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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

Z-IMPLICATION AND ITS APPLICATION

Speciality: 1203.01 – Computer Science
Field of science: Technical sciences

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Baku – 2025

The work was performed at the research laboratory "Intelligent Control and Decision-Making Systems in Industry and Economics" of Azerbaijan State Oil and Industry University.

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GENERAL CHARACTERISTICS OF THE RESEARCH

Relevance of the topic and degree of elaboration. In today's complex and data-rich environment, decision-making systems often need to effectively handle information that is uncertain, imprecise, and partially reliable. Traditional logic systems, built on binary values such as true or false, are inadequate for modeling such real-world complexities. Although fuzzy logic and probabilistic reasoning have made progress in representing imprecision and uncertainty, these approaches typically address these issues separately and cannot fully capture the dual nature of imperfect information -both its imprecision and degree of reliability.

This gap has driven research into Z-numbers and Z-fuzzy relations, this approach unifies imprecision and reliability within a single mathematical framework. While theoretical foundations and practical applications exist in fields like control systems, medicine, decision making and data analysis, fully developed and systematic methodologies for approximate reasoning based on Z-rules remain insufficient. Current approaches often face limitations in handling bimodal information-based rule bases. Considering these challenges, there is a need for new formal models that enable reasoning from Z-number-valued information, facilitating more human-like understanding and better management of uncertainty.

The above highlights the analysis of existing fuzzy and probabilistic implication tools, the elimination of their shortcomings, and justifies the need to develop approximate reasoning methods based on Z- number conception. This proves that the scientific problems considered in the dissertation are actual, and there is a need for new concepts and methods.

Approximate reasoning plays a crucial role in decision-making and expert systems, especially in situations where information is imprecise and uncertain. In this context, fuzzy implication models serve as a key mechanism for deriving conclusions based on conditional rules. While classical logic expresses "if... then..." statements rigidly, fuzzy logic enables more flexible and human-like interpretation through fuzzy implications.

However, existing fuzzy implications - such as those proposed by Mamdani, Gödel, Lukasiewicz, Zadeh, Aliev, Reichenbach, Kleene-Dienes, Goguen, Yager, Weber, Fodor and others face limitations in practical applications. These implications consider only imprecision, but fail to adequately address important aspects such as the reliability or degree of confidence in the information.

Therefore, there is a growing need for new approaches that extend the functional capabilities of fuzzy implication by extending it to Z-number-based implications. Z-numbers allow for the simultaneous modeling of both the vagueness of information and its trustworthiness (reliability or confidence). This leads to a more expressive and realistic reasoning results.

The actuality of this work lies in the fact that it investigates the shortcomings of existing fuzzy implication models, analyzes their current application potential, and justifies the development of new implications based on Z-numbers. This represents a timely and significant research direction, both theoretically and practically, for advancing modern fuzzy reasoning systems. The above-mentioned points determine the relevance of the dissertation work.

It should be noted that conditional reasoning is the foundation of control systems and decision-making systems. In this study we will also discuss this issue.

Object and subject of the study. The object of research is the control system. The subject of the research of dissertation is the creation Z-implication and Z-conditional reasoning approach.

Goals and objectives of the study. The objectives of the study is designing a controller using Z-conditional reasoning. The purpose of this dissertation is to formally define Z-implication, investigate its main properties, and explore its potential applications in control systems. To accomplish this goal, the following specific tasks have been defined and solved accordingly:

1) Defining drawbacks of existing conditional reasoning approaches;
2) Extending fuzzy implications to the Z-environment;

3) Creating an algorithm for Z-conditional reasoning
4) Designing a controller using Z-conditional reasoning;
5) Sensitivity analysis for the designed controller.

Research methods. To solve the problems given in the dissertation, fuzzy logic theory, Z-numbers theory, probability theory, approximate reasoning methods, aggregation and composition methods, and probabilistic implications were used. Experimental research methods were used together with mathematical and simulation modeling methods to confirm the obtained theoretical results.

The main provisions for defense. In the dissertation, the following clauses are submitted for defense:

1. Identifying shortcomings in existing conditional reasoning approaches and proposing a new approach;
2. Proposing the concept of Z-implication that satisfies the properties of implication functions;
3. Developing a conditional reasoning algorithm;
4. The synthesis of a controller based on the Z-conditional reasoning.

Scientific novelty of the research. The main scientific novelties presented in the dissertation are as follows:

1) First time formulation of the definition of Z-implication;
2) Extension of fuzzy implication to the Z-environment;
3) Creation of an algorithm for Z-conditional reasoning;
4) Extension of ALI-1 and ALI-4 logics to the Z-fuzzy environment.

Theoretical and practical significance of the research. Scientific Significance of the study. The study advances the theory of fuzzy logic and conditional inference by introducing the concept of Z-implication and formally defining it for the first time. This approach enables modeling of uncertainty and reliability in a unified framework. Thus, existing fuzzy logic systems and approximate inference methodologies are expanded and the groundwork is laid for building more comprehensive, expressive models.

Practical Significance of the study. Practically, the research enables the development of more robust control and decision making systems capable of handling uncertain, imprecise, and partially reliable information. The developed algorithms and software provide tools for real-world applications, improving decision-making accuracy and system reliability. The results of the dissertation are general in nature, the proposed algorithm can be applied in various fields of industry, economics, psychology, sociology, technical fields, etc.

Approbation of dissertation. The main scientific and practical results were discussed at the seminars of the scientific research laboratory "Intelligent control and decision-making systems in industry and economy", at the large seminar of ASOIU doctoral students dedicated to the "Science" Day, and presented at the following local and international conferences:

1.10th International Conference on Theory and Application of Soft Computing with Words and Perceptions, Pragua, Czech Republic, 2019;

2.11th World Conference “Intelligent System for Industrial Automation” (WCIS-2020), Tashkent, Uzbekistan,2020;

3.1-st UFAZ-ASOIU-UNISTRA scientific conference,Baku, 2021;

4.11th International Conference on Theory and Application of Soft Computing, Computing with Words and Perceptions and Artificial Intelligence , Antalya, Turkey, 2021;

5. The scientific conference of young researchers and PhD students dedicated to the 99th anniversary of the birth of the National Leader Heydar Aliyev, 2022.

6.15th International Conference on Applications of Fuzzy Systems, Soft Computing and Artificial Intelligence Tools, Budva, Montenegro, 2022;

7. The Republican Scientific Conference of young researchers and PhD students dedicated to the 100th anniversary of the birth of the National Leader Heydar Aliyev, Baku, Azerbaijan, 2023.

8.16-th International Conference on Applications of Fuzzy Systems, Soft Computing and Artificial Intelligence Tools,Antalya, Turkey,2023;

9. 12th International Conference on Theory and Application of Soft Computing, Computing with Words, Perceptions and Artificial Intelligence , Budva, Montenegro,2024;

10.III International Scientific-Practical Conference on Artificial Intelligence Technologies and Aerospace Issues, Baku, 2025;

11.17-th International Conference on Applications of Fuzzy Systems, Soft Computing and Artificial Intelligence Tools, İaşi, Romania, 2025.

The name of the organization where the dissertation work was performed. Azerbaijan State Oil and Industry University, “Intelligent Control and Decision-Making Systems in Industry and Economics” research laboratory.

Published scientific works. As a result of the conducted research, 15 works have been published, including: 12 without co-authorship, 9 of which were published abroad and are indexed in the SCOPUS database, and 1 in the Web of Science database; 7 conference paper (3 of them international), and 8 articles (7 published abroad and 1 in a local journal).

The volume of the dissertation’s structural sections separately and the general volume. The dissertation consists of an introduction (8737 characters), 5 chapters (Chapter I – 48893 characters, Chapter II – 11123 characters, Chapter III – 43758 characters, Chapter IV – 51316, Chapter V – 47123 characters), conclusion (1411 characters) and a list of references. The total volume of the dissertation is (212361 characters) and consists of 27 tables and 7 figures.

MAIN CONTENT OF THE WORK

The introduction presents the relevance of the topic in the field of research, the objectives set, research methods, theoretical and practical significance of the study.

In **the first chapter**, classical conditional reasoning methods based on fuzzy logic[1], as well as methods based on probabilistic logic, are analyzed. The main weaknesses (limitations) of both fuzzy and probabilistic approaches are identified. On the other hand, it is shown that there is a need to develop new concepts and approaches to overcome these shortcomings. Considering the increasing complexity and uncertainty in real-world problems, the importance of an approach capable of reasoning with information described by Z-numbers[2] has been demonstrated. It is substantiated that implication - based approaches provide a more suitable alternative for reasoning with Z-rules. In addition, this chapter presents a clear and concise explanation of the main purpose of the research and the problems that arise.

In the second chapter, the fundamental definitions of fuzzy logic and Z-number theory necessary for the formulation and development of Z-implication[3] are examined. Here are the basic concepts of fuzzy sets, operation over fuzzy sets, fuzzy relation, fuzzy composition, probabilistic set, aggregation probabilty distribution, a discrete Z-number and operation over them, Z-If-Then rules. The integration of different types of uncertainty is a widely explored direction in modern scientific literature. From this perspective, the Z-number concept is considered a potentially superior approach for the adequate integration of fuzzy and probabilistic uncertainties**.**

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1. Aliev, R.A. Soft Computing and its Application / Aliev, R.A., Aliev R.R. - Singapore: World Scientific, -2001.- 444 p.
2. Zadeh, L. A. A note on Z-numbers // Information sciences, – 2011,181(14), p. 2923-2932
3.Aliev, R.A., Ahmadov Sh.A., Gardashova L.A., Hueynov O. H.: Extension of ALI-I logic to Z-fuzzy environment. In proc.: //Lecture Notes in Networks and Systems, Springer, Cham, -2025. Vol. 1622.

The third chapter presents the first time formal definition of Z-implication. Its structural framework is justified through copula theory. It explains how classical fuzzy implications are adapted and modified to the Z-number environment. This chapter addresses a gap in the existing literature on Z-implication-based inference. The new type of implication, called Z-implication, is formed through the synergy of fuzzy and probabilistic implications. The proposed Z-implication is proved to be an implication and its main properties are analyzed Here we introduce a definition of ALI-1 and ALI-4 implication as a complex operator based on fuzzy and probabilistic implications. In a special case, the formulation is reduced to fuzzy or probabilistic implication. The basic definitions related to this problem are presented below:

Definition 1. Discrete Z-number[2]. A discrete Z-number is an ordered pair $Z=(A,B)$ of discrete fuzzy numbers *A* and *B*. *A* plays a role of a fuzzy constraint on values that a random variable X may take. *B* is a discrete fuzzy number with a membership function $\mu_B:\{b_1,...,b_n\}\to[0,1],\ \{b_1,...,b_n\}\subset[0,1]$, playing a role of a fuzzy constraint on the probability measure of *A*, *P(A)*.

$$P(A)=\sum_{i=1}^{n}\mu A(x_i)p(x_i)\,,P(A)\in supp(B)$$

Definition 2[1]. Representation of the fuzzy implication[1] Ali-1 is given below:

$$I(p,q)=\begin{cases}1-p, & \text{if } p<q\\ 1, & \text{if } p=q\\ q, & \text{if } p>q\end{cases}$$

The Ali-1 implication is a type of fuzzy implication and satisfies the properties of probabilistic implication.

Definition 3[1,4]. A function I: $[0,1]^2\to[0,1]$ is called a fuzzy implication: if it satisfies the following conditions for all p,p_1,p_2, $q,q_1,q_2\in[0,1]$, (I1) if $p_1\leq p_2$ then I(p_1,q)≥I(p_2,q), (I2) if $q_1\leq q_2$

4.Grzegorzewski, P.: Probabilistic implications// Fuzzy Sets and Systems.-2013, 226, p. 53–66.

then I(p, q_1)≤I(p, q_2), (I3) I(0,0)=1, (I4) I(1,1)=1, (I5) I(1,0)=0.

Definition 4[5]. Fuzzy composition. A max-min composition of a fuzzy set and a fuzzy relation is described as follows:

$$\mu_Q(q) = \max_q \min[\mu_P(p), \mu_R(p,q)],$$

where $\mu_R(p,q)$ is the membership function of fuzzy relation, $\mu_P(p)$ - membership function of a fuzzy set of a current input and $\mu_Q(q)$ - membership function of the output.

Definition 5[4]. A copula (specifically a 2-copula) is a function C: $[0,1]^2 \rightarrow [0,1]$ which satisfies the following conditions:

(a) C(p,0) = C(0,q) = 0 for every p,q∈ [0,1],
(b) C(p,1) = p for every p ∈ [0,1],
(c) C(1,q) =q for every q ∈ [0,1],
(d) for every p_1, p_2 , q_1, q_2 ∈ [0,1] such that $p_1 \leq p_2$ and $q_1 \leq q_2$
C(p_2 ,q_2) − C(p_2, q_1)−C(p_1, q_2) +C(p_1, q_1) ≥ 0.

For any copula C and for all p,q∈ [0,1] W(p,q) ≤ C(p,q) ≤ M(p,q), where W(p,q)=max{p +q −1,0}, M(p,q)=min{p,q} are also copulas.

Definition 6[4]. A function I_C :$[0,1]^2 \rightarrow [0,1]$ given by I_C(p,q) = 1 if p=0, C(p,q)/p, if p>0, where C is a copula, is called a probabilistic implication.

Consider a problem of formulating an implication denoted I_Z (Z-implication) given two Z-numbers $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$:

Definition 7[3]. A Z-implication I_Z may be described in terms of mapping between pairs of Z- sets $(A_1, B_1), (A_2, B_2)$ taking into account underlying sets of cumulative probability distributions:

$$G_1 = \{p: \int p_1 = p, \sum p_1 \mu_{A_1} \in B_1\},$$

5.Aliev, R. Fuzzy Process Control and Knowledge Engineering in Petrochemical and Robotic manufacturing / Aliev, R.,Aliev, F., Babaev,M. -Koln, Germany:Verlag TUV Rheinland, -1991.-146 p.

$$G_2 = \{q: \int p_2 = q, \sum p_2 \mu_{A_2} \in B_2\}\ .$$

Consequently, ALI-1 fuzzy implication (Def. 3) and probabilistic implication (Def. 6) would be a basis for formulating Z-implication I_Z as follows.

Z-implication I_Z is a vector-valued function:

$$I_Z = I_{FC}\big(k_1 I_F\big(\mu_{A_1}, \mu_{A_2}\big), k_2\{I_C(p,q): p \in G_1, q \in G_2\}\big),$$

where I_F and I_C are ALI-1 fuzzy implication and probabilistic implication respectively. We can get different probabilistic implications depending on copulas type. The second component of the vector-valued function $I_{FC} = (I_F(\mu_1, \mu_2), \{I_C(p,q)\})$ is a set of probabilistic implications $\{I_C(p,q)\}$ induced by sets of cumulative distributions G_1, G_2.

K is a binary two-dimensional column vector $K = (k_1, k_2)^T$, $k_1, k_2 \in \{0,1\}$. As special cases of Z-implication one has:

If $k_1 = 1, k_2 = 0$ then I_Z is a fuzzy implication.

If $k_1 = 0, k_2 = 1$ then I_Z is a set of probabilistic implications.

Z-implication is obtained in general case when $k_1 = 1, k_2 = 1$.

Definition 8[6] . An implication is a continuous function I from [0,1] × [0,1] into [0,1] such that $\forall p$, $p\ '$, q, $q\ '$, $r \in [0,1]$ the following properties are satisfied :

(I1) IF $p \le p\ '$ THEN I(p,q) ≥ I($p\ '$,q) (antitone in first argument),

(I2) IF q ≤ $q\ '$ THEN I(p,q) ≤ I(p,q') (monotone in second argument),

(I3) I(0,q) = 1 (falsity),

(I4) I(1,q) ≤ q (neutrality), (1)

(I5) I(p, I(q,r)) = I(q, I(p,r)) (exchange),

(I6) I(p,q) = I(n(q),n(p)) (contra positive symmetry),

where n (·) – is a negation, which could be defined as n(q) = T(¬Q) = 1- T(Q), where T(Q) is truth value of Q.

6.Aliev, R. A. Tserkovny, A.: Systemic approach to fuzzy logic formalization for approximate reasoning // Information Sciences,-2011, 181, -p.1045–1059.

Definition 9[4]. A probabilistic implication $I_c: [0,1]^2 \rightarrow [0,1]$ defined by Def. 6 satisfies the following conditions:
I(0,0)=1;
I(1,1)=1;
I(1,0)=0;
IF $q_1 \leq q_2$ THEN I(p, q_1) ≤ I(p, q_2) (2)

Definition 10[3]. Aggregation of cumulative probability distributions (probability aggregation method performed by computing the weighted sum):

$$p = \sum_{i-1}^{n} w_i \, p_i.$$

In the equation, p_i are cumulative probability distributions, w_i are positive weighting factors: in order to have a meaningful global probability, their sum must be equal to 1.

Let us consider the implication operation as

$$I(p,q) = \begin{cases} 1 - F(p,q)^{norm}, p > q \\ 1, \quad p \leq q \end{cases} \tag{3}$$

Theorem 1[6]. Let a continuous function I(p, q) is

$$I(p,q) = \begin{cases} 1 - F(p,q)^{norm}, p > q \\ 1, p \leq q \end{cases} = \begin{cases} \frac{1-p+q}{2}, p > q \\ 1, p \leq q \end{cases} \tag{4}$$

Then axioms (I1) – (I6) in (1) are satisfied and therefore this formula is an implication operation.

Proof.

(I1) $\forall p, p' \in [0,1]$ $p' \geq p > q \Longrightarrow I(p, q) - I(p', q) = 1 - p + q - 1 + p' - q = p' - p \geq 0 \Longrightarrow I(p, q) \geq I(p', q)$

Where as $q \geq p' \geq p \Longrightarrow I(p, q) - I(p', q) \equiv 0 \Longrightarrow I(p, q) \equiv I(p', q)$.

(I2) $\forall q, q' \in [0,1]$ $q \leq q' < p \Longrightarrow I(p, q) - I(p, q') = = 1 - p + q - 1 + p - q' = q - q' \leq 0 \Longrightarrow I(p, q) \leq I(p, q')$

Where as $p \leq q \leq q' \Longrightarrow I(p, q) - I(p, q') \equiv 0 \Longrightarrow I(p, q) \equiv I(p, q')$.

(I3) $I(0, q) \equiv 1$, $q \geq 0$.

(I4) $I(1, q) = \begin{cases} \frac{q}{2}, q \neq 1, \\ 1, q = 1, \end{cases} \Longrightarrow I(1, q) \leq q$.

Notice that since

(I5) $p \rightarrow q = k(1 - p + q)$, where $k = 0.5| \, p > q$ or $1 \; p \leq q$, then $\forall p, q, r \in [0,1] \mid p > q > r, \Longrightarrow p \longrightarrow (q \longrightarrow r) = k \, (p \longrightarrow (q \longrightarrow r)) = k(1 - p + 1 - q + r) = k(2 - p - q + r)$

Whereas

$k(q \longrightarrow (p \longrightarrow r)) = k(q \longrightarrow (1 - p + r)) = k(1 - q + 1 - p + r) = k(2 - p - q + r)$.

(I6) $I \; (n(q), \; n(p)) = (1 - q) \longrightarrow (1 - p) = \begin{cases} \frac{q+1-p}{2}, 1 - q > 1 - p \\ 1, 1 - q \leq 1 - \text{p} \end{cases} = \begin{cases} \frac{1-p+q}{2}, p > q \\ 1, p \leq \text{q} \end{cases}$

Theorem 2. Implication I(p, q) (4) satisfies properties of probabilistic implication.

Proof. As it is shown in (2) a probabilistic implication

$I_C : [0,1]^2 \rightarrow [0,1]$ satisfies the following conditions:

I(0,0)=1; I(1,1)=1; I(1,0)=0 (5) – (7)

IF $q_1 \leq q_2$ THEN $I(p, q_1) \leq I(p, q_2)$ (8)

Conditions (5) - (7) are satisfied (these are among conditions of fuzzy implications). Consider condition (8). In case $p > q$, (8) is satisfied because $I(p, q) = (1 - p + q)/2$ is monotonically non-decreasing w.r.t. q. In case $p \leq$ q (8) is also satisfied because $I(p, q)$ is constant.

Ali-4 (4) is Z-implication. Let's check on an example whether the Ali-4 implication is a Z-implication. Ali-4 is a probabilistic implication and satisfies the conditions (2).

Example. Properties of probabilistic implication: Let p=0 and q=0, then

$$\text{I}(0,0) = \begin{cases} 1, \quad \text{if } 0 \leq 0 \\ \\ (1\text{-p+q})/2, \qquad \text{if p} > q \end{cases}$$

The (I1) and (I2) (see matrix) properties are satisfied.

For example, assume that the created probability matrix is given as follow(Table 1):

Table 1

Probability matrix

p\q	***0,388148***	***0,709923***	***0,726526***	***0,726526***	***0,726526***	***0,726526***	***1,000000***
0,552072	0,418038	1,000000	1,000000	1,000000	1,000000	1,000000	1,000000
0,783175	0,302486	0,463374	0,471675	0,471675	0,471675	0,471675	1,000000
0,840490	0,273829	0,434716	0,443018	0,443018	0,443018	0,443018	1,000000
0,840490	0,273829	0,434716	0,443018	0,443018	0,443018	0,443018	1,000000
0,840490	0,273829	0,434716	0,443018	0,443018	0,443018	0,443018	1,000000
0,848778	0,269685	0,430572	0,438874	0,438874	0,438874	0,438874	1,000000
1,000000	0,194074	0,354961	0,363263	0,363263	0,363263	0,363263	1,000000

Let p=1 and q=0, then (I3) property:

$$I(1,0)=\begin{cases} 1, & \text{if } p\leq q \\ (1\text{-}1+0)/2=0, & \text{if } 1>0 \end{cases}$$

is satisfied.

Property 4(I4). Let $q_1 = 0.388148$ and q_2= 0.709923 then as seen from the matrix, the properties are satisfied:

If $q_1 \leq q_2$ then I(p, q_1) ≤ I(p, q_2)= 0.418038≤1.00000.

ALI-4 -fuzzy implication satisfiers (I1)-(I5) conditions (see Def.3). Let us check properties of fuzzy implication. Assume, that created relational matrix by using ALI-4 implication as follow(Table 2):

Table 2

Created relational matrix (by using ALI-4 implication)

p\q	***0,5***	***0,67***	***0,92***	***1***	***0,92***	***0,67***	***0,5***
0,2	1	1	1	1	1	1	1
0,34	1	1	1	1	1	1	1
0,73	0,385	0,47	1	1	1	0,47	0,385
1	0,25	0,335	0,46	1	0,46	0,335	0,25
0,73	0,385	0,47	1	1	1	0,47	0,385
0,34	1	1	1	1	1	1	1
0,2	1	1	1	1	1	1	1

Properties (I1)-(I5) are satisfied:

(I1) if $p_1 \leq p_2$ then I(p_1,q)≥I(p_2,q): 0.34≤0.73,

then 1≥0.385, 1≥0.47, …,1≥1.
(I2) if $q_1 \leq q_2$ then I(p, q_1)≤I(p, q_2): 0.67≤0.92, then 1≤1, 0.47≤1,0.335≤0.46, 0.47≤1.

(I3) I(0,0)=1: $\begin{cases} 1, & \text{if } p \leq q \\ (1\text{-}p+q)/2, & \text{if } p > q \end{cases}$

(I4) I(1,1)=1: $\begin{cases} 1, & \text{əgər } p \leq q \\ (1\text{-}p+q)/2, & \text{if } p > q \end{cases}$

(I5) I(1,0)=0:

$$I(1,0) = \begin{cases} 1, & \text{əgər } p \leq q \\ (1\text{-}1+0)/2=0, & \text{if } 1 > 0 \end{cases}$$

Ali-1 is Z implication.

Example. Assume, that Ali-1 implication and 1 rule IF E THEN U are given:

$$\mu_E(e) = 1.00/-10 + 0.73/-7 + 0.34/-3 + 0.20/0 + 0.13/3 + 0.08/7 + 0.06/10$$

$$\mu_U(u) = 1.00/-1 + 0.92/-0.7 + 0.67/-0.3 + 0.50/0 + 0.37/0.3 + 0.26/0.7 + 0.20/1$$

The created fuzzy relation matrix over Ali-I implication are below(Table 3):

Table 3

Fuzzy relation matrix R

	1	***0.92***	***0.67***	***0.5***	***0.37***	***0.26***	***0.2***
1	1	0.92	0.67	0.5	0.37	0.26	0.2
0.73	0.27	0.27	0.67	0.5	0.37	0.26	0.2
0.34	0.66	0.66	0.66	0.66	0.66	0.26	0.2
0.2	0.8	0.8	0.8	0.8	0.8	0.8	1
0.13	0.87	0.87	0.87	0.87	0.87	0.87	0.87
0.08	0.92	0.92	0.92	0.92	0.92	0.92	0.92
0.06	0.94	0.94	0.94	0.94	0.94	0.94	0.94

Let's look at the verification of the fuzzy implication properties below.

Example .

(I1): if $p_1 \le p_2$ then I(p_1,q)≥I(p_2,q),

Value of variables and implication are: p_1=0.13, p_2=0.2,

I(p_1,q)=0.87, I(p_2,q)=0.8

0.13<=0.2 and 0.87>= 0.8

it satisfies the (I1) conditions for all p, p_1, p_2,q, q_1, q_2∈[0,1] (see Table 3)

(I2): if $q_1 \le q_2$ then I(p, q_1)≤I(p, q_2),

Let q_1=0.67 and q_2= 0.92, I(p, q_1)=0.67 and I(p, q_2)=0.92,

q_1=0.67≤q_2= 0.92 then I(p, q_1)=0.67≤I(p, q_2)=0.92,

it satisfies the (I2) conditions for all p,p_1, p_2,q, q_1,q_2 ∈[0,1] (see Table 3)

(I3): I(0,0)=1,

Assume that p_1=0 and q_1=0 then

In Ali-1 $I(p,q)=\begin{cases} 1-p, & \text{if } p<q \\ 1, & \text{if } p=q \\ y, & \text{if } p>q \end{cases}$ implication if p=0 and q=0 Then

I (p,q)= 1or I(0,0)=1

it satisfies the (I3) conditions for all p,p_1, p_2,q, q_1,q_2 ∈[0,1]

(I4): I(1,1)=1,

p=1 and q=1

it satisfies the (I4) conditions for all p,p_1, p_2,q, q_1,q_2 ∈[0,1], (see Table 3)

(I5): I(1,0)=0,

if p=1 and q=0 , p>q then I(1,0)=0

it satisfies the (I5) conditions for all p,p_1, p_2,q, q_1,q_2 ∈[0,1].

We are getting: If $k_1 = 1, k_2 = 0$ then I_z is a fuzzy implication.

Example .

If $k_1 = 0, k_2 = 1$ then I_z is a probabilistic implication.

Let's look at the verification of the probabilistic implication properties below.

Assume that there are 2 distributions:
(0,0.07564,0.567815,0.348348, 0.008197,0,0) and
(0.164606, 0.313403, 0.276772. 0.217191, 0.028028, 0).

In accordance, cumulative probability on distributions are:

p=0.8(input):	q=0.8(output):
0	
0.07564	0.164606
0.643455	0.478009
0.991803	0.754781
1	0.971972
1	1
1	1

The copula constructed using the distributions is given in Table 4.

Table 4

Copula

p\q	***0***	***0.164606***	***0.478009***	***0.754781***	***0.971972***	***1***	***1***
0	0	0	0	0	0	0	0
0.07564	0	0	0	0	0.047612	0.07564	0.07564
0.643455	0	0	0.121464	0.398236	0.615427	0.643455	0.643455
0.991803	0	0.156409	0.469812	0.746584	0.963775	0.991803	0.991803
1	0	0.164606	0.478009	0.754781	0.971972	1	1
1	0	0.164606	0.478009	0.754781	0.971972	1	1
1	0	0.164606	0.478009	0.754781	0.971972	1	1

Example .

Let p=0.07564 and q=0.478009.
Using copula equation $W(p,q)=\max\{p+q-1,0\}$, we are getting:
$W(p,q)=\max\{0.07564+0.478009-1,0\}=\max\{0.553649-1,0\}=0$
$M(p,q)=\min\{p,q\})=(0.07564,0.478009)=0.07564$
$W(p,q) \le C(p,q) \le M(p,q)$ satisfiers: $0 \le 0 \le 0.07564$.

(d): for every $p_1, p_2, q_1, q_2 \in [0,1]$ such that $p_1 \leq p_2$ and $q_1 \leq q_2$ C(p_2, q_2) − C(p_2, q_1) −C(p_1, q_2) +C$(p_1, q_1) \geq 0$.
If p_1=0.643455, p_2=0.991803, q_1=0.478009, q_2=0.754781 then we obtain C(p_2, q_2) − C(p_2, q_1) −C(p_1, q_2) +C(p_1, q_1)=0.746584 - 0.469812 -0.398236 + +0.121464 >=0

An Ali-1 probabilistic implication $I_C : [0,1]^2 \rightarrow [0,1]$ for any copula C satisfies the following conditions:

1) I(0,0)=1
2) I(1,1)=1
3) I(1,0)=0
4) If $q_1 \leq q_2$ then I(p, q_1) ≤ I(p, q_2).

Example. Consider relation matrix (see Figure 1).
If q_1=0.102587, q_2=0.25778 then we get
I(p, q_1) ≤ I(p, q_2):
1≤1;
0.102587≤0.864042
0.102587≤0.25778

	0.102587	0.25778	0.467194	0.672929	0.851013	0.992232	1
0	1	1	1	1	1	1	1
0.135958	0.102587	0.864042	0.864042	0.864042	0.864042	0.864042	0.864042
0.583634	0.102587	0.25778	0.467194	0.416366	0.416366	0.416366	0.416366
0.892009	0.102587	0.25778	0.467194	0.672929	0.851013	0.107991	0.107991
1	0.102587	0.25778	0.467194	0.672929	0.851013	0.992232	1
1	0.102587	0.25778	0.467194	0.672929	0.851013	0.992232	1
1	0.102587	0.25778	0.467194	0.672929	0.851013	0.992232	1

Figure 1.Relation matrix

Example.
If q_1=0.467194, q_2=0.672929 then we get
I(p, q_1) ≤ I(p, q_2):
1≤1;
0.864042≤0.864042
0.467194≤0.672929

So, Ali-1 implication satisfiers the properties of probabilistic implication.

Fourth chapter considers Z-conditional algorithm and development of software. Currently, there is almost no information in the scientific literature about Z-conditional reasoning**,** especially regarding logical inference based on Z-implication**.** Processing of Z-rules requires the use of a new type of implication**.** However, this problem has not been discussed in the scientific literature**.** The new type of implication, called Z-implication, is formed through the synergy of fuzzy (Ali-1) and probabilistic implications (see chapter 3). The related algorithm that enables inference using Z-valued If–Then rules is proposed. For simplicity let us consider reasoning for SISO model.Assume If-Then rules and current observation are given:

Rule i: If X is $Z_{ix}(A_{ix}, B_{ix})$ Then Y is $Z_{iy}(A_{iy}, B_{iy}) i = \overline{1,n}$,

X is $Z_x'(A'_x, B'_x)$

The problem is to compute a Z-number-based value of Y: $Z_y'\left(A'_y, B'_y\right)$.

The algorithm[7] for solving this problem is described below:

Step 1. Using Def. 7, compute a relation between Z- input Z(A_{ix}, B_{ix}) and Z-output Z(A_{iy}, B_{iy}) for each rule i=1,...,n:

Step1.1. Set k_1=1, k_2=0, to apply fuzzy implication $I_{Fi}\left(\mu_{A_{ix}}, \mu_{A_{iy}}\right)$ (Def. 2) for each rule i=1,...,n.

Step1.2 Set k_1=0, k_2=1 to apply probabilistic implications $\left\{I_{Ci}(p,q): p \in G_{ix}, q \in G_{iy}\right\}$. For this purpose, at first it is needed to extract sets of cumulative probability distributions for Z(A_{ix}, B_{ix}) and Z(A_{iy}, B_{iy}):

$$G_{ix} = \{p: \int p_{ix} = p, \sum p_{ix}\mu_{A_{ix}} \in B_{ix}\},$$

7.Ahmadov, S.A. Z-implication and its application // III International Scientific and Practical Conference on Artificial Intelligence Technologies and Aerospace, Baku, -2025, - p.3-9

$$G_{iy} = \{q: \int p_{iy} = q, \sum p_{iy}\mu_{A_{iy}} \in B_{iy}\}$$

Next, apply probabilstic implication $I_{Ci}(p,q)$ for each

$p \in G_{ix}, q \in G_{iy}$.

Step 2. Aggregate the computed fuzzy and probabilistic matrices of all the rules i=1,...,n:

Step 2.1. Compute the union of matrices of all the rules by using disjunction operator of Ali-1 logic:

$$I_F = \cup_i I_{Fi}\left(\mu_{A_{ix}}, \mu_{A_{iy}}\right).$$

Step 2.2 Aggregate probabilistic matrices of all the rules by using Def. 10: $\{I_C = \sum_i^n w_i I_{Ci}(p,q)\ |p \in G_{i1}, q \in G_{i2},\ \sum_i^n w_i = 1\}$.

Thus, the aggregation matrices are obtained: $(I_F, \{I_C\})$.

Step 3. Perform composition operation of given current input Z(A'_x, B'_x) and obtained aggregated matrices $(I_F, \{I_C\})$:

Step 3.1. Perform composition for A part: $A'_y = A'_x \circ I_F$, where $\circ$ is disjunction- conjunction composition (Ali-1 logic).

Step 3.2. Given the set of distributions

$$G'_x = \{p': \int p'_x = p', \sum p'_x \mu_{A'_x} \in B'_x\}$$

for Z(A'_x, B'_x), and the set of aggregated probabilistic implications $\{I_C\}$, compute the set of distributions G'_y:

Step 3.2.1. For each aggregated matrix of cumulative probability distributions $I_C(p,q) \in \{I_C\}$ obtain the corresponding matrix of probability distributions $I_p(p_x, p_y)$.

Step 3.2.2. Compute G'_y given $\{I_p\}$ and G'_x:

$$G'_y = \{p'_y = p'_x \circ I_{Cp}\left(p_x, p_y\right): p'_x \in G'_x, I_p\left(p_x, p_y\right) \in \{I_p\}\},$$

where $\circ$ is disjunction- conjunction composition (Ali-1 logic).

Step 4. Given A'_y and G'_y, compute B'_y to find Z(A'_y, B'_y):

$$B' = \{(\mu_{b'_y}(b'_y), (b'_y)): b'_y = P\left(A'_y\right) = \sum \mu_{A'_y} p'_y, p'_y \in G'_y\}.$$

Membership function $\mu_{b'_y}$ is obtained based on $\mu_{b'_x}$ and $\mu_{b_{ix}}$,i=1,…,n by using Zadeh's extension principle.

Thus, $Z'_y(A'_y, B'_y)$ is computed.

Example. An application of the concept to If-Then rule-based control problem is considered. In the problem, SISO model with Z-number-based antecedent and consequent is used. For the sake of simplicity, it is shown the use of the approach for computation of the resulting Z^+-number-based output given Z^+-number-based input.

Z-number based control system is represented by production rules, consisted of one input and one output.

Let's assume that 7 rules[5] are involved in Z-valued control system:

1.If the error e is [negative big (NB), Very sure] THEN the control action u is [negative big (NB), Very sure];

2.If the error e is [negative medium (NM), Very sure] THEN the control action u is [negative medium (NM), Very sure];

3.If the error e is [negative small (NS), Very sure] THEN the control action u is [negative small (NS), Very sure];

4.If the error e is [zero (ZE), Very sure] THEN the control action u is [zero, Very sure];

5.If the error e is [positive small (PS), Very sure] THEN the control action u is [positive small (PS), Very sure];

6.If the error e is [positive medium (PM), Very sure] THEN the control action u is [positive medium (PM), Very sure];

7.If the error e is [positive big (PB), Very sure] THEN the control action u is [positive big (PB), Very sure];

where Very sure= 0.2/0.8+1/0.9+0.2/1.

Error of Z-valued control system is denoted e, control actions are described as u. By membership functions rule base of control system is described as:

Rule[5] 1

$$\mu_{E1}(e) = ((1.00/-10 + 0.73/-7 + 0.34/-3 + 0.20/0 + 0.13/3 + 0.08/7 + 0.06/10)), (0.2/0.8 + 1/0.9 + 0.2/1))$$

$\mu_{U1}(u) = ((1.00/{-1} + 0.92/{-0.7} + 0.67/{-0.3} + 0.50/0 + 0.37\,/0.3 + 0.26/0.7 + 0.20/1), (0.2/0.8 + 1/0.9 + 0.2/1)\ )$

… … ….

Rule[5] 7:

$\mu_{E7}(e) = ((0.06/{-10} + 0.08/{-7} + 0.13/{-3} + 0.20/0 + 0.34/3 + 0.73/7 + 1.00/10),), (0.2/0.8 + 1/0.9 + 0.2/1)\ )$

$\mu_{U7}(u) = ((0.20/{-1} + 0.26/{-0.7} + 0.37/{-0.3} + 0.50/0 + 0.67/0.3 + 0.92/0.7 + 1.00/1), (0.2/0.8 + 1/0.9 + 0.2/1)\ )$

Problem is to perform Z inference given a new input.

Assume new input:

Z=$\mu_E(e_{ic}^{c}) = (0.11./{-10} + 0.17/{-7} + 0.34/{-3} + 0.61/0 + 0.96/3 + 0.73/70.41/10, very\ sure)$

Fragments of created fuzzy relation matrices using ALI-I implication are represented in Tables 5-7.

Table 5

Fuzzy relation matrix R1

	1	***0.92***	***0.67***	***0.5***	***0.37***	***0.26***	***0.2***
1	1	0.92	0.67	0.5	0.37	0.26	0.2
0.73	0.27	0.27	0.67	0.5	0.37	0.26	0.2
0.34	0.66	0.66	0.66	0.66	0.66	0.26	0.2
0.2	0.8	0.8	0.8	0.8	0.8	0.8	1
0.13	0.87	0.87	0.87	0.87	0.87	0.87	0.87
0.08	0.92	0.92	0.92	0.92	0.92	0.92	0.92
0.06	0.94	0.94	0.94	0.94	0.94	0.94	0.94

Table 6

Fuzzy relation matrix R2

	0.91	***1***	***0.86***	***0.67***	***0.5***	***0.34***	***0.26***
0.73	0.27	0.27	0.27	0.67	0.5	0.34	0.26
...	...	...	...	...	...	...	...
0.11	0.89	0.89	0.89	0.89	0.89	0.89	0.89
0.08	0.92	0.92	0.92	0.92	0.92	0.92	0.92

Table 7

Fuzzy relation matrix R7

	0.2	***0.26***	***0.37***	***0.5***	***0.67***	***0.92***	***1***
0.06	0.94	0.94	0.94	0.94	0.94	0.94	0.94
0.08	0.92	0.92	0.92	0.92	0.92	0.92	0.92
...	...	...	...	...	...	...	...
1	0.2	0.26	0.37	0.5	0.67	0.92	1

Aggregation of relations is realized by using logical connective of Ali-1[1]. The result is given in Table 8.

Table 8

Aggregated fuzzy relation matrix

0.92	0.92	0.92	0.92	0.92	0.92	0
0.89	0.89	0.89	0.89	0.89	0.89	0
...	...	...	...	...	...	...
0.37	0.5	0.39	0.67	0.86	1	0.91
0.66	0	0.5	0.67	0.66	0.66	0.66

The next step is calculating B part of the Z-number-based output. For the sake of simplicity, we will construct Z^+-number-based output (instead of B part, we consider one distribution).

In this case we define one probability distribution over input and output for all rules.

Probability distributions is calculated using goal programming method. Fragment of the obtained distributions over inputs and outputs of two rules are given below(Tables 9,10):

Table 9

Obtained distributions over rule 1

input	output	Cum Prob.	
0.787008	0	0.787008	0
0	0.791118	0.787008	0.791117
0	0	0.787008	0.791117
0	0	0.787008	0.791117
0	0	0.787008	0.791117
0	0	0.787008	0.791117
0.212992	0.208883	1	1

Table 10

Obtained distributions over rule 7

input	output	Cum.prob	Cum.prob	New input	Cum.prob
0.212992	0.208884	0.212992	0.208884	0	0
0	0	0.212992	0.208884	0	0
…	…	…	…	…	…
0.787008	0	1	1	0	1

Now we need to construct probabilistic implications for each rule.

For example, for distributions of rule, the probabilistic implication is shown in Table 11:

Table 11

Probabilistic implication for rule 1

-1E-06	0.212992	0.212992	0.212992	0.212992	0.212992	0.212992
-1E-06	0.212992	0.212992	0.212992	0.212992	0.212992	0.212992
...	...	...	...	...	...	...
-1E-06	0.791117	0.791117	0.791117	0.791117	0.791117	1

The result of aggregation of probabilistic implications for all the rules is shown in Table 12:

Table 12

Aggregated of probabilistic implications

-1E-06	0.066675	0.904096	0.904096	0.904096	0.904096	0.904096
-1E-06	0.066675	0.904096	0.904096	0.904096	0.904096	0.904096
...	...	...	...	...	...	...
-1E-06	0.212992	0.212992	0.475534	0.630216	0.913349	0.05621
-1E-06	0.34696	0.467194	0.475534	0.630216	0.960138	1

Next, given the new input, aggregated fuzzy matrix (Table 8) and aggregated probabilistic implications (Table 12), we need to compute the resulting $Z^{+}(A,p)$.

For this purpose, the formulated Z-implication is used. A part is computed by using Ali-I implication, and *p* is computed by probabilistic implication.

As the latter produced cumulative probability distribution (0, 0.34696, 0.467194, 0.657509, 0.657509, 0.960138,1), it is converted to probabilistic distribution (0, 0.346961, 0.120234, 0.190315, 0,0.302629, 0.039862). The obtained result is given as follows:

$Z^{+}(A,p)$=((0.5/-1,0.61/-0.7,0.5/-0.3,0.67/0,0.73/0.3,0.73/0.7,0.73/1), (0, 0.346961, 0.120234, 0.190315,0,0.302629,0.039862))

Also, in chapter 4 is discussed testing of software.To demonstrate the practical applicability of the proposed Z-implication based reasoning model, a software prototype was developed using Microsoft Excel.

In the fifth chapter, we will design controller using on base of complex Z-approach. Namely, the application of Z-conditional reasoning to control systems and sensitivity analysis of the given model was considered.

A control system consists of a controller, an object, and a comparison block (Figure 2). Let's consider proportional controller. It is characterized by two parameters:

error (e=[-10,10]) and control action (u=[-1 1]).

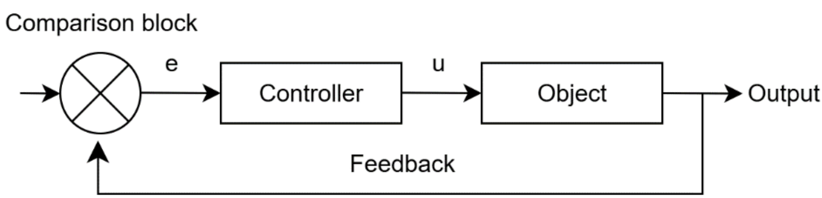

Figure 2. Structure of a control system

Let us presume that the controller is described by the following three Z-rules :

1.If the error *e* is [negative small (NS), very sure] THEN the control action *u* is [negative small (NS),very sure].

2.If the error *e* is [zero (ZE),not very sure] THEN the control action *u* is [zero, not very sure].

3.If the error e is [positive small (PS),sure] THEN the control action u is [positive small (PS),not very sure].

$$not\ \ very\ \ sure = \left\{\frac{0.1}{0.6}, \frac{1}{0.7}, \frac{0.5}{0.8}\right\},\ \ sure = \left\{\frac{0.2}{0.75}, \frac{1}{0.8}, \frac{0.4}{0.9}\right\},$$
$$very\ sure = \left\{\frac{0.3}{0.8}, \frac{1}{0.85}, \frac{0.3}{0.9}\right\}$$

The input parameter is error of control system e, the control actions is u. Rules through Z-information can be represented as follows:

Given 3 rules:

$$\begin{array}{l} IF\ \text{e is } Z_1 \text{ THEN u is } Z_2 \\ IF\ \text{e is } Z_3 \text{ THEN u is } Z_4 \\ IF\ \text{e is } Z_5 \text{ THEN u is } Z_6 \end{array} \quad \text{or} \quad \begin{array}{l} IF\ \text{e is (NS, VS) THEN u is (NS, VS)} \\ IF\ \text{e is (Z, NVS) THEN u is (Z, NVS)} \\ IF\ \text{e is (PS, S) THEN u is (PS, NVS)} \end{array}$$

where Z-numbers are:

$$Z_1 = \left(\left\{\frac{0.29}{-10}, \frac{0.56}{-7}, \frac{1}{-3}, \frac{0.68}{0}, \frac{0.36}{3}, \frac{0.15}{7}, \frac{0.08}{10}\right\}, \left\{\frac{0.3}{0.8}, \frac{1}{0.85}, \frac{0.3}{0.9}\right\}\right)$$

$$Z_2 = \left(\left\{\frac{0.56}{-1}, \frac{0.81}{-0.7}, \frac{1}{-0.3}, \frac{0.87}{0}, \frac{0.69}{0.3}, \frac{0.45}{0.7}, \frac{0.32}{1}\right\}, \left\{\frac{0.3}{0.8}, \frac{1}{0.85}, \frac{0.3}{0.9}\right\}\right)$$

$$Z_3 = \left(\left\{\frac{0.15}{-10}, \frac{0.29}{-7}, \frac{0.68}{-3}, \frac{1}{0}, \frac{0.68}{3}, \frac{0.29}{7}, \frac{0.15}{10}\right\}, \left\{\frac{0.1}{0.6}, \frac{1}{0.7}, \frac{0.5}{0.8}\right\}\right)$$

$$Z_4 = \left(\left\{\frac{0.45}{-1}, \frac{0.62}{-0.7}, \frac{0.87}{-0.3}, \frac{1}{0}, \frac{0.87}{0.3}, \frac{0.62}{0.7}, \frac{0.45}{1}\right\}, \left\{\frac{0.1}{0.6}, \frac{1}{0.7}, \frac{0.5}{0.8}\right\}\right)$$

$$Z_5 = \left(\left\{\frac{0.08}{-10}, \frac{0.15}{-7}, \frac{0.36}{-3}, \frac{0.68}{0}, \frac{1}{3}, \frac{0.56}{7}, \frac{0.29}{10}\right\}, \left\{\frac{0.2}{0.75}, \frac{1}{0.8}, \frac{0.4}{0.9}\right\}\right)$$

$$Z_6 = \left(\left\{\frac{0.32}{-1}, \frac{0.45}{-0.7}, \frac{0.69}{-0.3}, \frac{0.87}{0}, \frac{1}{0.3}, \frac{0.81}{0.7}, \frac{0.62}{1}\right\}, \left\{\frac{0.1}{0.6}, \frac{1}{0.7}, \frac{0.5}{0.8}\right\}\right)$$

The problem is to perform Z-conditional reasoning. Let a new input be:

$$e \text{ is } \left(\left\{\frac{0.09}{-10}, \frac{0.15}{-7}, \frac{0.32}{-3}, \frac{0.59}{0}, \frac{0.94}{3}, \frac{0.71}{7}, \frac{0.39}{10}\right\}, \left\{\frac{0.2}{0.75}, \frac{1}{0.8}, \frac{0.4}{0.9}\right\}\right)$$

For this Z-number based input, find the Z-value of u.

According to the algorithm (Chapter 4), we compute a relation between Z- input $Z(A_{ix}, B_{ix})$ and Z-output $Z(A_{iy}, B_{iy})$ for each rule

i=1,...,3 (step 1). Namely, we apply fuzzy and probabilistic implications. The fuzzy matrix obtained for the first rule (step 1.1) is shown in Table 13.

Table 13

Fuzzy relation matrix R1

	0,56	***0,81***	***1***	***0,87***	***0,69***	***0,45***	***0,32***
0,29	0,71	0,71	0,71	0,71	0,71	0,71	0,71
0,56	1	0,44	0,44	0,44	0,44	0,45	0,32
1	0,56	0,81	1	0,87	0,69	0,45	0,32
0,68	0,56	0,32	0,32	0,32	0,32	0,45	0,32
0,36	0,64	0,64	0,64	0,64	0,64	0,64	0,32
0,15	0,85	0,85	0,85	0,85	0,85	0,85	0,85
0,08	0,92	0,92	0,92	0,92	0,92	0,92	0,92

Further, probabilistic implications $\{I_{Ci}(p,q): p \in G_{ix}, q \in G_{iy}\}$ are applied (Step1.2). At first, we need to extract probability distributions for Z-number-based values of e and u. Given a Z-number Z=(A,B) it is needed to find probability distributions p satisfying: $\sum_i p(x_i)\mu_A(x_i) = b$,where $b \in B$. The selected support points $b \in B$ of the inputs and outputs are (0.8, 0.85,0.9) and (0.8,0.85,0.9) respectively.

The distributions are obtained for these points by solving goal programming problem. The obtained distrubutions for the first and the third rules are given below (Table 14,15) :

Table 14

The obtained distrubutions for the first rule

Giriş üçün paylamalar, p_x			**Çıxış üçün paylamalar, p_y**		
0,8	**0,85**	**0,9**	**0,8**	**0,85**	**0,9**
0	0	0	0,052444903	0	0
0,133333	0,075641	0,017949	0,175624926	0,164 606	0,046413
0,50785	0,567814	0,694231	0,253120487	0,313403	0,448053
0,275965	0,348347	0,287821	0,229063992	0,276772	0,364077
0,082852	0,008198	0	0,18992893	0,217191	0,141458
0	0	0	0,099816762	0,028028	-1E-06
-1E-06	0	0	0	0	0

Table 15

The obtained distrubutions for the third rule

Giriş üçün paylamalar, p_x			Çıxış üçün paylamalar, p_y		
0,75	**0,8**	**0,9**	**0,6**	**0,7**	**0,8**
0	0	0	0,275779388	0,175553	0,052519
0	0	0	0,04048479	0,045768	0,06187
0,123583	0,06589	0	0,067849702	0,096793	0,146992
0,271424	0,309886	0,287821	0,08837339	0,135062	0,210834
0,413968	0,490892	0,694231	-1E-06	0,169435	0,261308
0,191026	0,133332	0,017949	0,225121918	0,186951	0,231467
0	0	0	0,302392813	0,190437	0,03501

For the obtained probability distributions, the cumulative probability distributions p, q are constructed and the probabilstic implications $I_{Ci}(p, q), p \in G_{ix}, q \in G_{iy}$ are applied.

Some of the obtained matrices (Figure 3) for the first rule are given below(using cumulative probability):

	0,052445	**0,22807**	**0,48119**	**0,710254**	**0,900183**	**1**	**1**
0	1	1	1	1	1	1	1
0,133333	0,052445	0,866667	0,866667	0,866667	0,866667	0,866667	0,866667
0,641183	0,052445	0,22807	0,48119	0,358817	0,358817	0,358817	0,358817
0,917149	0,052445	0,22807	0,48119	0,710254	0,900183	0,082851	0,082851
1,00000	0,052445	0,22807	0,48119	0,710254	0,900183	1	1
1,00000	0,052445	0,22807	0,48119	0,710254	0,900183	1	1
1	0,052445	0,22807	0,48119	0,710254	0,900183	1	1

	0	**0,164606**	**0,478009**	**0,754781**	**0,971972**	**1**	**1**
0	1	1	1	1	1	1	1
0,075641	0	0,924359	0,924359	0,924359	0,924359	0,924359	0,924359
	...	...	...	...	...	...	...
1	0	0,164606	0,478009	0,754781	0,971972	1	1
1	0	0,164606	0,478009	0,754781	0,971972	1	1

	0,052519	**0,114389**	**0,261381**	**0,472215**	**0,733522**	**0,96499**	**1**
0	1	1	1	1	1	1	1
	...	...	...	...	...	...	...
0,866668	0,052519	0,114389	0,261381	0,472215	0,733522	0,133332	0,133332
1	0,052519	0,114389	0,261381	0,472215	0,733522	0,96499	1
1	0,052519	0,114389	0,261381	0,472215	0,733522	0,96499	1

Figure 3. Obtained matrices for the first rule

At step 2, the computed fuzzy and probabilistic matrices of all the rules i=1,...,3 are aggregated. According to step 2.1, the aggregated fuzzy matrix is obtained by using disjunction operation of Ali-1 logic (Table 16):

$$\vee_{ALI-1}(p,q)=\begin{cases} p & p+q<1 \\ 1 & \mathrm{p+q=1} \\ \mathrm{q} & \mathrm{p+q>1} \end{cases}$$

Table 16

The aggregated fuzzy matrix

0,92	0,92	0,92	0,92	0,92	0,92	0,92
0,85	0,85	0,85	0,85	0,85	0,85	0,85
...	...	...	...	...	...	...
0,32	0,45	0,44	0,44	0,44	0,44	0,44
0,71	0,71	0,71	0,71	0,71	0,71	0,71

Further, using the weighted sum method (Def. 3), we aggregate the probability matrices over all rules (step 2.2). The results are shown in (Table 17)

Table 17

Aggregated probability matrices over all rules

0,992473238	0,992473238	0,992473238	0,992473238	0,992473238	0,992473238	0,992473238
0,620184764	0,891592198	0,891592198	0,891592198	0,891592198	0,891592198	0,891592198
...	...	...	...	...	...	...
0,095283049	0,223077958	0,417143989	0,608879896	0,777472565	0,865697794	0,412306073
0,095283049	0,223077958	0,417143989	0,608879896	0,777472565	0,865697794	0,679696207
0,095283049	0,223077958	0,417143989	0,608879896	0,777472565	0,865697794	1
...	...	...	...	...	...	...
0,03501265	0,136532111	0,311757832	0,53332271	0,767565766	0,66943672	0,058410378
0,03501265	0,136532111	0,311757832	0,53332271	0,767565766	0,946655855	1
0,03501265	0,136532111	0,311757832	0,53332271	0,767565766	0,946655855	1

At step 3, composition operation is to be performed for the given current input $\mathrm{Z}(A'_e, B'_e)$ and aggregated matrices $(I_F, \{I_C\})$. A part of Z-valued output is calculated according to step 3.1 as (Table 18):

$$\mu_U(u) = \max_u \min[\mu_E(e_{ic}^c), \mu_R(u,e)]$$

Table 18

Result of composition operation

e1(new input)							
0,09	0,09	0,09	0,09	0,09	0,09	0,09	0,09
0,15	0,15	0,15	0,15	0,15	0,15	0,15	0,15
...	...	...	...	...	...	...	...
0,39	0,39	0,39	0,39	0,39	0,39	0,39	0,39
	0,45	***0,45***	***0,69***	***0,87***	***0,94***	***0,81***	***0,44***
Result	***-1***	***-0,7***	***-0,3***	***0***	***0,3***	***0,7***	***1***

Thus, the result is: $A_u = \left(\left\{\frac{0.45}{-1}, \frac{0.45}{-0.7}, \frac{0.69}{-0.3}, \frac{0.87}{0}, \frac{0.94}{0.3}, \frac{0.81}{0.7}, \frac{0.44}{1}\right\}\right)$.

Now, by using the max–min composition of the aggregated probability matrices and the probability distributions of the new current input (step 3.2, Table 19), we computed probability distributions of the resulting output (Table 20).

Table 19

New input

Paylanmalar				Kumulyativ paylanmalar		
0,7	**0,8**	**0,9**		**0,7**	**0,8**	**0,9**
0	0	0		0,0000	0,0000	0,0000
0,042575	-1E-06	-1E-06		0,0426	0,0000	0,0000
0,116257	0,038604	0		0,1588	0,0386	0,0000
0,199474	0,274913	0,154821		0,3583	0,3135	0,1548
0,334469	0,513237	0,819908		0,6928	0,8268	0,9747
0,264072	0,173247	0,025272		0,9568	1,0000	1,0000
0,043152	0	0		1,0000	1,0000	1,0000

Table 20

Computed probability distributions of the resulting output

P1:	0,187848375	0,11958306	0,14496842	0,137132092	0,075242536	0,130243982	0,204981535
P2:	0,095283049	0,127794909	0,194066031	0,191735907	0,168592669	0,088225228	0,134302206
P3:	0,03501265	0,101519461	0,175225721	0,221564879	0,234243056	0,179090089	0,053344145

At step 4, calculation of B part is performed. For computed probability distributions (Table 20) and A part

$A_u = \left(\left\{\frac{0.45}{-1}, \frac{0.45}{-0.7}, \frac{0.69}{-0.3}, \frac{0.87}{0}, \frac{0.94}{0.3}, \frac{0.81}{0.7}, \frac{0.44}{1}\right\}\right)$ of the resulting output, the corresponding B part is defined:

$$B_u = \left(\left\{\frac{0.1}{0.62409476}, \frac{1}{0.690133396}, \frac{0.3}{0.76382951220365}, \right)\right.$$.

Thus, $Z(A_u, B_u)$ is computed.

Also in this chapter design and application of 7 rules-based control system is discussed(using Ali-4 Z implication). Let's assume that these following 7 rules[5] are involved in Z-control system:

If the error e is [negative big (NB),small sure] THEN the control action u is [negative big (NB), almost sure];

If the error e is[negative medium (NM), sure] THEN the control action u is [negative medium (NM),sure];

If the error e is [negative small (NS), very sure] THEN the control action u is [negative small(NS),very sure];

If the error e is [zero (ZE),moderately sure] THEN the control action u is [zero,moderately sure];

If the error e is [positive small (PS),sure] THEN the control action u is [positive small (PS), moderately sure];

If the error e is [positive medium (PM),very sure] THEN the control action u is [positive medium (PM), sure];

If the error e is [positive big (PB),moderately sure] THEN the control action u is[positive big (PB), sure].

Here, errors for Z-control system are defined with *e*, control actions are described as *u*. Through rules membership function can be computed as follows:

almost sure $= 0.3/0.7 + 1/0.73 + 0.2/0.75$;

moderately sure $= 0.2/0.5 + 1/0.6 + 0.4/0.8$;

sure $= 0.1/0.7 + 1/0.8 + 0.3/0.9$;

very sure $= 0.2/0.8 + 1/0.9 + 0.2/1$;

Values of e :

negative big=1.00/-10+0.73/-7+0.34/-3+0.20/0+0.13/3+

+0.08/7+0.06/10;
negative medium=0.73/-10+1/-7+0.61/-3+0.34/0+0.20/3+ 0.11/7+ +0.08/10;
negative small=0.34/-10+0.61/-7+1/-3+0.73/0+0.41/3+0.20/7+ +0.13/10;
zero=0.20/-10+0.34/-7+0.73/-3+1/0+0.73/3+0.34/7+0.20/10;
positive small=0.13/-10+0.20/-7+0.41/-3+0.73/0+1/3+0.61/7+ +0.34/10;
positive medium=0.08/-10+0.11/-7+0.20/-3+0.34/0+0.61/3+1.00/7+ +0.73/10;
positive big=0.06/-10+0.08/-7+0.13/-3+0.20/0+0.34/3+0.73/7+ +1.00/10.

Values of u:
negative big=1.00/-1+0.92/-0.7+0.67/-0.3 + +0.50/0+0.37/0.3+ +0.26/0.7 +0.20/1;
negative medium= 0.91/-1+1/-0.7+0.86/-0.3+ +0.67/0+0.5/0.3+ +0.34/0.7+0.26/1;
negative small= 0.61/-1+0.86/-0.7+1/-0.3+ +0.92/0+0.74/0.3+ +0.5/0.7+0.37/1;
zero= 0.50/-1+0.67/-0.7+0.92/-0.3+ +1.00/0+0.92/0.3+ +0.67/0.7+0.50/1;
moderately= 0.37/-1+0.50/-0.7+0.74/-0.3+ +0.92/0+1/0.3+ +0.86/0.7+0.67/1;
positive medium= 0.26/-1+0.34/-0.7+0.50/-0.3+ +0.67/0+0.86/0.3+ +1.00/0.7+0.91/1;
positive big = 0.20/-1+0.26/-0.7+0.37/-0.3+ +0.50/0+0.67/0.3+ +0.92/0.7+1.00/1.

For given 7 rules (errors and control actions) membership functions were evaluated. Problem is to perform the output using 7 rules and ALI-4 implication.

New input: $Z_{\mathrm{E}} =$

$$(\mu_{\mathrm{E}}(e_{ic}^{c}) = 0.11/-10 + 0.17/-7 + 0.34/-3 + 0.61/0 + 0.96/3 + 0.73/7 + 0.41/10, \mathit{sure})$$

Computer simulation results step by step is discussed chapter 5.

Obtained result is given below. Probabilistic matrices are created for 7 rules and aggregated probabilistic matrix is determined (Table 21)

Table 21

Aggregated probabilistic matrix

.914693	0.92802	1	1	1	1	1
0.738237	0.830239	0.846479	1	1	1	1
…	…	…	…	…	…	….
0.481938	0.611951	0.742803	0.846316	0.924793	0.926892	1
0.364251	0.424492	0.561735	0.607711	0.846115	0.926138	1
0.225287	0.285528	0.422771	0.468746	0.637164	0.761984	1
...	...	...	...	...	...	...
0,203301	0,243951	0,429466	0,567629	0,67483	0,767178	1
0,137032	0,177683	0,363198	0,501361	0,66999	0,762338	1
0,054055	0,094706	0,217379	0,361198	0,529827	0,67839	1

New input is given in Table 22.

Table 22

New inputs and their cumulative probabilities

0.7	**0.8**	**0.9**
0	0	0
0.042575	-1E-06	-1E-06
0.116257	0.038604	0
0.199474	0.274913	0.154821
0.334469	0.513237	0.819908
0.264072	0.173247	0.025272
0.043152	0	0

0.7	**0.8**	**0.9**
0	0	0
0.042575	-1E-06	-1E-06
0.158832	0.038603	-1E-06
0.358307	0.313516	0.15481959
0.692776	0.826753	0.974727898
0.956848	1	1
1	1	1

Using the new distributions (cumulative distributions), aggregated matrix and maxmin aggregation we obtain three output distributions(p_1, p_2, p_3).

We transform obtained cumulative probabilities to standard probabilities and results are as follow(Table 23):

Table 23

Obtained probabilities

p_1:	0,358307	0	0	0,099989	0,112701	0,110875	0,318129
p_2:	0,313516	0	0,109255	0,045975	0,168417	0,124821	0,238016
p_3 :	0,203301	0,04065	0,185515	0,138163	0,107201	0,092348	0,232822

Using new inputs and aggregated probabilistic distributions we define the following result:

$$Z(A,B)=\left(\left\{\frac{0.61}{-1},\frac{0.61}{-0.7},\frac{0.7825}{-0.3},\frac{0.9325}{0},\frac{0.96}{0.3},\frac{0.895625}{0.7},\frac{0.862188}{1}\right\},\left\{\frac{0.1}{0.793588},\frac{1}{0.798296},\frac{0.2}{0.809171}\right\}\right)$$

Defuzzification of the obtained Z-output is carried out in a way known from the scientific literature.

MAIN SCIENTIFIC RESULTS OF THE WORK

The results obtained in the dissertation are as follows:

1.Formal Definition of Z-Implication: For the first time, the concept of Z-implication was formally introduced and defined. This new model allows reasoning with both the imprecision and the reliability of information, overcoming limitations of traditional fuzzy and probabilistic implications.

2.Extension of Fuzzy Implication to Z-Environment: Classical fuzzy implications, such as those of Al1 and Ali-4 , were extended to the Z-environment, enabling more expressive conditional reasoning that incorporates confidence levels in rule-based systems.

3.Development of a Z-Conditional Reasoning Algorithm: An original algorithm was developed for performing approximate reasoning based on Z-numbers. This algorithm serves as the computational core of the proposed decision-making model.

4.Controller Design: The controller was designed using Z-conditional reasoning. This demonstrates the applicability of the proposed method to control systems, particularly where input information is uncertain and imprecise.
5.Extension of ALI-1, Ali-4 Logic to Z-Fuzzy Environment: The ALI-1 implication logic was adapted and extended to operate under Z-number-based reasoning, expanding its applicability to more complex and uncertain environments.

These results provide a strong foundation for future research and practical implementation of intelligent systems that require reasoning with both vagueness and partial trustworthiness of data.

The main content of the dissertation has been published in the following works:

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Systems, Soft Computing and Artificial Intelligence Tools, Iaşi,Romania, -2025, -11 p. https://icafs2025.az/

Personal contribution of the applicant in the works published with co-authorship:

[1] – Author of the main idea, statement of the problem, analysis of the results.
[9] – Statement of the problem and solution.
[10] – Algorithm development, computer simulation.

The defense will be held on 28 October 2025 at 16:00 pm at the meeting of the Dissertation council ED 2.48 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at Azerbaijan State Oil and Industry University.

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Dissertation is accessible at the Azerbaijan State Oil and Industry University Library

Electronic version of the abstract is available on the official website of the Azerbaijan State Oil and Industry University

Abstract was sent to the required addresses on 25 September 2025.

Signed for print: 23.09.2025

Paper format: A5

Volume: 38256

Number of hard copies: 70