$$

Here, z_0 is value of α -cut. λ -value , α and β are left and right spreads and the lower and upper bounds of the set of fuzzy numbers, respectively.
Rule 3. The best A interval approximation is calculated as follows:

$$I_{z_0} = \left[z_0, z_0 + W_A\right] \quad (53)$$

Rule 4. In these calculations we will take into account that

$$\int_{-\infty}^{\infty} \mu_A(x)\,dx = W_A < \infty \quad (54)$$

In this selection the fuzzy utility function is used.

$$U\left(A^*\right) = \max U\left(A_i\right); \quad A_i \in A \tag{55}$$

The solution of the problem is as follows:

For " Very good "

$$L = 1 - \left(\frac{0.5 - x}{0.5 - 0.4}\right); \quad R = 1 - \left(\frac{x - 0.6}{0.7 - 0.6}\right)$$

For "Good "

$$L = 1 - \left(\frac{0.7 - x}{0.7 - 0.6}\right); \quad R = 1 - \left(\frac{x - 0.8}{0.9 - 0.8}\right)$$

For " Avarage "

$$L = 1 - \left(\frac{0.9 - x}{0.9 - 0.8}\right); \quad R = 1 - \left(\frac{x - 1}{1 - 1}\right)$$

$$\lambda = \int_0^1 L(t)dt = \int_0^1 (1 - t)dt = 1 - \frac{1}{2} - 0 = 0.5$$

$$\rho = \int_0^1 R(t)dt = \int_0^1 (1 - t)dt = 0.5$$

For " Very good "

$$W_A = (1 - 0.9) + 0.1 * 0.5 = 0.1 + 0.05 = 0.15$$

$$z_0 = \underline{a} - \alpha * \lambda = 0.9 - 0.1 * 0.5 \ = 0.85$$

$$\left[z_0, z_0 + W_A\right] = \left[0.85, 0.85 + 0.15\right] = \left[0.85, 1\right]$$

For "Good "

$$W_A = (0.8 - 0.7) + 0.1 * 0.5 + 0.1 * 0.5 = 0.1 + 0.1 = 0.2$$

$$z_0 = \underline{a} - \alpha * \lambda = 0.7 - 0.1 * 0.5 = 0.65$$

$$\left[z_0, z_0 + W_A\right] = \left[0.65, 0.65 \ + 0.2\right] = \left[0.65, 0.85\right]$$

For " Average "

$$W_A = (0.6 - 0.5) + 0.1 * 0.5 + 0.1 * 0.5 = 0.1 + 0.1 = 0.2$$

$$z_0 = \underline{a} - \alpha * \lambda = 0.5 - 0.1 * 0.5 = 0.5 - 0.05 = 0.45$$

$$\left[z_0, z_0 + W_A\right] = \left[0.45, 0.45 \ + 0.2\right] \ = \left[0.45, 0.65\right]$$

In the last step, we calculate utility function for each of the 3 alternatives:

$$U_{A_1} = d_{11} * C_1 + d_{12} * C_2 + d_{13} * C_3 =$$

$$= [0.3, 0.3] + [0.2, 0.17] + [0.1, 0.2] + [0.06, 0.08] = [0.66, 0.75]$$

$$U_{A_2} = d_{21} * C_1 + d_{22} * C_2 + d_{23} * C_3 =$$

$$= [0.2, 0.3] + [0.2, 0.2] + [0.2, 0.2] + [0.04,\ 0.06] = [0.64, 0.76]$$

$$U_{A_3} = d_{31} * C_1 + d_{32} * C_2 + d_{33} * C_3 =$$

$$= [0.1, 0.2] + [0.2, 0.2] + [0.13,\ 0.17] + [0.06, 0.08] = [0.49, 0.65]$$

Thus, the utility function for each alternative is obtained as follows:

U_{A_1} = [0.66, 0.75]; U_{A_2} = [0.64, 0.76]; U_{A_3} = [0.49, 0.65]

We find the distances by comparing the defined intervals.

Comparing the utility functions U_{A_1} and U_{A_2}, we obtain the following distance:

$$d(I, J) = \frac{0.66 - 0.64}{|(0.66 - 0.64) + (0.75 - 0.76)|} = \frac{0.02}{0.01} = 2$$

Comparing the utility functions U_{A_2} and U_{A_3}, we obtain the following distance:

$$d(I, J) = \frac{0.64 - 0.49}{|(0.64 - 0.49) + (0.76 - 0.65)|} = \frac{0.15}{0.26} = 0.57$$

Comparing the utility functions U_{A_1} and U_{A_3}, we obtain the following distance:

$$d(I, J) = \frac{0.66 - 0.49}{|(0.66 - 0.49) + (0.75 - 0.65)|} = \frac{0.17}{0.27} = 0.63$$

As a result of the comparison, the maximum value of distance $d(I, J)$ is obtained by comparing the utility functions U_{A_1} and U_{A_2}.

This shows that the best alternative for equipment selection is A_1.

Demand forecasting in production through Z- information. To gain a competitive advantage, manufacturers must adapt to the rapidly changing world, produce new products and provide proper customer service. Therefore, forecasting demand is one of the important issues. Forecasting demand is important for the supply chain because it significantly affects developing operational strategies. Forecasting demand is considered as the basis hypothesis of strategic business activities and the starting point for most supply chain processes such as raw material planning, procurement, internal logistics, cash flow and production. It also facilitates critical business activities such as demand forecasting, financial planning, production planning, risk assessment and raw material procurement. Most importantly, forecast accuracy allows retailers to avoid stock sales and stockpiling, improve production times, minimize costs, increase operational efficiency and improve customer experience. Forecasting demand is about identifying production capacity, resources that meet customer expectations, optimizing resources, reducing production costs, etc. This allows us to identify situations that will arise in the planning of production processes and take measures to better meet the goals of companies. Despite the rapid data transfer, the sales forecast is much more than simple use of past sales data to determine future consumer demand. The process can be divided into qualitative and quantitative forecasting, both of which rely on different sources and data sets to extrapolate useful sales data. There are several different methods you can use to forecast demand in the field of quantity and quality forecasting. In this section, the issue of determining demand in production is explored in a new theory called Z-numbers, and the forecasting model is based on the Z-regression model. Assume that demand for a product depends on three variables under uncertainty: price, quality and waiting time. Input and output information provided for demand forecasting is expressed in Z-numbers. The key is to establish a relationship between demand (D), price (P), quality (Q) and waiting time (L) using Z-number data. For this purpose we will use the concept of Z-regression. Using approach given in chapter 5 to show the dependence between the variables, set the function F:

$$Z_D = F(Z_P, Z_Q, Z_L) \tag{56}$$

Suppose that the function F is expressed as follows:

$$Z_D = C_1 Z_P + C_2 Z_Q + C_3 Z_L \tag{57}$$

where, VL is expressed as low, SL as slightly low, L as low, A as average, H as high, SH as slightly high and VH as very high.

SS - slightly shure, S - shure, VS - very shure.

SS = [0.5, 0.6, 0.7] S = [0.6, 0.7, 0.8] VS = [0.7, 0.8, 0.9] .

Z -data fragment shown in table 4:

Table 4. Z-data fragment

Rule №	(İf) part			(Then) part
	P (Price)	W (Warranty)	L (lead time)	D -Demand
1	VL, S	VL, S	VL, VS	H, S
2	VL, S	VL, VS	VL, VS	SH, VS
3	VL, S	A, S	VL, S	VH, S
…	…	…	…	…
6	VL, SS	VL, VS	L, VS	H, S
7	VL, SS	L, VS	L, S	H, VS
…	…	…	…	…

For price criteria: VL =[11,12], L =[12,14], A=[14,16], H =[16,18], VH =[18,19]

For warranty criteria: VL =[7,9], L =[9,13], A=[13,17], H =[17,21], VH =[21,25]

For lead time criteria: VL =[2,4], L =[4,8], A=[8,12], H =[12,16], VH =[16,18]

For demand criteria: VL =[1,1.5], SL =[1.5,2.5], L =[2.5,3.5], A =[3.5,4.5], H =[4.5,5.5], SH =[5.5,6.5], VH =[6.5,7]

The Z-number linear regression function is constructed as $Z_{Y^M}(Z_{X_1}, Z_{X_2}, ..., Z_{X_N}) = Z_{C_0} + \sum_{i=1}^{N} Z_{C_i} Z_{X_i}$, and the construction of the regression model is based on calculations on a large number of Z-numbers. To determine Z_{Y^M} , we define the values of the parameters C_1, C_2, C_3, so that the error $\sum_{k=1}^{K} \left| Z_{Y,k}^M - Z_{Y,k} \right| \to \min$. These operations are performed by the method of interval approximation. If all the variables and coefficients of the regression model are Z-numbers, we obtain the following values: $C_{1,}$ = 0.25, C_2 =0,237, C_3 = -0.02

The distance (accuracy) for the data used to build the model is calculated as follows:

$$D = (0.62 + 0.95 + 0.73 + 1.27 + 1.78 + 0.68 + 0.91 + 0.68 + 1.69 + 1.54) / 10 = \mathbf{1.08}$$

The data fragment for the test is shown in Table 5.

Table 5. Z-data for testing

Rule №	İf part			Then part
	P	W	L	D
1	VL, S	VL, S	A, VS	H, S
2	VL, S	A, VS	A, S	H, VS
3	VL, VS	A, S	A, S	SH, S
4	VL, VS	H, S	A, VS	SH, S
5	VL, S	VH, S	A, SS	SH, SS

Distance for testing is D_T = (1.38+0.55+1.44+1.15+1.21)/5=**1.14**

Optimization of coke production process based on fuzzy dynamic programming. The research on the creation of control system for the delayed coking complex, which is a dynamic system, focuses on the identification and optimal management of coke production, taking into account the specific information conditions, i.e the uncertainty of the environment[12]. Delayed coking is a thermal cracking process used in petroleum refineries to upgrade and convert petroleum residuum (bottoms from vacuum distillation of crude oil) into liquid and gas product streams leaving behind a solid concentrated carbon material—petroleum coke. All of the heat necessary for coking is provided in the heater, whereas coking takes place in the coke drum; hence, the process is called “delayed coking.” Coking is the heating of coal in the absence of oxygen to a temperature above 600 °C to drive off the volatile components of the raw coal, leaving a hard, strong, porous material of high carbon content called coke. Coke consists almost entirely of hydrocarbons. Given the nature of the specific gravity and the effect of consumption of raw materials on the coking process, we express them in the form of $X = \{X_1, X_2\}$ vector, where X_1 is the specific gravity of raw materials, X_2 is the amount of primary raw materials consumed. The process of coke formation in the reaction chamber is controlled by changing the composition and temperature of the secondary raw material, as well as the reflux temperature supplied to this chamber. The composition of the secondary raw material depends on the composition of the primary raw material and the regime at the bottom of the distillation column and is indirectly characterized by the recirculation coefficient-K. The temperature regime in the reaction chamber also depends on the weight of the delayed coking process, the temperature of the reflux at the outlet of the furnace, the temperature of the upper part of the reaction chamber and, to some extent, the pressure in the system. The recirculation coefficient, the temperature of the recycled raw material, the temperature of the light flare and the temperature of the upper part of the reaction chamber are considered as control parameters. When determining the fuzzy model of the coking process, the required

[12] Mamedova G.A., Aliyeva K.R. Fuzzy optimal control of coke production. Journal “Oil and gas”, Moscow, 1987.

number of observations are made in the coking process to identify the object. The results of the observation are shown below:

N	X_1	X_2	u_{11}	u_{12}	u_{13}	u_{14}	u_2	u_3	u_4	Y
1	0.9453	70	26.8	26.8	25.6	25.2	490	515	420	0
2	0.9453	80	26.0	26.0	24.8	25.6	492	515	425	23.1
3	0.9453	80	26.6	26.6	25.6	25.2	485	515	425	48.2
4	0.9453	76	26.4	25.0	25.0	26.4	486	512	425	74.0
5	0.9453	76	26.4	25.0	25.0	26.4	498	520	425	99.6
6	0.9453	70	26.6	25.6	25.6	25.2	485	515	425	48.2
7	0.9456	70	27.0	25.0	25.0	25.6	498	520	425	147.4
8	0.9456	70	27.0	25.0	25.0	28.4	498	520	425	169.5
9	0.9456	70	27.0	25.0	25.0	26.4	498	520	425	192.9
10	0.9456	70	26.4	25.0	25.0	25.6	498	520	425	216.0
11	0.9456	76	26.4	25.4	25.6	24.4	490	510	424	239.2
12	0.9456	76	26.4	26.4	25.6	24.0	495	510	423	264.0
13	0.9436	76	26.4	26.4	25.6	24.0	492	510	420	268.6
14	0.9436	76	26.4	26.4	25.6	24.0	490	510	420	313.4
15	0.9436	76	26.4	26.4	25.6	24.0	490	520	410	338.3
16	0.9436	76	26.8	26.4	25.4	24.8	490	520	410	363.3
17	0.9436	76	26.8	26.8	24.0	24.8	490	500	430	385.9
18	0.9436	76	26.8	26.8	24.2	24.8	490	515	430	408.5
19	0.9436	76	26.8	26.8	24.0	24.8	492	510	430	431.5
⋮										

Here, X_1 - specific gravity of raw materials, X_2 - consumption of primary raw material, u_{11} - re-consumption (flow I), u_{12} - re-consumption (flow II), u_{13} - re-consumption (flow III), u_{14} - re-consumption (flow IV), u_2 - re-consumption temperature, u_3 - light gas oil temperature, u_4 - reaction chamber temperature, Y - coke output.

The result of the process and the sampling depth K of the final product depend on both the composition of the raw material and the quality of process control. Depth of acquisition of the final product - U characterizes the economic performance of the complex and is considered the output parameter. Given the physical characteristics of the coking process, a fuzzy linear difference equation was used to describe the situation:

$$Y_{k+1} = a_0 \oplus Y_k \otimes A_1 \oplus X_{1k} \otimes A_2 \oplus \cdots \oplus B_1 \otimes U_{1k} \oplus \cdots \oplus B_4 \otimes u_{4k} \qquad (58)$$

Thus, the solution of the problem of identification of the coking process is solved by the method of identification of fuzzy dynamic models and based on the determination of coefficients $A_i(i=\overline{0,2})$ and $B_j(j=\overline{1,4})$.

A_0 =0.00005

A_1 =0.5/13.327+0.8/16.5935+1.0/24.763+0.8/26.4177+0.5/19.9155

A_2 =0.5/0.0035+0.8/0.0152+1.0/-0.0236+0.8/-0.0232+0.5/-0.0065

B_1 =0.5/-0.8245+0.8/-0.0465+1.0/-0.8241+0.8/-0.8205+0.5/-0.6164

B_2 =0.5/0.0052+0.8/0.004+1.0/0.0038+0.8/0.0041+0.5/0.0077

B_3 =0.5/0.005+0.8/0.0051+1.0/0.005+0.8/0.0051+0.5/0.007

B_4 =0.5/0.0041+0.8/0.006+1.0/0.0053+0.8/0.0045+0.5/0.0092

Thus, the fuzzy dynamic model of the coke plant will be as follows:

$$
\begin{aligned}
Y_{k+1} &= 0.00005Y \oplus \\
&\oplus (0.5/13.327+0.8/16.593+1.0/24.76+0.8/26.4177+0.5/19.915)\, x_{1k} \\
&\oplus (0.5/0.0035+0.8/0.015+1.0/{-0.0236}+0.8/{-0.023}+0.5/{-0.0065})\, x_{2k} \\
&\oplus (0.5/{-0.8245}+0.8/{-0.0465}+1.0/{-0.824}+0.8/{-0.820}+0.5/{-0.616})\, U_{1k} \\
&\oplus (0.5/0.0052+0.8/0.004+1.0/0.0038+0.8/0.0041+0.5/0.0077)\, U_{2k} \\
&\oplus (0.5/0.005+0.8/0.0051+1.0/0.005+0.8/0.0051+0.5/0.007)\, U_{3k} \\
&\oplus (0.5/0.0041+0.8/0.006+1.0/0.0053+0.8/0.0045+0.5/0.0092)\, U_{4k}
\end{aligned}
$$

The problem of fuzzy optimal control of coke production is as follows: Given the specific gravity of the given (X_1) raw material and (X_2) technological constraints within the consumption of the primary raw material, determine the values of the recirculation coefficient (U_1), the temperature (U_2) for the secondary raw material, the temperature of the light gas oil (U_3) and the temperature (U_4) at the top of the reaction chamber to ensure the maximum amount of coke during the transition of the chamber from one state to another during stable operation. Mathematically, this problem can be expressed as follows: at given values of X_1 and X_2 it is required to find such a fuzzy control sequence

$$\{U_0 = (U_{10},....,U_{40}),\ U_1 = (U_{11},....,U_{41}),....,(U_{N1},....,U_{4N})\} \tag{59}$$

that

Y_{k+1}=0.00005Y $\oplus$
$\oplus$ (0.5/13.327+0.8/16.5935+1.0/24.763+0.8/26.4177+0.5/19.915) Z_{1k}
$\oplus$ (0.5/0.0035+0.8/0.0152+1.0/-0.0236+0.8/-0.0232+0.5/-0.0065) Z_{2k}
$\oplus$ (0.5/-0.8245+0.8/-0.0465+1.0/-0.8241+0.8/-0.820+0.5/-0.616) U_{1k}
$\oplus$ (0.5/0.0052+0.8/0.004+1.0/0.0038+0.8/0.0041+0.5/0.0077) U_{2k}
$\oplus$ (0.5/0.005+0.8/0.0051+1.0/0.005+0.8/0.0051+0.5/0.007) U_{3k}
$\oplus$ (0.5/0.0041+0.8/0.006+1.0/0.005+0.8/0.0045+0.5/0.009) U_{4k}

Restrictions are as follows:

U_{1k} - approximately, should be in the range (1.2; 1.8);
U_{2k} - approximately, should be in the range (495; 520) (60)
U_{3k} - approximately, should be in the range (515; 520)
U_{4k} = 455

The mathematical description of the optimization problem is as follows:

$$J = Y_N \tag{61}$$

$$Y_{k+1} = f\left(X_{1k}, X_{2k}, U_{1k}, \cdots, U_{4k}, Y\right) \tag{62}$$

$$Y_0 = 0$$

$$U_{1k} \in (1.2;1.8); \quad U_{2k} \in (495;520); \quad U_{3k} \in (515;520); \quad U_{4k} = 455.$$

The problem of optimal control of the coking process[13] is solved as follows. First, we define the membership functions of fuzzy sets that express fuzzy goals and constraints. Set the membership function as follows:

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[13] Mamedova G.A., Aliyeva K.R. Fuzzy optimal coke production control. Journal "Oil and gas", Moscow,1988.

$$\mu_{G^N}(Y_N) = \exp(-\tau_0 |Y_N - b_0|) \tag{63}$$

Analogically, $U_i(i=\overline{1,4})$, $\mu_u(U_{ik})$ construct the membership function of a fuzzy constraint: $\mu_u(U_{ik}) = \{1 + d_t(U_{ik} - b_i)^{r_i}\}^{-1}, r_i > 0$

$d_i(i=1,4)$=const, $r_j(j=1,4)$=const

In this case, the solution to the optimization problem is defined as follows:

$$D = U_1 \bigcap U_2 \bigcap \cdots U_{N-1} \bigcap G^N \tag{64}$$

The solution of the problem in terms of membership functions can be expressed as follows:

$$\mu_D(U_D, \cdots, U_{N-1}) = \min\{(\mu_u(U_0), \cdots, \mu_{N-1}(U_{N-1}), \mu_{G^N}(Y_N)\} \tag{65}$$

In (65), the value of Y_N can be expressed in terms of Y_0, $U_0, \cdots, U_{N-1}$. Then the above problem can be expressed as follows: Find the sequence $\mu_{u_0}, \cdots, \mu_{u_{N-1}}$ that maximizes μ_D. It is convenient to express the solution of equation (65) in the form $U_K^* = \prod(Y_K)$, $K = 0, \overline{N-1}$.

This solves the following problem:

$$\begin{gathered} \mu_D(U_0^*, \cdots, U_{N-1}^*) = \max u_0, \cdots, \max u_{N-1} \min\{(\mu_{u_0}(U_0), \cdots, \\ \mu_{U_{N-2}}(U_{N-2}), \mu_{U_{N-1}}(U_{N-1}), \mu_{G^N}(f(Y_{N-1}, U_{N-1}))\} \end{gathered} \tag{66}$$

Fuzzy dynamic programming method is used to solve equation (66). In this case, the system of recurrent equations takes the following form:

$$\mu_{G^{N-\vee}}(Y_{N-\vee}) = \max U_{N-\vee} \min\{\mu_{U_{N-\vee}}(U_{N-\vee}), \mu_{G^{N-\vee+1}}(Y_{N-\vee+1})\},$$

$$Y_{N-\vee+1} = f(Y_{N-\vee}, U_{N-\vee}), \vee = \overline{1, N}$$

Equation (66) is solved by a fuzzy dynamic programming method. The calculation of the optimal values of the control parameters in the coking process is determined at the beginning of the reactor operation. These results reflect both the current and recommended optimal values of the coking process parameters (Table 6).

Table 6. Current and recommended optimal values of coking process parameters

N	Fuel dansıt.	Fuel consum.	Recir. coeffi.	Recir. temp.	Gas oil temp.	Top col. temp	Coke production
Current mode							
23	0.987	165	1.14	505.0	489.0	445	
Optimal solution							
13	0.995	160	1.35	496.0	515.0	448	Approx. 25
15	0.995	160	1.35	496.0	515.0	448	Approx. 25
17	0.995	160	1.35	496.0	515.0	448	Approx. 66
19	0.995	160	1.35	496.0	515.0	448	Approx. 66
21	0.995	160	1.35	496.0	515.0	448	Approx.106
23	0.987	165	1.35	496.0	515.0	448	Approx.146
1	0.987	165	1.35	501.0	485.0	441	Approx.186
3	0.987	165	1.35	501.0	485.0	441	Approx.226
5	0.987	165	1.35	501.0	485.0	441	Approx.266
7	0.987	165	1.35	501.0	485.0	441	Approx.306
9	0.987	165	1.35	501.0	485.0	441	Approx.346
11	0.987	165	1.35	501.0	485.0	441	Approx.386
13	0.987	165	1.35	501.0	485.0	441	Approx.426
15	0.987	165	1.35	501.0	485.0	441	Approx.466

Results

The main **scientific results** obtained in the dissertation are as follows:

1. The adequacy of the preference of the decision-maker or a group in decision-making theory depends on taking into account existing uncertainty. In this regard, in order to take into account the higher level of uncertainty, the formulation of fuzzy type-2 preferences has been studied and applied in various real-world problems in the dissertation;
2. In the existing scientific literature, it is generally assumed that the available or obtained information is completely reliable. However, in complex real decision-making problems, both the opinions of experts and the knowledge obtained from data are often partially reliable. The formation and analysis of Z-number-valued preferences on the basis of Z-number theory in the context of partially reliable information is considered;
3. A system of linear equations widely used in Leontyev economic input-output model, decision-making based on linear programming was analyzed for the bimodal environment and exact and approximate solution methods were proposed;
4. As a series of decision-making methods is based on the aggregation of existing knowledge, a method of aggregation of bimodal information has been proposed;
5. A method of constructing a regression model with Z-variables and coefficients has been proposed to model decision-making and control objects characterized by fuzzy and probabilistic uncertainties. The advantage of the method is the reduced computational complexity;
6. Although the classical Delphi method is useful for dealing with completely reliable and precise information, and is mainly based on the knowledge of a group of experts, the transformation of this knowledge to numerical information prevents its widespread use. With this in mind, the fuzzy Delphi method was proposed for the first time. The fact that this method is useful and accurate relies on more than 3,300 reads of this method in Research Gate;
7. The problem of multi-stage decision-making was considered on the basis of fuzzy dynamic programming and a solution procedure was proposed;

8. The scientific findings of the dissertation have been applied to a wide range of business, economic, industrial decision-making and multi-stage control problems. The practical results confirmed the usefulness, practicality and validity of the proposed decision-making tools.

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Personal contribution of the applicant in the works published in co-authorship.

[1]- problem statement, mathematical modeling, and analysis of results;
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