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ABSTRACT

of the dissertation for the degree of Doctor of Science

Z AND U NUMBERS-BASED DECISION MAKING METHODS

Speciality: 3338.01 – System analysis, control and information processing (control and decision making)

Field of science: Technical sciences

Applicant: **Konul Imran Jabbarova**

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The work was performed at the research laboratory "Intelligent Control and Decision-Making Systems in Industry and Economics" of Azerbaijan State Oil and Industry University.

Scientific consultant: Corresponding member of ANAS, Doctor of Technical Sciences, professor
Rafik Aziz Aliev

Official opponents: Academician of ANAS, Doctor of Technical Sciences, professor
Ali Mammad Abbasov

Corresponding member of ANAS, Doctor of Technical Sciences, professor
Ismayil Mahmud Ismayilov

Doctor of Technical Sciences, professor
Mahammad Nurmahammad Nuriyev

Doctor of Technical Sciences, professor
Alakbar Ali Aga Aliyev

Dissertation council ED 2.48 of Supreme Attestation Commission under the President of the Republic of Azerbaijan operating at the Azerbaijan State Oil and Industrial University

Chairman of the Dissertation council: Corresponding member of ANAS, Doctor of Technical Sciences, professor
Rafik Aziz Aliev

Scientific secretary of the Dissertation council: PhD, Associate Professor
Akif Vali Alizadeh

Chairman of the Scientific seminar: Doctor of Technical Sciences, professor
Kamala Rafiq Aliyeva

GENERAL CHARACTERISTICS OF THE WORK

Relevance of the topic and degree of elaboration. In contemporary times, Artificial Intelligence (AI) has increasingly found broad application across all domains of human activity. The investigation of the fundamental foundations of AI theory, including its various components, represents one of the major challenges facing contemporary science. Among the core, and arguably dominant, components underlying AI theory is the emergence of scientific problems related to decision-making under uncertainty and learning in such environments. This dissertation is primarily devoted to the scientific investigation of these two issues, focusing on modern approaches to multi-objective and multi-attribute decision-making methods based on Z-numbers and U-numbers, as well as the principles of learning in bimodal environments. In this regard, the dissertation is positioned as a study dedicated to addressing pressing and highly relevant problems of modern science.

This study proposes methods for decision-making problems characterized by Z-number-based decision relevant information on criteria and constraints. A computational approach has been developed for the case of Z-number antecedents and consequents in If–Then rules used to represent expert knowledge. In addition, a learning approach designed for Z-number-based information environments is introduced. The proposed methods can be applied to various practical problems by simultaneously considering both fuzzy and probabilistic information (i.e., bimodal information) without reducing it exclusively to either precise or fuzzy forms.

Object and subject of the research. The object of this dissertation encompasses problems in artificial intelligence (AI), whereas its subject focuses on decision-making and learning processes within bimodal information environments.

The goals and objectives of the dissertation work. The aim of this dissertation is to examine decision analysis and learning principles, which constitute core areas of Artificial Intelligence, within a novel framework. Specifically, the study focuses on developing decision-making methods grounded in Z- and U-number theories and on proposing a learning principle applicable under conditions of Z-information.

Research methods. The research methods employed in this study include fuzzy logic theory, the extended concepts of Z- and U-numbers, learning theory as a foundational element of Artificial Intelligence, and linear programming theory.

Main provisions for Defense. The following provisions are submitted for defense in the dissertation:

- Development of a linear programming method in a bimodal environment and its application to decision-making;
- Extension of the multi-criteria decision-making theory under Z- and U-information conditions.;
- Proposal and development of the decision-making concept in hierarchical systems under Z-information environment;
- Development of a learning concept under fuzzy and probabilistic uncertainty.

Scientific novelty of the research. The scientific contributions of this dissertation are as follows:

- A linear programming method with an objective function and constraints expressed through Z-numbers has been proposed, which is recognized as one of the most cited and widely applied methods in the scientific literature;
- An interpolation-based approach has been developed for solving decision-making problems using a Z-number-based "If...Then..." rule base, introducing a novel methodology;
- A Z-information-based decision-making method has been proposed for hierarchical systems characterized by uncertainty;
- Efficient decision-making methods have been developed based on Z- and U-numbers;
- A reinforcement learning method under bimodal information conditions has been formulated.

Theoretical and practical significance of research. The theoretical significance of the study lies in extending decision-making and learning problems, which constitute the primary complexities of

AI, to extended fuzzy environments, namely Z- and U-environments. Its practical significance stems from the applicability of the proposed methods as effective tools for conditions of profound uncertainty.

Approval and application. The main scientific and practical results of the dissertation were discussed in Research laboratory "Intelligent control and decision-making systems in industry and economics" of Azerbaijan State Oil and Industry University as well as in scientific seminars organized at international conferences:

– Eighth International Conference on Soft Computing, Computing with Words and Perceptions in System Analysis, Decision and Control. 3-4 September, 2015Antalya, Turkey;

– Ninth World Conference "Intelligent Systems for Industrial Automation", WCIS-2016, 25-27 October 2016, Tashkent, Uzbekistan;

– Tenth World Conference "Intelligent Systems for Industrial Automation", WCIS-2018, 25-26 October 2018, Tashkent, Uzbekistan;

– 11th World Conference on Intelligent systems for industrial automation – WCIS-2020, 26-28 November, Tashkent, Uzbekistan.

– 12th World Conference on Intelligent systems for industrial automation – WCIS-2022, 25-26 November, Tashkent, Uzbekistan;

– Sumqayıt Dövlət Universiteti, Beynəlxalq Konfrans - 2023, 25-26 Aprel;

–13th World Conference on Intelligent systems for industrial automation – WCIS-2024, 24-25 October, Tashkent, Uzbekistan.

The name of the institution where the dissertation work was performed. Azerbaijan State Oil and Industry University, Research laboratory "Intelligent control and decision-making systems in industry and economics".

The structure of the dissertation. The dissertation consists of an introduction, seven chapters, a conclusion, and a list of references.

Publications. In connection with the topic of the dissertation, 34 scientific works have been published, including 9 articles in WOS-indexed journals and 7 articles in SCOPUS-indexed journals, as well as in journals recommended by the Higher Attestation Commission.

MAIN CONTENTS OF THE WORK

The introduction presents the relevance of the research topic, the goals and objectives of the research, the principal statements submitted for defense, the research methodology and the theoretical and practical significance of the research.

The first chapter is dedicated to the analysis of decision-making methods under conditions of uncertainty. A review of the existing literature has identified several shortcomings, which are as follows:

1. The insufficient investigation of hierarchical multi-criteria decision-making problems;
2. The insufficient study of optimization problems, particularly those in which variables, objective functions, and constraints are represented using Z-numbers;
3. The insufficient treatment of machine learning problems under Z-information conditions in the scientific literature;
4. The lack of comprehensive studies on decision-making problems that take into account the aforementioned aspects.

This chapter also examines the application of Z-number and U-number theory to decision-making problems in various fields.

The second chapter presents preliminary information on fuzzy numbers and arithmetic operations involving fuzzy numbers. It also explores various mathematical operations of the Relative Distance Measure (RDM) interval algebra, superiority of intervals, and measures of interval specificity. The chapter also analyzes information on Z-numbers, discrete Z-numbers and their mathematical operations, the distance between Z-numbers, and the comparison of Z-numbers. In addition, it provides information on U-numbers, which are a special case of Z-numbers, as well as computational methods applied to U-numbers.

Definition 1. A discrete Z-number [1,2,3]. A discrete Z-number is an ordered pair $Z = (A, B)$ where A is a discrete fuzzy number playing a role of a fuzzy constraint on values of a random variable X: $X\ is\ A$. B is a discrete fuzzy number with a membership function $\mu_B: \{b_1, \dots, b_n\} \to [0,1]$, $\{b_1, \dots, b_n\} \subset [0,1]$, playing a role of a fuzzy constraint on the probability measure of A:

$$P(A) = \sum_{i=1}^{n} \mu_A(x_i) p(x_i) isB. \qquad (1)$$

Definition 2. U-numbers[4,5]. Let *X* be a random variable and *A* be a fuzzy number playing a role of fuzzy constraint on values that the random variable may take: *X* is *A*. The definition of a usual value of *X* may be expressed in terms of the probability distribution of *X* as follows:

$$usually(X\ is\ A) = \mu_{most}\left(\sum_i p(x_i)\mu_A(x_i)\right) \qquad (2)$$

Here, the measure of usuality is expressed through the terms *usually* and *most*.The expression represented by U-numbers, "usually the professor's income is moderate," is presented in Figure 1.

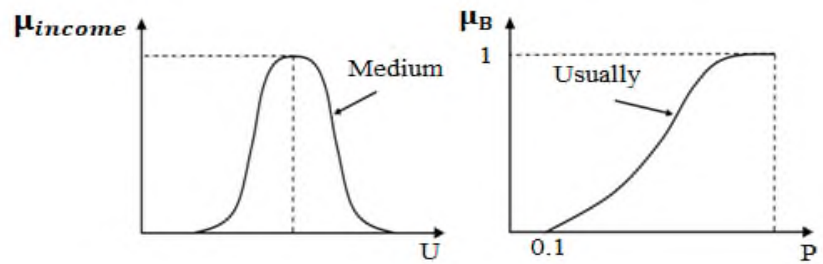

Figure 1. An example of U-number

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[1] Aliev, R. A., Alizadeh A. V., Huseynov O. H. The arithmetic of discrete Z-numbers // Information Sciences, - 2015. 290(1), - p. 134-155.

[2] Aliev, R.A., Huseynov O.H. Decision theory with imperfect information. Singapoure: World Scientific, - 2014, 444 p.

[3] Aliev, R. A. The Arithmetic of Z-numbers / R.A. Aliev, O.H. Huseynov, R.R. Aliyev [et al.]. Theory and Applications. New York, London, Singapore: World Scientific, - 2015.

[4] Aliev, R. A. Approximate arithmetic operations of U-numbers // Procedia Computer Science, - 2016. 102, - p. 378 – 384.

[5] Aliev, R. A. Introduction To U-Number Calculus // Intelligent Automation and Soft Computing. – 2017. – p. 1-6.

The "usuality" will be a composite term characterized by fuzzy quantities as *always, usually, frequently/often, seldom, almost always, never, rarely*. The codebook for 'usuality' is provided in Figure 2.

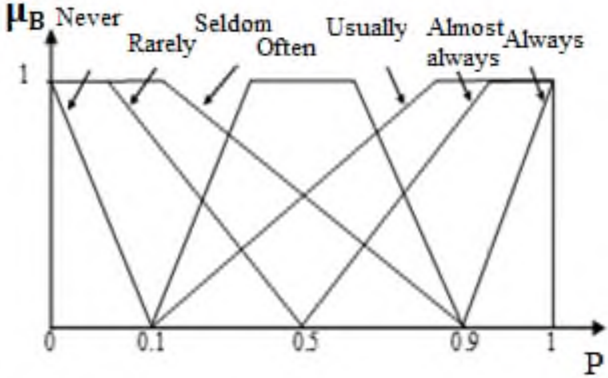

Figure 2. The codebook of the fuzzy quantifiers of usuality

Definition 3. **Fuzzy Pareto optimality (FPO) principle based comparison of Z-numbers**[6]. The Fuzzy Pareto Optimality (FPO) principle enables the evaluation of the Pareto optimality degree of multi-criteria alternatives. For this purpose, comparisons of Z-numbers are performed and the indicators $do(Z_1)$ and $do(Z_2)$ reflect the overall degree of their optimality. These degrees vary within the interval [0, 1] and as a result, one Z-number may be considered superior to another; specifically, if the condition $do(Z_1) > do(Z_2)$ holds, then Z_1 is regarded as preferable.

Definition 4. Distance between Z-numbers[7]**.** Since a Z-number is characterized by the fuzzy number A, the fuzzy number B, and the set of probability distributions G, it is proposed to define the distance

[6] Aliev, R.A.; Huseynov, O.H.; Serdaroglu, R. Ranking of Z-Numbers and Its Application in Decision Making // International Journal of Information Technology & Decision Making,- 2019. 15(6), - p. 1503-1519.

[7] Aliev, R.A. Z-relation equation-based decision making / Guirimov, B. G. , Huseynov, O. H. Aliyev, R. R. // Expert System with Applications, - 2021. 184, - p. 115387.

$D(Z_1, Z_2)$ between Z-numbers as follows.

The distance between A_1 and A_2 is defined as

$$D(A_1, A_2) = sup_{\alpha \in (0,1]} D\,(A_1^\alpha, A_2^\alpha), \quad (3)$$

$$D(A_1^\alpha, A_2^\alpha) = \left| \frac{A_{11}{}^\alpha + A_{12}{}^\alpha}{2} - \frac{A_{21}{}_1^\alpha + A_{22}{}^\alpha}{2} \right|. \quad (4)$$

Here, A_1^α and A_2^α denote the corresponding α-cuts of A_1 and A_2, $A_{11}{}^\alpha$ denote lower and upper bounds of A_1^α ($A_{12}{}^\alpha$ and $A_{22}{}^\alpha$ are those of A_2^α).

A distance is defined between the sets G_1 and G_2, which correspond to the probability distributions p_1 and p_2 associated with Z_1 and Z_2, respectively. The distance between p_1 and p_2 is expressed as follows

$$D(G_1, G_2) = inf_{p_1 \in G_1, p_2 \in G_2} \left\{ (1 - \int_R (p_1 p_2)^{\frac{1}{2}} dx)^{\frac{1}{2}} \right\} \quad (5)$$

If $D(A_1, A_2)$, $D(B_1, B_2)$ and $D(G_1, G_2)$ are known, the distance between the Z-numbers Z_1 and Z_2 is defined as

$$D(Z_1, Z_2) = \beta D(A_1, A_2) + (1 - \beta) D_{total}(B_1, B_2). \quad (6)$$

$D_{total}(B_1, B_2)$ is defined as

$$D_{total}(B_1, B_2) = w D(B_1, B_2) + (1 - w) D(G_1, G_2), \quad (7)$$

Chapter 3 presents the formulation of a Z-number-based linear programming problem that considers the reliability of solutions obtained in linear programming, as well as a solution algorithm using the Differential Evolution (DE) optimization method, which enables the determination of a global solution.

The general formulation of a linear programming problem based on Z-information is expressed as follows [8]:

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[8] Aliev, R. A. Z-number based Linear Programming / R. A. Aliev, A. V.Alizadeh, K.I. Jabbarova // International Journal Of Intelligent Systems, - 2015. 30, - p. 563–589.

$$Z_f(Z_{x_1}, Z_{x_2}, \dots, Z_{x_n}) = Z_{c_1}Z_{x_1} + Z_{c_2}Z_{x_2} + \dots + Z_{c_n}Z_{x_n} \rightarrow min \quad (8)$$

subject to

$$\begin{aligned} &Z_{a_{11}}Z_{x_1} + Z_{a_{12}}Z_{x_2} + \dots + Z_{a_{1n}}Z_{x_n} \preceq Z_{b_1}, \\ &Z_{a_{21}}Z_{x_1} + Z_{a_{22}}Z_{x_2} + \dots + Z_{a_{2n}}Z_{x_n} \preceq Z_{b_2}, \\ &\dots \\ &Z_{a_{m1}}Z_{x_1} + Z_{a_{m2}}Z_{x_2} + \dots + Z_{a_{mn}}Z_{x_n} \preceq Z_{b_m}, \end{aligned} \quad (9)$$

$$Z_{x_1}, Z_{x_2}, \dots, Z_{x_n} \succeq 0^Z \quad (10)$$

Based on the Z-inequalities (8)-(10), the following problem can be formulated:

$$Z_f(Z_{x_1}, Z_{x_2}, \dots, Z_{x_n}) = Z_{c_1}Z_{x_1} + Z_{c_2}Z_{x_2} + \dots + Z_{c_n}Z_{x_n} \rightarrow max \quad (11)$$

Constraint conditions

$$\begin{aligned} &Z_{a_{11}}Z_{x_1} + Z_{a_{12}}Z_{x_2} + \dots + Z_{a_{1n}}Z_{x_n} \leq Z_{b_1}, \\ &Z_{a_{21}}Z_{x_1} + Z_{a_{22}}Z_{x_2} + \dots + Z_{a_{2n}}Z_{x_n} \leq Z_{b_2}, \\ &\dots \\ &Z_{a_{m1}}Z_{x_1} + Z_{a_{m2}}Z_{x_2} + \dots + Z_{a_{mn}}Z_{x_n} \leq Z_{b_m}, \end{aligned} \quad (12)$$

$$Z_{x_1}, Z_{x_2}, \dots, Z_{x_n} \geq 0^Z. \quad (13)$$

In this context, the decision variables and parameters are represented as Z-numbers:

$$Z_{x_i} = (\tilde{A}_{x_i}, \tilde{B}_{x_i}), Z_{c_i} = (\tilde{A}_{c_i}, \tilde{B}_{c_i}), Z_{a_{ij}} = (\tilde{A}_{a_{ij}}, \tilde{B}_{a_{ij}}),$$
$$Z_{b_j} = (\tilde{A}_{b_j}, \tilde{B}_{b_j}), i = 1, \dots, n, j = 1, \dots, m.$$

The Differential Evolution (DE) optimization algorithm is used to solve this optimization problem.

Before initiating the optimization process, all decision variables were preliminarily evaluated by randomly selecting values from the interval Z_X [-1,1]. Prior to commencing the optimization, the parameters of the Differential Evolution (DE) algorithm are established, the DTO fitness function with the objective function Z_f given in (11) is defined and the population size is selected (as a rule, it is ten times greater than the number of optimization parameters, i.e., *10N*$_{par}$*).* Then, the Differential Evolution Optimization process starts.

Initially, the template parameters (Z_x) of the measurement *(*N_{par}*)* are set to determine the decision variables *(*Z_x*).*

Next, the algorithm parameters are defined: the mutation rate (F), the crossover rate (CR), and the population size (PN). The fitness function is then calculated as the objective function.

PN parameter vectors are randomly generated (e.g., from the feasible parameter space [-1, 1]), creating the population *P*=$\{Z_{X_1}, Z_{X_2}, \ldots, Z_{X_{ps}}\}$.

If the final result (either the predetermined number of generations is reached or the required error level is achieved) is not as expected, a new set of parameters should be generated. The next vector is selected: Z_{X_i} (*i*=1,...,*PopSize*). Then, three distinct trial vectors, $Z_{X_{r1}}, Z_{X_{r2}}, Z_{X_{r3}}$, are selected from P, each of which is different from the current vector Z_{X_i}. The test vector is generated:$Z_{X_t} = Z_{X_{r1}} + F \cdot (Z_{X_{r2}} - Z_{X_{r3}})$.$Z_{X_t}$. A new vector is generated from the test vector. The individual vector parameters of Z_{X_t}, together with the probability of the crossover operator, are used to transform Z_{X_i} into a new vector. If the fitness function value of $Z_{X_{new}}$ is better (i.e., lower) than the minimization criterion of Z_{X_i}, the current $Z_{X_i}$is replaced by $Z_{X_{new}}$. in the population *P* . Then, from the population *P*, the parameter vector $Z_{X_{best}}$ with the best fitness (objective) function value is selected. Subsequently, the vectors of decision variables are separated from $Z_{X_{best}}$.

Subsequently, a new vector is generated from the test vector Z_t. The individual vector parameters of Z_t are inherited together with the probability of the mutation norm and assigned to the Z_{new} vector. If the fitness function of Z_{new} is better (i.e., lower) than that of Z_i, the current $Z_i$is replaced by Z_{new} in the population *P*. Next, from the population *P*, the parameter vector Z_{best} (representing the optimal decision variables) with the highest fitness function (Z_f) is selected. Now, all decision variables are extracted from Z_{best}.

Numerical Example

Let us consider a linear programming problem with two decision variables. The objective function is:

$$Z_{c_1}Z_{x_1} + Z_{c_2}Z_{x_2} \to min$$

Constraint conditions

$$Z_{a_{11}}Z_{x_1} + Z_{a_{12}}Z_{x_2} \leq Z_{b_1},$$
$$Z_{a_{21}}Z_{x_1} + Z_{a_{22}}Z_{x_2} \leq Z_{b_2},$$
$$Z_{x_1}, Z_{x_2} \geq 0^Z.$$

Here, $Z_{x_1} = \left(\tilde{A}_{x_1}, \tilde{B}_{x_1}\right), Z_{x_2} = (\tilde{A}_{x_2}, \tilde{B}_{x_2})$ və $Z_{c_1} = \left(\tilde{A}_{c_1}, \tilde{B}_{c_1}\right)$, $Z_{c_2} = \left(\tilde{A}_{c_2}, \tilde{B}_{c_2}\right)$, $Z_{a_{11}} = \left(\tilde{A}_{a_{11}}, \tilde{B}_{a_{11}}\right)$, $Z_{a_{12}} = (\tilde{A}_{a_{12}}, \tilde{B}_{a_{12}})$, $Z_{a_{21}} = \left(\tilde{A}_{a_{21}}, \tilde{B}_{a_{21}}\right), Z_{a_{22}} = \left(\tilde{A}_{a_{22}}, \tilde{B}_{a_{22}}\right)$,

$Z_{b_1} = (\tilde{A}_{b_1}, \tilde{B}_{b_1}), Z_{b_2} = (\tilde{A}_{b_2}, \tilde{B}_{b_2})$.

The values of the parameters are given below:

Z-number $Z_{c_1} = (\tilde{A}_{c_1}, \tilde{B}_{c_1})$:

$$\tilde{A}_{c_1} = \frac{0.01}{0} + \frac{0.13}{0.1} + \frac{0.61}{0.2} + \frac{1}{0.3} + \frac{0.61}{0.4} + \frac{0.14}{0.5} + \frac{0.01}{0.6} + \frac{0}{0.7} + \\ + \frac{0}{0.8} + \frac{0}{0.9} + \frac{0}{1},$$

$$\tilde{B}_{c_1} = \frac{0}{0} + \frac{0}{0.1} + \frac{0}{0.2} + \frac{0.01}{0.3} + \frac{0.14}{0.4} + \frac{0.60}{0.5} + \frac{1}{0.6} + \frac{0.61}{0.7} + \\ + \frac{0.14}{0.8} + \frac{0.01}{0.9} + \frac{0}{1}.$$

Z-number $Z_{c_2} = (\tilde{A}_{c_2}, \tilde{B}_{c_2})$:

$$\tilde{A}_{c_2} = \frac{0}{0} + \frac{0.01}{0.1} + \frac{0.14}{0.2} + \frac{0.61}{0.3} + \frac{1}{0.4} + \frac{0.61}{0.5} + \frac{0.14}{0.6} + \frac{0.01}{0.7} + \\ + \frac{0}{0.8} + \frac{0}{0.9} + \frac{0}{1},$$

$$\tilde{B}_{c_2} = \frac{0}{0} + \frac{0}{0.1} + \frac{0.01}{0.2} + \frac{0.14}{0.3} + \frac{0.61}{0.4} + \frac{1}{0.5} + \frac{0.61}{0.6} + \frac{0.14}{0.7} + \frac{0.01}{0.8} +$$

$$+\frac{0}{0.9}+\frac{0}{1}.$$

Z-number $Z_{a_{11}} = (\tilde{A}_{a_{11}}, \tilde{B}_{a_{11}})$:

$$\tilde{A}_{a_{11}} = \frac{0.14}{0} + \frac{0.61}{0.1} + \frac{1}{0.2} + \frac{0.61}{0.3} + \frac{0.14}{0.4} + \frac{0.01}{0.5} + \frac{0}{0.6} + \frac{0}{0.7} + \\ + \frac{0}{0.8} + \frac{0}{0.9} + \frac{0}{1},$$

$$\tilde{B}_{a_{11}} = \frac{0}{0} + \frac{0}{0.1} + \frac{0}{0.2} + \frac{0}{0.3} + \frac{0}{0.4} + \frac{0}{0.5} + \frac{0.01}{0.6} + \frac{0.14}{0.7} + \\ + \frac{0.61}{0.8} + \frac{1}{0.9} + \frac{0.61}{1}.$$

Z-number $Z_{a_{12}} = (\tilde{A}_{a_{12}}, \tilde{B}_{a_{12}})$:

$$\tilde{A}_{a_{21}} = \frac{0.61}{0} + \frac{1}{0.1} + \frac{0.61}{0.2} + \frac{0.14}{0.3} + \frac{0.01}{0.4} + \frac{0}{0.5} + \frac{0}{0.6} + \frac{0}{0.7} + \\ + \frac{0}{0.8} + \frac{0}{0.9} + \frac{0}{1},$$

$$\tilde{B}_{a_{21}} = \frac{0}{0} + \frac{0}{0.1} + \frac{0}{0.2} + \frac{0}{0.3} + \frac{0}{0.4} + \frac{0}{0.5} + \frac{0.01}{0.6} + \frac{0.14}{0.7} + \\ + \frac{0.61}{0.8} + \frac{1}{0.9} + \frac{0.61}{1}.$$

For simplicity, $Z_{a_{21}} = (\tilde{A}_{a_{21}}, \tilde{B}_{a_{21}})$ and $Z_{a_{22}} = (\tilde{A}_{a_{22}}, \tilde{B}_{a_{22}})$ are selected as singletons:

$\tilde{A}_{a_{12}} = 1, \tilde{B}_{a_{12}} = 1;$

$\tilde{A}_{a_{22}} = 1, \tilde{B}_{a_{22}} = 1.$

Z-number $Z_{b_1} = (\tilde{A}_{b_1}, \tilde{B}_{b_1})$:

$$\tilde{A}_{b_1} = \frac{0.14}{0} + \frac{0.61}{0.1} + \frac{1}{0.2} + \frac{0.61}{0.3} + \frac{0.14}{0.4} + \frac{0.01}{0.5} + \frac{0}{0.6} + \frac{0}{0.7} + \\ + \frac{0}{0.8} + \frac{0}{0.9} + \frac{0}{1},$$

$$\tilde{B}_{b_1} = \frac{0}{0} + \frac{0}{0.1} + \frac{0}{0.2} + \frac{0}{0.3} + \frac{0}{0.4} + \frac{0}{0.5} + \frac{0.01}{0.6} + \frac{0.14}{0.7} + \frac{0.61}{0.8} + \\ + \frac{1}{0.9} + \frac{0.61}{1}.$$

Z-number $Z_{b_2} = (\tilde{A}_{b_2}, \tilde{B}_{b_2})$:

$$\tilde{A}_{b_2} = \frac{0.14}{0} + \frac{0.61}{0.1} + \frac{1}{0.2} + \frac{0.61}{0.3} + \frac{0.14}{0.4} + \frac{0.01}{0.5} + \frac{0}{0.6} + \frac{0}{0.7} + \\ + \frac{0}{0.8} + \frac{0}{0.9} + \frac{0}{1},$$

$$\tilde{B}_{b_2} = \frac{0}{0} + \frac{0}{0.1} + \frac{0}{0.2} + \frac{0}{0.3} + \frac{0}{0.4} + \frac{0}{0.5} + \frac{0.01}{0.6} + \frac{0.14}{0.7} + \frac{0.61}{0.8} + \\ + \frac{1}{0.9} + \frac{0.61}{1}.$$

By adding Z-valued slack variables, we obtain the following:

$$Z_{c_1}Z_{x_1} + Z_{c_2}Z_{x_2} \to min$$
$$Z_{a_{11}}Z_{x_1} + Z_{a_{12}}Z_{x_2} + Z_{x_3} = Z_{b_1},$$
$$Z_{a_{21}}Z_{x_1} + Z_{a_{22}}Z_{x_2} + Z_{x_4} = Z_{b_2},$$
$$Z_{x_1}, Z_{x_2}, Z_{x_3}, Z_{x_4} \geq Z_0.$$

Next, we obtain the equivalent form:

$$\mathrm{Z_{c_1}Z_{x_1} + Z_{c_2}Z_{x_2} + (Z_{b_1} - (Z_{a_{11}}Z_{x_1} + Z_{a_{12}}Z_{x_2} + Z_{x_3} + Z_{x_4})) +} \\ \mathrm{+(Z_{b_2} - (Z_{a_{21}}Z_{x_1} + Z_{a_{22}}Z_{x_2} + Z_{x_3} + Z_{x_4})) \to min}$$
$$Z_{x_1}, Z_{x_2}, Z_{x_3}, Z_{x_4} \geq^Z Z_0.$$

To solve this problem, we applied the Differential Evolution Optimization algorithm. The following parameter values of the Differential Evolution Optimization (DEO) algorithm were used: crossover probability CR=0.7, differential evolution step size 0.8, and population size NP=20. The optimal solution of the problem and the optimal value of the objective function are presented below.

The first decision variable $Z_{x_1} = (\tilde{A}_{x_1}, \tilde{B}_{x_1})$:

$$\tilde{A}_{x_1} = \frac{0.01}{0} + \frac{0.14}{0.1} + \frac{0.61}{0.2} + \frac{1}{0.3} + \frac{0.61}{0.4} + \frac{0.14}{0.5} + \frac{0.01}{0.6} + \frac{0}{0.7} + \\ + \frac{0}{0.8} + \frac{0}{0.9} + \frac{0}{1},$$

$$\tilde{B}_{x_1} = \frac{0}{0} + \frac{0}{0.1} + \frac{0}{0.2} + \frac{0}{0.3} + \frac{0.01}{0.4} + \frac{0.14}{0.5} + \frac{0.61}{0.6} + \frac{1}{0.7} + \\ + \frac{0.61}{0.8} + \frac{0.14}{0.9} + \frac{0.01}{1}.$$

The second decision variable $Z_{x_2} = (\tilde{A}_{x_2}, \tilde{B}_{x_2})$:

$$\tilde{A}_{x_2} = \frac{1}{0} + \frac{0.61}{0.1} + \frac{0.14}{0.2} + \frac{0.01}{0.3} + \frac{0}{0.4} + \frac{0}{0.5} + \frac{0}{0.6} + \frac{0}{0.7} + \frac{0}{0.8} + \\ + \frac{0}{0.9} + \frac{0}{1},$$

$$\tilde{B}_{x_2} = \frac{0}{0} + \frac{0.01}{0.1} + \frac{0.14}{0.2} + \frac{0.61}{0.3} + \frac{1}{0.4} + \frac{0.61}{0.5} + \frac{0.14}{0.6} + \frac{0.01}{0.7} + \\ + \frac{0}{0.8} + \frac{0}{0.9} + \frac{0}{1}.$$

The third slack decision variable $Z_{x_3} = (\tilde{A}_{x_3}, \tilde{B}_{x_3})$:

$$\tilde{A}_{x_3} = \frac{0}{0} + \frac{0}{0.1} + \frac{0}{0.2} + \frac{0}{0.3} + \frac{0}{0.4} + \frac{0.01}{0.5} + \frac{0.14}{0.6} + \frac{0.61}{0.7} + \\ + \frac{1}{0.8} + \frac{0.61}{0.9} + \frac{0.14}{1},$$

$$\tilde{B}_{x_3} = \frac{0}{0} + \frac{0}{0.1} + \frac{0}{0.2} + \frac{0}{0.3} + \frac{0}{0.4} + \frac{0}{0.5} + \frac{0}{0.6} + \frac{0.01}{0.7} + \frac{0.14}{0.8} + \\ + \frac{0.61}{0.9} + \frac{1}{1}.$$

The fourth slack decision variable $Z_{x_4} = (\tilde{A}_{x_4}, \tilde{B}_{x_4})$:

$$\tilde{A}_{x_4} = \frac{0.01}{0} + \frac{0.14}{0.1} + \frac{0.61}{0.2} + \frac{1}{0.3} + \frac{0.61}{0.4} + \frac{0.14}{0.5} + \frac{0.01}{0.6} + \frac{0}{0.7} + \\ + \frac{0}{0.8} + \frac{0}{0.9} + \frac{0}{1},$$

$$\tilde{B}_{x_4} = \frac{0}{0} + \frac{0}{0.1} + \frac{0}{0.2} + \frac{0.01}{0.3} + \frac{0.14}{0.4} + \frac{0.61}{0.5} + \frac{1}{0.6} + \frac{0.61}{0.7} + \\ + \frac{0.14}{0.8} + \frac{0.01}{0.9} + \frac{0}{1}.$$

The optimal value of the objective function $Z_f(Z_{x_1}, Z_{x_2}) = (\tilde{A}_{Z_f(Z_{x_1},Z_{x_2})}, \tilde{B}_{Z_f(Z_{x_1},Z_{x_2})})$:

$$\tilde{A}_{Z_f(Z_{x_1},Z_{x_2})} = \frac{0}{0} + \frac{0}{0.0} + \frac{0.01}{0.04} + \frac{0.14}{0.21} + \frac{0.61}{0.44} + \frac{1}{0.60} + \frac{0.61}{0.71} + \\ + \frac{0.14}{1.09} + \frac{0.01}{1.55} + \frac{0}{2.12} + \frac{0}{2.80},$$

$$\tilde{B}_{Z_f(Z_{x_1},Z_{x_2})} = \frac{0}{0.015} + \frac{0}{0.02} + \frac{0}{0.025} + \frac{0}{0.030} + \frac{0.01}{0.035} + \\ + \frac{0.14}{0.040} + \frac{0.61}{0.045} + \frac{1}{0.051} + \frac{0.61}{0.055} + + \frac{0.14}{0.17} + \frac{0.01}{0.32}.$$

The Z-evaluation-based linear programming problem used for the comparative analysis of the results is identical to the linear programming problem considered with the given generalized fuzzy number[9].

Chapter 4 examines the extension of the general interpolation method for fuzzy rules to Z-number-based rules [10][11].

The proposed approach is based on determining the distance between the current observation vector and the vectors of antecedents. The result is computed as the weighted sum of the rule consequents, with the interpolation weights determined based on the specified distance values.

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[9] Kumar, A., Singh, P., Kaur, J. Generalized Simplex Algorithm to Solve Fuzzy Linear Programming Problems with Ranking of Generalized Fuzzy Numbers // An Official Journal of Turkish Fuzzy Systems Association, - 2010. 1(2), - p. 80-103.

[10] Huang, Z. Rule Model Simplification / Doctor of Philosophy thesis / - University of Edinburgh, UK, - 2006. – 221 p.

[11] Jabbarova K.İ, Rzayeva U., Jabbarova A.İ. Development of the method of general interpolation for Z-number-valued if-then rules // Mathematics and Cybernetics, 2023. 4(124), - p.10-26.

Statement of the problem

The interpolation problem of Z-rules is formulated as follows:

Rule 1: If X_1 *is* $Z_{X_1,1} = (A_{X_1,1}, B_{X_1,1})$ *and, ..., and* X_m *is* $Z_{X_m,1} = (A_{X_m,1}, B_{X_m,1})$ *Then* Y *is* $Z_Y = (A_{Y,1}, B_{Y,1})$;

Rule 2: If X_1 is $Z_{X_1,2} = (A_{X_1,2}, B_{X_1,2})$ *and, ..., and* X_m *is* $Z_{X_m,2} = (A_{X_m,2}, B_{X_m,2})$ *Then* Y *is* $Z_Y = (A_{Y,2}, B_{Y,2})$;

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Rule n: If $X_1 Z_{X_1,n} = (A_{X_1,n}, B_{X_1,n})$ *and, ..., and* X_m *is* $Z_{X_m,n} = (A_{X_m,n}, B_{X_m,n})$ *Then* Y is $Z_Y = (A_{Y,n}, B_{Y,n})$.

and the current inputs

$$X_1 \text{ is } Z'_{X_1} = (A'_{X_1}, B'_{X_1}) \text{ and, ..., and } X_m \text{ is } Z'_{X_m} = (A'_{X_m}, B'_{X_m}),$$

find the Z-value of *Y*.

The main idea remains the same: If the components of the current observation vector $Z' = (Z'_{X_1}, \ldots Z'_{X_m})$ lie "between" the components of the vectors of two antecedents rules, $Z_1 = (Z_{X_1,1}, \ldots, Z_{X_m,1})$ and $Z_2 = (Z_{X_1,2}, \ldots, Z_{X_m,2})$ the corresponding output is calculated as a linear combination of the consequents. The coefficients of this combination represent the impact of each consequent on the resulting output:

Step 1. For the current observation $Z' = (Z'_{X_1}, \ldots Z'_{X_m})$ and the antecedents rules vectors $Z_1 = (Z_{X_1,1}, \ldots, Z_{X_m,1})$ and $Z_2 = (Z_{X_1,2}, \ldots, Z_{X_m,2})$, the satisfaction of the conditions is verified:

$Z_{X_1,1} \leq Z'_{X_1} \leq Z_{X_1,2}$ (or $Z_{X_1,2} \leq Z'_{X_1} \leq Z_{X_1,1}$),..., $Z_{X_m,1} \leq Z'_{X_m} \leq Z_{X_m,2}$ (or $Z_{X_m,2} \leq Z'_{X_m} \leq Z_{X_m,1}$).

These conditions can be described as follows:

$D\left(Z_{X_{1},1}, {Z_1}^*\right) \geq D\left(Z'_{X_1}, {Z_1}^*\right) \geq D\left(Z_{X_{1},2}, {Z_1}^*\right)$ (or $D\left(Z_{X_{1},1}, {Z_1}^*\right) \leq D\left(Z'_{X_1}, {Z_1}^*\right) \leq D\left(Z_{X_{1},2}, {Z_1}^*\right)$),…,$D\left(Z_{X_{m},1}, {Z_m}^*\right) \geq D\left(Z'_{X_m}, {Z_m}^*\right) \geq D\left(Z_{X_{m},2}, {Z_m}^*\right)$ (or $D(Z_{X_{m},1}, {Z_m}^*) \leq D(Z'_{X_m}, {Z_m}^*) \leq$
$\leq D(Z_{X_{m},2}, {Z_m}^*)$) (14)

where ${Z_1}^*$,…,${Z_m}^*$ are the ideal Z-numbers.

Step 2. If the conditions given in (14) are satisfied, the distances $D_v(Z', Z_1)$ and $D_v(Z', Z_2)$ between the current observation vector Z' and the vectors of the two the antecedents rules are calculated as follows::

$$D_v(Z', Z_1) = \sqrt{D^2(Z'_{X_1}, Z_{X_{1},1}) + \ldots + D^2(Z'_{X_m}, Z_{X_{m},1})},$$

$$D_v(Z', Z_2) = \sqrt{D^2(Z'_{X_1}, Z_{X_{1},2}) + \ldots + D^2(Z'_{X_m}, Z_{X_{m},2})} \qquad (15)$$

where *D* is the distance between Z-numbers (Definition 4).

Step 3. The distances $D_v(Z', Z_1)$ and $D_v(Z', Z_2)$ calculated in Step 2 are used to determine the interpolation coefficients (weights) $w_j, j = 1,2$. The smaller the distance between the current observation vector Z' and the antecedent observation vector Z_j, the larger the corresponding weight $w_j, j = 1,2$. Simultaneously, the weights $w_j, j = 1,2$ must satisfy the conditions $w_1, w_2 \in [0,1]$ and $w_1 + w_2 = 1$. Accordingly, the following formula can be used:

$$\mathrm{w_j} = 1 - \frac{\mathrm{D_v}(\mathrm{Z'}, \mathrm{Z_j})}{\mathrm{D_v}(\mathrm{Z'}, \mathrm{Z_1}) + \mathrm{D_v}(\mathrm{Z'}, \mathrm{Z_2})}, \mathrm{j} = 1,2, \qquad (16)$$

It is evident that $w_1, w_2 \in [0,1]$ and $w_1 + w_2 = 1$.

Step 4. The final output is calculated as the weighted sum of the consequents of the first and second rules:

$$Z' = w_1 Z_{Y,_1} + w_2 Z_{Y,_2}, \qquad (17)$$

where $Z_{Y,j}$ is the Z-valued consequent of the *j*-th rule, $w_j, j = 1,2$ are the weights of the linear interpolation computed in Step 3.

Next, we consider two examples to illustrate the proposed approach.

Example 1. The following Z-rules are considered:

Rule 1: If X_1 is $Z_{X_1,1} = (\mathrm{L}, \mathrm{P})$ and X_2 is $Z_{X_2,1} = (L, \mathrm{U})$ and X_3 is $Z_{X_3,1} = (\mathrm{H}, \mathrm{P})$ Then Y is $Z_{Y,1} = (A, \mathrm{A})$

Rule 2: If X_1 is $Z_{X_1,2} = (\mathrm{M}, \mathrm{U})$ and X_2 is $Z_{X_2,2} = (H, \mathrm{R})$ and X_3 is $Z_{X_3,2} = (\mathrm{L}, \mathrm{U})$ Then Y is $Z_{Y,2} = (\mathrm{VH}, \mathrm{P})$.

The codebook of linguistic terms (levels of satisfaction – linguistic values) described by triangular fuzzy numbers (TFNs) is given below.

Linguistic terms for the A part of the Z-nuimber:

Very Low (VL) - (1,1,2),

Low (L) - (1,2,3),

Medium (M)- (2,3,4),

High (H)- (3,4,5),

Very High (VH)- (4,5,5).

Linguistic terms for the B component of the Z-number (Reliability):

Rare (R)– (0.05, 0.25, 0,5),

Plausible (P)– (0.25, 0.5, 0,85),

Usual (U)– (0.5, 0.85, 1).

The current inputs are represented by Z-numbers composed of components based on triangular fuzzy numbers (TFNs):

$$X_1 \text{ - } Z'_{X_1} = ((1.5,2.5,3.5)(0.35,0.65,0.95)),$$
$$X_2\text{-}Z'_{X_2} = ((2.5,3.5,4.5)(0.2,0.4,0.6)) \ ,$$
$$X_3\text{- } Z'_{X_3} = ((2.5,3.5,4.5)(0.35,0.65,0.95)).$$

The vectors of antecedents rules are given below:

$\mathrm{Z}_1 = (\mathrm{Z}_{\mathrm{X}_1,1} = (\mathrm{L}, \mathrm{P}), \mathrm{Z}_{\mathrm{X}_2,1} = (\mathrm{L}, \mathrm{U}), \mathrm{Z}_{\mathrm{X}_3,1} = (\mathrm{H}, \mathrm{R})),$

$\mathrm{Z}_2 = (\mathrm{Z}_{\mathrm{X}_1,2} = (\mathrm{M}, \mathrm{U}), \mathrm{Z}_{\mathrm{X}_2,2} = (\mathrm{H}, \mathrm{R}), \mathrm{Z}_{\mathrm{X}_3,2} = (\mathrm{L}, \mathrm{U})).$

Find the Z-value of Y .

In Step 1, the satisfaction of the sequence of conditions is verified using (14):

$D(Z_{X_1,2}, Z*) = 1.17 \leq D(Z'_{X_1}, Z*) = 1.6 \leq D(Z_{X_1,1}, Z*) = 1.99,$
$D(Z_{X_2,2}, Z*) = 0.69 \leq D(Z'_{X_2}, Z*) = 1.6 \leq D(Z_{X_2,1}, Z*) = 1.87,$
$D(Z_{X_3,1}, Z*) = 0.59 \leq D(Z'_{X_3}, Z*) = 0.9 \leq D(Z_{X_3,2}, Z*) = 1.87.$

Thus, the sequence of conditions is satisfied.

In Step 2, according to (15), the values of the distances $D_v(Z', Z_1)$ and $D_v(Z', Z_2)$ are calculated:

$$D_v(Z', Z_1) = \sqrt{D(Z'_1, Z_{X_1,1})^2 + D(Z'_2, Z'_{X_2,1})^2 + D(Z'_3, Z_{X_3,1})^2} =$$
$$= \sqrt{0.43^2 + 1.24^2 + 0.43^2} = 1.38$$
$$D_v(Z', Z_2) = \sqrt{D(Z'_1, Z_{X_{1,},2})^2 + D(Z'_2, Z_{X_2,2})^2 + D(Z'_3, Z_{X\,3,2})^2} =$$
$$= \sqrt{0.43^2 + 0.45 + 1.13^2} = 1.29$$

In the third step, the interpolation weights are determined using (16):

$$w_1 = 1 - \frac{\mathrm{D_v(Z',Z_1)}}{\mathrm{D_v(Z',Z_1)+D_v(Z',Z_2)}} = 1 - \frac{1.38}{1.38+1.29} = 0.48,$$
$$w_2 = 1 - \frac{D_v(Z',Z_2)}{D_v(Z',Z_1)+D_v(Z',Z_2)} = 1 - \frac{1.29}{1.38+1.29} = 0.52.$$

In Step 4, the final output is obtained using (17):

$$\mathrm{Z_y = w_1 Z_{y,_1} + w_2 Z_{y,_2}} = 0.48 * (\mathrm{L, U}) + 0.52 * (\mathrm{VH, P}) =$$
$$= 0.48 * ((1,2,3)(0.5,0.85,1) + 0.52 * ((4,5,5)(0.25,0.5,0.85)) =$$
$$= (0.48,0.96,1.44)(0.5,0.85,1) + (2.08,2.6,2.6)(0.25,0.5,0.85) =$$
$$= (2.56,3.56,4.04)(0.43,0.7,0.79)$$

This can be represented as a linguistic Z-number (M,U).

Example 2. Consider the following Z-number-based rules:
If $\mathrm{X_1}$ is $\mathrm{Z_{X_1,1}} = (\mathrm{M, U})$ and $\mathrm{X_2}$ is $\mathrm{Z_{X_2,1}} = (\mathrm{H, R})$ and $\mathrm{X_3}$ is $\mathrm{Z_{X_3,1}} = (\mathrm{L, U})$ Then $\mathrm{Y_1}$ is $\mathrm{Z_{Y,1}} = (\mathrm{VH, P})$.

If X_1 is $Z_{X_{2},2} =$ (H, U) X_2 is $Z_{X_{2},2} =$ (L, U) and X_3 is $Z_{X_{3},2} =$ (M, U) Then Y_2 is $Z_{Y,2} =$ (M, U).

The codebook of linguistic terms is given above (Example 1).
Let the current inputs be as follows:

X_1 is $Z'_{X_1} = ((2.5,3.5,4.5)(0.5,0.85,1))$,
X_2 is $Z'_{X_2} = ((2,3,4)(0.2,0.4,0.6))$,
X_3 is $Z'_{X_3} = \left((1.5,2.5,3.5)(0.5,0.85,1)\right)$.
Then, we find the Z-value of Y.

In accordance with the sequence performed in Example 1, the condition sequence is verified based on (14) (Step 1), the distance between the current observation vector and vectors of rule antecedents is calculated (Step 2), and the interpolation weights are determined (Step 3).
$w_1 = 0.52, w_2 = 0.48.$
In Step 4, the final output is calculated based on (17):

$$Z_y = w_1 Z_{y,_1} + w_2 Z_{y,_2} = 0.52 * (VH, R) + 0.48 * (M, U) =$$
$$= (3.04,4.04,4.52)(0.44,0.7,0.79)$$

The obtained Z-number can be represented linguistically as (H,U).

Example 3. Application of the proposed approach to the evaluation of job satisfaction.

The evaluation of job satisfaction is an important issue. It is characterized by vague and partially reliable information related to the dependence between the overall level of job satisfaction and its specific aspects. The problem is that the data reflect psychological factors, perceptions, and other related aspects. Consequently, such data are generally described linguistically. Let us consider the Z-number-based “IF...THEN...” rules that explain the impact of 20 factors on job satisfaction[12][13]. Questionnaires completed by experts are used as the source for constructing the Z-number-valued "IF...THEN..." rule base.

Rule 1: IF X_1 (Activity) is $Z_{X_1,1} = (\mathrm{VS}, \mathrm{H}), X_2$ (Independence) is $Z_{X_2,1} = (\mathrm{S}, \mathrm{H})$, X_3 (Variety) is $Z_{X_3,1} = (VS, \mathrm{H})$, X_4 (Social status) is $Z_{X_4,1} = (\mathrm{VS}, \mathrm{H})$, X_5 (Supervision-Human relations) is $Z_{X_5,1} = (\mathrm{VS}, \mathrm{H})$, X_6 (Supervision-technical) is $Z_{X_6,1} = (\mathrm{S}, \mathrm{H})$, X_7 (Moral values) is $Z_{X_7,1} = (\mathrm{QS}, \mathrm{H})$, X_8 (Security) is $Z_{X_8,1} = (VS, \mathrm{H})$, X_9 (Social service) is $Z_{X_9,1} = (\mathrm{VS}, \mathrm{H})$, X_{10} (Authority) is $Z_{10,1} = (\mathrm{S}, \mathrm{H})$, X_{11} (Ability) is $Z_{X_{11},1} = (\mathrm{VS}, \mathrm{H})$, X_{12} (Company policies and practices) $Z_{X_{12},1} = (\mathrm{QS}, \mathrm{H})$, X_{13} (Compensation) is $Z_{X_{13},1} = (\mathrm{S}, \mathrm{H})$, X_{14} (Advancement) is $Z_{X_{14},1} = (\mathrm{VS}, \mathrm{H})$, X_{15} (Responsibility) is $Z_{X_{15},1} = (\mathrm{VS}, \mathrm{H})$, X_{16} (Creativity) is $Z_{X_{16},1} = (\mathrm{VS}, \mathrm{H})$, X_{17} (Working conditions) is $Z_{X_{17},1} = (\mathrm{S}, \mathrm{H})$, X_{18} (Co-workers) is $Z_{X_{18},1} = (\mathrm{QS}, \mathrm{H})$, X_{19} (Recognition) is $Z_{X_{19},1} = (\mathrm{VS}, \mathrm{H})$, X_{20} (Achievement) is $Z_{X_{20},1} = (\mathrm{VS}, \mathrm{H})$ THEN Y_1 (Overall Job Satisfaction) is $Z_{Y,1} = (\mathrm{S}, \mathrm{H})$;

Rule 2: IF X_1 (Activity) is $Z_{X_1,1} = (\mathrm{S}, \mathrm{H}), X_2$ (Independence) is $Z_{X_2,1} = (\mathrm{S}, \mathrm{H})$, X_3 (Variety) is $Z_{X_3,1} = (\mathrm{S}, \mathrm{H})$, X_4 (Social status) is $Z_{X_4,1} = (\mathrm{S}, \mathrm{H})$, X_5 (Supervision-Human relations) is $Z_{X_5,1} = (\mathrm{OM}, \mathrm{Y})$, X_6 (Supervision-technical) is $Z_{X_6,1} = (\mathrm{QS}, \mathrm{H})$, X_7 (Moral values) is $Z_{X_7,1} = (\mathrm{QS}, \mathrm{H})$, X_8 (Security) is $Z_{X_8,1} = (\mathrm{QS}, \mathrm{H})$, X_9 (Social service) is $Z_{X_9,1} = (\mathrm{S}, \mathrm{H})$, X_{10} (Authority) is $Z_{10,1} = (\mathrm{S}, \mathrm{H})$, X_{11} (Ability) is $Z_{X_{11},1} = (\mathrm{S}, \mathrm{H})$, X_{12} (Company policies and

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[12] Aliev, R.A. Approximate Reasoning on a Basis of Z-Number-Valued If-Then Rules / Pedrycz W., Huseynov O.H., Eyupoglu S.Z. //IEEE Transactions on Fuzzy Systems, - 2017. 25(6), -p. 1589-1600.

[13] Jabbarova, K.I., Eyupoglu, S.Z., Aliyeva, K.R. The Identification of Job Satisfaction under Z-Information // Intelligent Automation & Soft Computing, - 2017. №24, - p.1-5.

practices) is $Z_{X_{12},1} = (\mathrm{LS,M})$, X_{13} (Compensation) is $Z_{X_{13},1} = (\mathrm{LS,M})$, X_{14} (Advancement) is $Z_{X_{14},1} = (\mathrm{QS,H})$, X_{15} (Responsibility) is $Z_{X_{15},1} = (\mathrm{S,H})$, X_{16} (Creativity) is $Z_{X_{16},1} = (\mathrm{S,H})$, X_{17} (Working conditions) is $Z_{X_{17},1} = (\mathrm{LS,M})$, X_{18} (Co-workers) is $Z_{X_{18},1} = (\mathrm{S,H})$, X_{19} (Recognition) is $Z_{X_{19},1} = (\mathrm{QS,H})$, X_{20} (Achievement) is $Z_{X_{20},1} = (\mathrm{QS,H})$ THEN Y_1 (Overall Job Satisfaction) is $Z_{Y,1} = (\mathrm{QS,H})$;

Rule 3: IF X_1 (Activity) is $Z_{X_1,1} = (\mathrm{QS,H}), X_2$ (Independence) is $Z_{X_2,1} = (\mathrm{S,H})$, X_3 (Variety) is $Z_{X_3,1} = (\mathrm{M,Y})$, X_4 (Social status) is $Z_{X_4,1} = (\mathrm{QS,H})$, X_5 (Supervision-Human relations) is $Z_{X_5,1} = (\mathrm{S,H})$, X_6 (Supervision-technical) is $Z_{X_6,1} = (\mathrm{S,H})$, X_7 (Moral values) is $Z_{X_7,1} = (\mathrm{S,H})$, X_8 (Security) is $Z_{X_8,1} = (\mathrm{QS,H})$, X_9 (Social service) $Z_{X_9,1} = (\mathrm{S,H})$, X_{10} (Authority) is $Z_{10,1} = (\mathrm{QS,H})$, X_{11} (Ability) is $Z_{X_{11},1} = (\mathrm{QS,H})$, X_{12} (Company policies and practices) is $Z_{X_{12},1} = (\mathrm{QS,H})$, X_{13} (Compensation) is $Z_{X_{13},1} = (\mathrm{QS,H})$, X_{14} (Advancement) is $Z_{X_{14},1} = (\mathrm{QS,H})$, X_{15} (Responsibility) is $Z_{X_{15},1} = (\mathrm{QS,H})$, X_{16} (Creativity) is $Z_{X_{16},1} = (\mathrm{S,H})$, X_{17} (Working conditions) is $Z_{X_{17},1} = (\mathrm{QS,H})$, X_{18} (Co-workers) is $Z_{X_{18},1} = (\mathrm{QS,H})$, X_{19} (Recognition) is $Z_{X_{19},1} = (\mathrm{S,H})$, X_{20} (Achievement) $Z_{X_{20},1} = (\mathrm{QS,H})$ THEN Y_1 (Overall Job Satisfaction) is $Z_{Y,1} = (\mathrm{QS,H})$;

Rule 4: IF X_1 (Activity) is $Z_{X_1,1} = (\mathrm{S,H}), X_2$ (Independence) is $Z_{X_2,1} = (\mathrm{S,H})$, X_3 (Variety) is $Z_{X_3,1} = (\mathrm{LS,M})$, X_4 (Social status) is $Z_{X_4,1} = (\mathrm{S,H})$, X_5 (Supervision-Human relations) is $Z_{X_5,1} = (\mathrm{U,M})$, X_6 (Supervision-technical) is $Z_{X_6,1} = (\mathrm{U,M})$, X_7 (Moral values) is $Z_{X_7,1} = (\mathrm{U,M})$, X_8 (Security) is $Z_{X_8,1} = (\mathrm{S,H})$, X_9 (Social service) is $Z_{X_9,1} = (\mathrm{QS,H})$, X_{10} (Authority) is $Z_{10,1} = (\mathrm{U,M})$, X_{11} (Ability) is $Z_{X_{11},1} = (\mathrm{LS,M})$, X_{12} (Company policies and practices) is $Z_{X_{12},1} = (\mathrm{LS,M})$, X_{13} (Compensation) is $Z_{X_{13},1} = (\mathrm{QS,H})$, X_{14}

(Advancement) is $Z_{X_{14},1} = (\mathrm{S}, \mathrm{H})$, X_{15} (Responsibility) is $Z_{X_{15},1} = (\mathrm{LS}, \mathrm{M})$, X_{16} (Creativity) is $Z_{X_{16},1} = (\mathrm{LS}, \mathrm{M})$, X_{17} (Working conditions) is $Z_{X_{17},1} = (\mathrm{QS}, \mathrm{H})$, X_{18} (Co-workers) is $Z_{X_{18},1} = (\mathrm{S}, \mathrm{H})$, X_{19} (Recognition) is$Z_{X_{19},1} = (\mathrm{VS}, \mathrm{H})$, X_{20} (Achievement) is $Z_{X_{20},1} = (\mathrm{S}, \mathrm{H})$ THEN Y_1 is (Overall Job Satisfaction) is $Z_{Y,1} = (\mathrm{LS}, \mathrm{M})$;

Rule 5: IF X_1 (Activity) is $Z_{X_1,1} = (\mathrm{QS}, \mathrm{H}), X_2$ (Independence) $Z_{X_2,1} = (\mathrm{QS}, \mathrm{H})$, X_3 (Variety) is $Z_{X_3,1} = (\mathrm{QS}, \mathrm{H})$, X_4 (Social status) is $Z_{X_4,1} = (\mathrm{QS}, \mathrm{H})$, X_5 (Supervision-Human relations) is $Z_{X_5,1} = (\mathrm{S}, \mathrm{H})$, X_6 (Supervision-technical) is $Z_{X_6,1} = (\mathrm{S}, \mathrm{H})$, X_7 (Moral values) is $Z_{X_7,1} = (\mathrm{QS}, \mathrm{H})$, X_8 (Security) is $Z_{X_8,1} = (\mathrm{QS}, \mathrm{H})$, X_9 (Social service) is $Z_{X_9,1} = (\mathrm{QS}, \mathrm{H})$, X_{10} (Authority) is $Z_{10,1} = (\mathrm{QS}, \mathrm{H})$, X_{11} (Ability) is $Z_{X_{11},1} = (\mathrm{QS}, \mathrm{H})$, X_{12} (Company policies and practices) is $Z_{X_{12},1} = (\mathrm{LS}, \mathrm{M})$, X_{13} (Compensation) is $Z_{X_{13},1} = (\mathrm{LS}, \mathrm{M})$, X_{14} (Advancement) is $Z_{X_{14},1} = (\mathrm{QS}, \mathrm{H})$, X_{15} (Responsibility) is $Z_{X_{15},1} = (\mathrm{S}, \mathrm{H})$, X_{16} (Creativity) is $Z_{X_{16},1} = (\mathrm{QS}, \mathrm{H})$, X_{17} (Working conditions) is $Z_{X_{17},1} = (\mathrm{LS}, \mathrm{M})$, X_{18} (Co-workers) is $Z_{X_{18},1} = (\mathrm{S}, \mathrm{H})$, X_{19} (Recognition) is $Z_{X_{19},1} = (\mathrm{QS}, \mathrm{H})$, X_{20} (Achievement) is $Z_{X_{20},1} = (\mathrm{QS}, \mathrm{H})$ THEN Y_1 (Overall Job Satisfaction) is $Z_{Y,1} = (\mathrm{QS}, \mathrm{H})$;
Rule 6: IF X_1 (Activity) is $Z_{X_1,1} = (\mathrm{S}, \mathrm{H}), X_2$ (Independence) is $Z_{X_2,1} = (\mathrm{QS}, \mathrm{H})$, X_3 (Variety) is $Z_{X_3,1} = (\mathrm{LS}, \mathrm{M})$, X_4 (Social status) is $Z_{X_4,1} = (\mathrm{S}, \mathrm{H})$, X_5 (Supervision-Human relations) is $Z_{X_5,1} = (\mathrm{S}, \mathrm{H})$, X_6 (Supervision-technical) is $Z_{X_6,1} = (\mathrm{S}, \mathrm{H})$, X_7 (Moral values) is $Z_{X_7,1} = (\mathrm{LS}, \mathrm{M})$, X_8 (Security) is $Z_{X_8,1} = (\mathrm{LS}, \mathrm{M})$, X_9 (Social service) is $Z_{X_9,1} = (\mathrm{S}, \mathrm{H})$, X_{10} (Authority) is $Z_{10,1} = (\mathrm{S}, \mathrm{H})$, X_{11} (Ability) is $Z_{X_{11},1} = (\mathrm{S}, \mathrm{H})$, X_{12} (Company policies and practices) is $Z_{X_{12},1} = (\mathrm{S}, \mathrm{H})$, X_{13} (Compensation) is $Z_{X_{13},1} = (\mathrm{LS}, \mathrm{M})$, X_{14} (Advancement) is $Z_{X_{14},1} = (\mathrm{QS}, \mathrm{H})$, X_{15} (Responsibility) is $Z_{X_{15},1} = (\mathrm{S}, \mathrm{H})$, X_{16} (Creativity) is $Z_{X_{16},1} = (\mathrm{S}, \mathrm{H})$, X_{17} (Working conditions) is $Z_{X_{17},1} = (\mathrm{QS}, \mathrm{H})$, X_{18} (Co-workers) is $Z_{X_{18},1} = (\mathrm{S}, \mathrm{H})$, X_{19} (Recognition) is

$Z_{X_{19},1} = (\mathrm{S},\mathrm{H})$, X_{20} (Achievement) is $Z_{X_{20},1} = (\mathrm{S},\mathrm{H})$ THEN Y_1 (Overall Job Satisfaction) is $Z_{Y,1} = (\mathrm{S},\mathrm{H})$;

The codebook for the terms used in the rules is presented below: Linguistic terms for the A component of a Z-number (*Level of satisfaction – Linguistic value*)

Unsatisfied (U) - (1,1,2),
Less Satisfied (LS)- (1,2,3)
Quite Satisfied (QS) - (2,3,4),
Satisfied (S)- (3,4,5),
Very satisfied (VS)- (4,5,5).

Linguistic terms for the B component of a Z-number
Low (L)- (0.05,0.25,0.5),
Medium (M)- (0.25,0.5,0.75),
High (H)- (0.5,0.75,1).

Suppose that the following observations of the factors are given:
$X_1$is $Z'_{X_1} = ((3.5,4.5,5)(0.5,0.75,1))$,
$X_2$is $Z'_{X_2} = ((3,4,5)(0.5,0.75,1))$,
$X_3$is$Z'_{X_3} = ((3.5,4.5,5)(0.5,0.75,1))$,
$X_4$is $Z'_{X_4}((3.5,4.5,5)(0.5,0.75,1))$,
$X_5$is$Z'_{X_5} = ((2.5,3.5,4.5)(0.5,0.75,1))$,
$X_6$is$Z'_{X_6} = ((2.5,3.5,4.5)(0.5,0.75,1))$,
$X_7$is $Z'_{X_7} = ((2,3,4)(0.5,0.75,1))$,
$X_8$is $Z'_{X_8} = ((2,3,4)(0.5,0.75,1))$,
$X_9$is $Z'_{X_9} = ((3.5,4.5,5)(0.5,0.75,1))$,
$X_{10}$is$Z'_{X_{10}} = ((3,4,5)(0.5,0.75,1))$,
$X_{11}$is$Z'_{X_{11}} = ((3.5,4.5,5)(0.5,0.75,1))$,
$X_{12}$is $Z'_{X_{12}} = ((1.5,2.5,3.5)(0.35,0.65,0.95))$,
$X_{13}$is $Z'_{X_{13}} = ((2.5,3.5,4.5)(0.35,0.65,0.95))$,
$X_{14}$is $Z'_{X_{14}} = ((2.5,3.5,4.5)(0.5,0.75,1))$,

$X_{15}$is $Z'_{X_{15}} = ((2.5{,}3.5{,}4.5)(0.5{,}0.75{,}1))$,
$X_{16}$is $Z'_{X_{16}} = ((2.5{,}3.5{,}4.5)(0.5{,}0.75{,}1))$,
$X_{17}$is$Z'_{X_{17}} = ((2.5{,}3.5{,}4.5)(0.35{,}0.65{,}0.95))$,
$X_{18}$is $Z'_{18} = ((2{,}3{,}4)(0.5{,}0.75{,}1))$,
$X_{19}$is$Z'_{X_{19}} = ((2.5{,}3.5{,}4.5)(0.5{,}0.75{,}1))$,
$X_{20}$is $Z'_{X_{20}} = ((2.5{,}3.5{,}4.5)(0.5{,}0.75{,}1))$.

Using the method described above, we proceed to calculate the corresponding value of job satisfaction. In Step 1, the conditions are verified according to (14), and the results show that Rules 1 and 2 are applicable for interpolation. In Step 2, the distances are computed as follows.

$$\mathrm{D_v(Z', Z_1)} = \sqrt{\mathrm{D(Z'_1, Z_{X_{1,1}})^2 + D(Z'_2, Z_{X_{2,1}})^2 + \cdots + D(Z'_{20}, Z_{X_{20,1}})^2}}$$
$$= \sqrt{0.23^2 + 0.23^2 + 0.23^2 + \ldots + 0.43^2} = 1.93,$$

$$D_v(Z', Z_2) = \sqrt{D(Z'_1, Z_{X_{1,2}})^2 + D(Z'_2, Z_{X_{2,2}})^2 + \ldots + D(Z'_{20}, Z_{X_{20,2}})^2}$$
$$= \sqrt{0.23^2 + 0.23^2 + 0.23^2 + \ldots + 1.15^2} = 1.96.$$

In Step 3, the corresponding interpolation weights are calculated as $w_1 \approx 0.5$, $w_2 \approx 0.5$.

At Stage 4, the overall level of job satisfaction is calculated using (17):

$$Z_y = w_1 Z_{y,_1} + w_2 Z_{y,_2} = 0.5 * (\mathrm{S, H}) + 0.5 * (\mathrm{QS, H}) =$$
$$= 0.5 * ((3{,}4{,}5)(0.5{,}0.75{,}1) + 0.5 * \big((2{,}3{,}4)(0.5{,}0.75{,}1)\big) =$$
$$= (1.5{,}2{,}2.5)(0.5{,}0.75{,}1) + (1{,}1.5{,}2)(0.5{,}0.75{,}1) =$$
$$= (2.5{,}3.5{,}4.5)(0.36{,}0.62{,}0.96).$$

According to the codebook, the overall level of job satisfaction is (QS, H).

Let us examine another data sample related to the factors:
$X_1$is Z'_{X_1}=((3,4,5)(0.5,0.75,1)),
$X_2$is Z'_{X_2}=((3,4,5)(0.5,0.75,1),
X_3isZ'_{X_3}=((2.5,3.5,4,5)(0.35,0.65,0.95))
X_4isZ'_{X_4}=((3,4,5)(0.5,0.75,1)),
X_5isZ'_{X_5}=((1.5,2.5,3.5)(0.35,0.65,0.95)),
X_6isZ'_{X_6}=((1.5,2.5,3.5)(0.35,0.65,0.95),
X_7isZ'_{X_7}=((1.5,2.5,3.5)(0.35,0.65,0.95)),
X_8isZ'_{X_8}=((2.5,3.5,4.5)(0.5,0.75,1)),
$X_9$is Z'_{X_9}=((2.5,3.5,4.5)(0.5,0.75,1)),
$X_{10}$is$Z'_{X_{10}}$=((2.5,3.5,4.5)(0.35,0.65,0.95)),
$X_{11}$is $Z'_{X_{11}}$ =((2.5,3.5,4.5)(0.35,0.65,0.95)),
$X_{12}$is $Z'_{X_{12}}$ = ((1,2,3)(0.25,0.5,0.75)),
$X_{13}$is$Z'_{X_{13}}$=((1.5,2.5,3.5)(0.35,0.65,0.95)),
$X_{14}$is$Z'_{X_{14}}$=((2.5,3.5,4.5)(0.5,0.75,1)),
$X_{15}$is$Z'_{X_{15}}$=((2.5,3.5,4.5)(0.35,0.65,0.95),
$X_{16}$is$Z'_{X_{16}}$=((2.5,3.5,4.5)(0.35,0.65,0.95)),
$X_{17}$is$Z'_{X_{17}}$=((1.5,2.5,3.5)(0.35,0.65,0.95)),
$X_{18}$is$Z'_{X_{18}}$=((3,4,5)(0.5,0.75,1),
$X_{19}$is$Z'_{X_{19}}$=(2.5,3.5,4.5)(0.5,0.75,1)),
$X_{20}$is $Z'_{X_{20}}$=((2.5,3.5,4.5)(0.5,0.75,1)).

According to condition (14), Rules 2 and 4 can be used for interpolation. The distance values are:

$D_v(Z', Z_1) = 1.58$, $D_v(Z', Z_2) = 3.76$

Interpolation weights:
$w_1 = 0.7$, $w_2 = 0.3$.

Overall level of job satisfaction:

$Z_y = w_1 Z_{y,_1} + w_2 Z_{y,_2}$ =0.7·(QS,H)+0.3·(LS,M)=
=0.7·((2,3,4)(0.5,0.75,1))+0.3·((1,2,3)(0.25,0.5,0.75))=
=((1.4,2.1,2.8)(0.5,0.75,1))+((0.3,0.6,0.9)(0.25,0.5,0.75))=
=((1.7,2.7,3.7)(0.19,0.41,0.73)).

According to the codebook, the overall job satisfaction can be represented as (QS,M). In this case, it can be seen that job satisfaction is lower than the first one (QS,H).

This chapter also introduces a new method for solving decision-making problems with a hierarchical structure under Z-information conditions. The formulation of such problems and their solution algorithm are presented below. As is well known, hierarchical multi-attribute decision models are designed for the classification and/or assessment of objects defined within the attribute–value domain [14]. They are based on the division of a complex decision problem into smaller and less complex subproblems.

Statement of the problem

Suppose that a set of alternatives $\{f_1, \ldots, f_m\}$ evaluated on the criteria $\{C_1, \ldots, C_n\}$ is given. Each criterion C_j , $,j = 1, \ldots, n$ is defined by subcriteria $\left\{C_{j_1}, \ldots, C_{j_{n_j}}\right\}$, where n_j represents the number of subcriteria corresponding to criterion C_j. Thus, for each alternative $f_i, i = 1, \ldots, m,$ corresponding to criterion $C_j, j = 1, \ldots, n,$ the vector of values for the subcriteria $\left\{C_{j_1}, \ldots, C_{j_{n_j}}\right\}$ is predefined as

$$(X_{ij_1}, \ldots, X_{ij_{n_j}}).$$

Thus, $X_{ij_1}, \ldots, X_{ij_{n_j}}$ represent the values of alternative f_i according to the subcriteria $\left\{C_{j1}, \ldots, C_{jn_j}\right\}$. Due to the imprecision and partial

[14] Bohanec, M., Zupan, B., Rajkovic V. Hierarchical Multi-Attribute Decision Models and Their Application in Health Care // Studies in Health Technology and Informatics, - 1999. 68, - p. 670-675.

reliability of the corresponding information, $X_{ij_1}, \dots, X_{ij_{n_j}}$ are represented by Z-numbers:

$$X_{ij_1} = (A_{ij_1}, B_{ij_1}), \dots, X_{ij_{n_j}} = (A_{ij_1}, B_{ij_1}).$$

The problem under consideration is to determine the optimal alternative f^* based on the set of criteria$\{C_1, \dots, C_n\}$:

$$f * \in \{f_1, \dots, f_m\}, f * \underset{\sim}{\succ} f_i, i = 1, \dots, m.$$

Hence, the problem is characterized by a hierarchical structure, consisting of alternatives at the first level, sub-criteria at the second level, and criteria at the third level. The proposed solution approach involves calculating the overall criterion evaluation for each alternative f_i as a Z-number Y_i, and by comparing the values of Y_i. Accordingly, at the fourth level of the hierarchy, the overall Z-valued scores Y_i for the alternatives will be derived through the aggregation of the criteria. All computations and comparisons will be conducted within the framework of Z-number arithmetic:

The problem is solved in the following sequence:

1. In the first stage, for each criterion $C_j, j = 1, \dots, n$ the aggregation of the corresponding sub-criteria values

$$X_{ij} = (A_{ij_1}, B_{ij_1}), \dots, X_{ij_{n_1}} = \left(A_{ij_{n_j}}, B_{ij_{n_j}}\right)$$

is performed for each alternative f_i in order to obtain the Z-valued scalar evaluation $Y_{ij} = (A_{ij}, B_{ij})$.

A weighted average aggregation can be applied here:

$$Y_{ij} = w_{ij_1} X_{ij_1} + \dots + w_{ij_{n_j}} X_{ij_{n_j}}.$$

In this context, $w_{j_1}, \dots, w_{j_{n_j}}$ denote the importance weights associated with the sub-criteria $C_{j_1}, \dots, C_{j_{n_j}}$, respectively. Thus, Y_{ij} represents the scalar evaluation of alternative f_i with respect to the sub-criteria of $C_j, j = 1, \dots, n$. Consequently, the criterion evaluation matrix (decision table) can be constructed:

	C_1	$\cdots$	C_n
f_1	Y_{11}	$\cdots$	Y_{1n}
$\cdots$	$\cdots$	$\cdots$	$\cdots$
f_m	Y_{m_1}	$\cdots$	Y_{11}

2. For each alternativef_i the scalar evaluations Y_{ij} obtained in the preceding stage are aggregated. As a result, the overall scalar evaluation of f_i is calculated based on the set of Z-numbers $Y_i = (A_i, B_i)$. Here, a weighted average aggregation can also be applied:

$$Y_i = w_1 Y_{i_1} + \cdots + w_n Y_{i_n}.$$

In this context, $w_1, \dots, w_n$ denote the relative importance weights assigned to the sub-criteria $C_1, \dots, C_n$, respectively.

To rank the alternativesf_i the overall Z-valued evaluations $Y_i = (A_i, B_i)$ obtained in the previous stage are compared. The comparison is performed according to Definition 3. Thus, the alternative $f^* \in \{f_1, \dots, f_m\}$ that satisfies the condition

$$Y^* \geq Y_i, i = 1, \dots, m$$

is considered optimal.

Chapter 5 presents a review of studies devoted to learning with verification in classical and fuzzy environments. It also introduces the general formulation and solution algorithm of the learning with verification problem under Z-information conditions, along with its application to a real-world problem.

Machine learning is a core component of Artificial Intelligence and is widely applied. A major limitation of current AI systems is the lack of explainability of the results, since both classical machine learning and deep learning rely on neural networks. In this context, we consider the extension of the Q-learning problem to the Z-information setting under fuzzy information conditions.

Statement of the problem and solution

Let us consider the Q-learning method in which the objectives and constraints are formulated using Z-numbers. Accordingly, the value $ZQ(x,a)$ of action a in state x is also represented as a Z-number. The action a in state x has a constraint, which is represented by ZQ(x, a)[15]:

$$ZQ_{t+1}(x,a) = \\ =(1-\beta)ZQ_t(x,a)+\beta[(Zr_{t+1}+\gamma ZV(y))\wedge ZQ_C(x,a)] \quad (18)$$

Here, $ZQ_{t+1}(x,a)$ represents the Z-valued utility of action a - in state x . Zr_{t+1} is the Z-valued reward, $ZQ_C(x,a)$ is the Z-valued penalty of action a in state x, and $\wedge$- denotes the minimum operation over Z-numbers [16], γ is the discount factor, and β s the learning rate.

ZV(y) is the Z-value of the optimal action-state pair:

$$ZV(y) = \vee_{a\in A} ZQ_{t+1}(y,a) \quad (19)$$

Here, $\vee$ denotes the maximum operation over Z-numbers.

Thus, the optimal action is determined as follows:

$$a = arg\max_{a\in A} \vee ZQ(x,a) \quad (20)$$

Equations (18)–(20) are solved in the following steps:

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[15] Berenji, H. R. Fuzzy Reinforcement Learning and Dynamic Programming // Lecture Notes in Computer Science, -1994. 1-9.

[16] Aliev, R.A. Approximate Reasoning on a Basis of Z-Number-Valued If-Then Rules / Pedrycz W., Huseynov O.H., Eyupoglu S.Z. //IEEE Transactions on Fuzzy Systems, - 2017. 25(6), -p. 1589-1600.

Step 1: The discount factor γ, the learning rate β and the accuracy threshold indicating convergence ϵ have been set. . Set t=0

Episode t:

Step 2. Compute $ZQ_{t+1}(x, a)$ using (18).

Step 3. For each x, find the action a using (19)-(20), and compute the new value of $ZQ_{t+1}(x, a)$ using (18).

Step 4. If $\left(ZQ_{t+1}(x, a), ZQ_t(x, a)\right) > \epsilon$, return to Step 2; otherwise, proceed to Step 5.

Step 5. Record the optimal action a for each x identified in Step 3.

End.

Example. A Q-learning problem under fuzzy information is considered[17][18]. Let us consider the extension of this problem to the Z-information case. The problem is shown in Figure 3.

In this problem, $\{\alpha_1, \alpha_2\}$ is the set of actions, and $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ is the set of states. The constraints on the actions are given below:

$ZQ_C(\sigma_1, \alpha_1)$ = ((0.5,0.6,0.7)(0.8,0.9,1))
$ZQ_C(\sigma_1, \alpha_2)$ = ((0.9,1,1)(0.8,0.9,1))
$ZQ_C(\sigma_2, \sigma_2)$ = ((0.9,1,1)(0.8,0.9,1))
$ZQ_C(\sigma_2, \alpha_1)$ = ((0.7,0.8,0.9)(0.8,0.9,1))
$ZQ_C(\sigma_3, \alpha_1)$ = ((0.9,1,1)(0.8,0.9,1))
$ZQ_C(\sigma_3, \alpha_2)$ = ((0.6,0.7,0.8)(0.8,0.9,1))
$ZQ_G(\sigma_4) = ((0.9,1,1)(0.8,0.9,1))$
$ZQ_G(\sigma_5) = ((0.7,0.8,0.9)(0.8,0.9,1))$

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[17] Berenji, H. R. Fuzzy Reinforcement Learning and Dynamic Programming // Lecture Notes in Computer Science, -1994. 1-9.

[18] Jabbarova, K.I., Huseynov, O.H., Jabbarova, A.I.Toward Z-number valued reinforcement learning problem // Proceedings of the 12th World Conference on Intelligent System for Industrial Automation (WCIS-2022). Tashkent, Uzbekistan, - 2024, - p. 352–360.

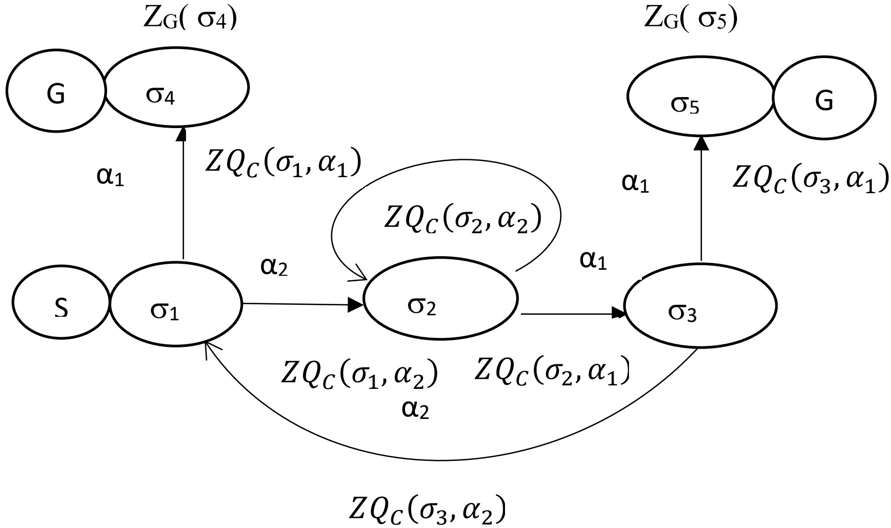

Figure 3. A multi-stage decision-making problem under Z-information

All rewards Zr_{t+1} are shown in Table 1.

Table 1. Reward Table

	1	2	3	4	5
1	((-0.2,-0.1,0) (0.8,0.9,1))	((0,0,0.1) (0.8,0.9,1))	((-0.2,-0.1,0) (0.8,0.9,1))	((0,0.1,0.2) (0.8,0.9,1))	((-0.2,0.1,0) (0.8,0.9,1))
2	((-0.2,-0.1,0) (0.8,0.9,1))	((0,0,0.1) (0.8,0.9,1))	((0,0,0.1) (0.8,0.9,1))	((-0.2,-0.1,0) (0.8,0.9,1))	((-0.2,-0.1,0) (0.8,0.9,1))
3	((0,0,0.1) (0.8,0.9,1))	((-0.2,-0.1,0) (0.8,0.9,1))	((-0.2,-0.1,0) (0.8,0.9,1))	(-0.2,-0.1,0) (0.8,0.9,1))	((0,0.1,0.2) (0.8,0.9,1))
4	((-0.2,-0.1,0) (0.8,0.9,1))	((-0.2,-0.1,0) (0.8,0.9,1))	((-0.2,-0.1,0) (0.8,0.9,1))	((-0.2,-0.1,0) (0.8,0.9,1))	((-0.2,-0.1,0) (0.8,0.9,1))
5	((-0.2,-0.1,0) (0.8,0.9,1))	((-0.2,-0.1,0) (0.8,0.9,1))	((-0.2,-0.1,0) (0.8,0.9,1))	((-0.2,-0.1,0) (0.8,0.9,1))	((-0.2,-0.1,0) (0.8,0.9,1))

Let the discount factor be $\gamma = 0.9$, the learning rate $\beta = 0.9$, and the accuracy threshold ϵ=0.02. According to the solution algorithm, the problem is solved as follows:

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$\mathrm{ZV}(\sigma_4) = \max\ (\mathrm{ZQ}(4,4)) = ((0.9,1,1)\ (0.8,0.9,1));$

$Zr_{t+1} + \gamma ZV(\sigma_4) = \big((0,0.1,0.2)(0.8,0.9,1)\big) + 0.9 \cdot$
$((0.9,1,1)(0.8,0.9,1)) = \big((0,0.1,0.2)(0.8,0.9,1)\big) +$
$+((0.81,0.9,0.9)(0.8,0.9,1)) = \big((0.81,1.1)(0.8,0.9,1)\big);$

$\beta\big[\big(Zr_{t+1} + \gamma ZV(\sigma_4)\big) \wedge ZQ_C(\sigma_1, \alpha_1)\big] = 0.9 \cdot$
$\cdot \big((0.81,1.1)(0.8,0.9,1)\big) \wedge \big((0.5,0.6,0.7)(0.8,0.9,1)\big)=$
$= 0.9 \cdot \big((0.5,0.6,0.7)(0.8,0.9,1)\big) = ((0.45,0.54,0.63)(0.8,0.9,1);$

$(1-\beta)ZQ_t(\sigma_1, \alpha_1) = 0.1 \cdot \big((0,0,0.1)(0.8,0.9,1)\big) =$
$= (0,0,0.01)(0.8,0.9,1);$

$((0.45,0.54,0.63)(0.8,0.9,1)) + ((0,0,0.01)(0.8,0.9,1)) =$
$= \big((0.45,0.54,0.73)(0.8,0.9,1)\big).$

$\boldsymbol{ZQ_C(\sigma_1, \alpha_1)}$ **= ((0.45,0.54,0.73)(0.8,0.9,1))**

$ZV(\sigma_2) = \max\big(\mathrm{ZQ}(2,2), \mathrm{ZQ}(2,3)\big) = \max\big(((0,0,0.1)(0.8,0.9,1)\big);$
$\big((0,0,0.1)(0.8,0.9,1))\big) = \big((0,0,0.1)(0.8,0.9,1)\big);$

$Zr_{t+1} + \gamma ZV(\sigma_2) = \big((0,0,0.1)(0.8,0.9,1)\big) + 0.9 \cdot$
$((0,0,0.1)(0.8,0.9,1)) = \big((0,0,0.1)(0.8,0.9,1)\big) +$
$+((0,0,0.09)(0.8,0.9,1)) = \big((0,0,0.19)(0.8,0.9,1)\big);$

$\beta\big[\big(Zr_{t+1} + \gamma ZV(\sigma_2)\big) \wedge ZQ_C(\sigma_1, \alpha_2)\big]$
$= 0.9 \cdot \big((0,0,0.19)(0.8,0.9,1)\big) \wedge \big((0.9,1,1)(0.8,0.9,1)\big) =$
$= 0.9 \cdot \big((0,0,0.19)(0.8,0.9,1)\big) = ((0,0,0.171)(0.8,0.9,1);$

$(1-\beta)ZQ_t(\sigma_1, \alpha_2) = 0.1 \cdot \big((0,0,0.1)(0.8,0.9,1)\big) =$
$= (0,0,0.01)(0.8,0.9,1);$

$((0,0,0.01)(0.8,0.9,1)) + ((0,0,0.171)(0.8,0.9,1)) =$
$= \big((0,0,0.181)(0.8,0.9,1)\big).$

$\boldsymbol{ZQ_C(\sigma_1, \alpha_2)}$ **= ((0,0,0.181)(0.8,0.9,1))**

$ZV(\sigma_3) = \max\big(\mathrm{ZQ}(3,5), \mathrm{ZQ}(3,1)\big) = \max\big(((0,0,0.1)(0.8,0.9,1)\big);$
$\big((0,0,0.1)(0.8,0.9,1))\big) = \big((0,0,0.1)(0.8,0.9,1)\big);$

$Zr_{t+1} + \gamma ZV(\sigma_3) = \big((0,0,0.1)(0.8,0.9,1)\big) + 0.9 \cdot$
$((0,0,0.1)(0.8,0.9,1)) = \big((0,0,0.1)(0.8,0.9,1)\big) +$
$+((0,0,0.09)(0.8,0.9,1)) = \big((0,0,0.19)(0.8,0.9,1)\big);$
$\beta\big[\big(Zr_{t+1} + \gamma ZV(\sigma_3)\big) \wedge ZQ_C(\sigma_2, \alpha_1)\big] =$
$= 0.9 \cdot \big((0,0,0.19)(0.8,0.9,1)\big) \wedge \big((0.7,0.8,0.9)(0.8,0.9,1)\big) =$
$= 0.9 \cdot \big((0,0,0.19)(0.8,0.9,1)\big) = ((0,0,0.171)(0.8,0.9,1);$
$(1-\beta)ZQ_t(\sigma_2, \alpha_1) = 0.1 \cdot \big((0,0,0.1)(0.8,0.9,1)\big) =$
$= (0,0,0.01)(0.8,0.9,1);$
$((0,0,0.01)(0.8,0.9,1)) + ((0,0,0.171)(0.8,0.9,1)) =$
$= \big((0,0,0.181)(0.8,0.9,1)\big).$
$\boldsymbol{ZQ_C(\sigma_2, \alpha_1) = ((0,0,0.181)(0.8,0.9,1))}$
$ZV(\sigma_1) = \max\big(\mathrm{ZQ}(1,4), \mathrm{ZQ}(1,4)\big) = \max\big(\big((0,0,0.1)(0.8,0.9,1)\big),$
$\big((0,0,0.1)(0.8,0.9,1)\big);$
$Zr_{t+1} + \gamma ZV(\sigma_1) = \big((0,0.1,0.2)(0.8,0.9,1)\big) + 0.9 \cdot$
$((0,0,0.1)(0.8,0.9,1)) = \big((0,0,0.1)(0.8,0.9,1)\big) +$
$+((0,0,0.9)(0.8,0.9,1)) = \big((0,0,1.0.19)(0.8,0.9,1)\big);$
$\beta\big[\big(Zr_{t+1} + \gamma ZV(y\sigma_1)\big) \wedge ZQ_C(\sigma_3, \alpha_2)\big] =$
$= 0.9 \cdot [\big((0,0,1.0.19)(0.8,0.9,1)\big) \wedge \big((0.6,0.7,0.8)(0.8,0.9,1)\big)] =$
$= 0.9 \cdot \big((0,0,1.0.19)(0.8,0.9,1)\big) =$
$(1-\beta)ZQ_t(\sigma_3, \alpha_2) = 0.1 \cdot \big((0,0,0.1)(0.8,0.9,1)\big) =$
$= (0,0,0.01)(0.8,0.9,1);$
$((0,0,0.01)(0.8,0.9,1)) + ((0,0,0.171)(0.8,0.9,1)) =$
$= \big((0,0,0.181)(0.8,0.9,1)\big)$
$ZQ_C(\sigma_3, \alpha_2) = \big((0,0,0.181)(0.8,0.9,1)\big)$
$ZV(\sigma_5) = \max\big(\mathrm{ZQ}(5,5)\big) = \big((0.7,0.8,0.9)(0.8,0.9,1)\big);$
$Zr_{t+1} + \gamma ZV(\sigma_5) = \big((0,0.1,0.2)(0.8,0.9,1)\big) + 0.9 \cdot$
$((0.7,0.8,0.9)(0.8,0.9,1)) = \big((0,0.1,0.2)(0.8,0.9,1)\big) +$
$+((0.63,0.72,0.81)(0.8,0.9,1)) = \big((0.63,0.82,1.01)(0.8,0.9,1)\big);$
$\beta\big[\big(Zr_{t+1} + \gamma ZV(\sigma_5)\big) \wedge ZQ_C(\sigma_3, \alpha_1)\big] = 0.9 \cdot$
$\cdot [\big((0.63,0.82,1.01)(0.8,0.9,1)\big) \wedge \big((0.9,1,1)(0.8,0.9,1)\big)] =$

$= 0.9 \cdot \big((0.63,0.82,1.01)(0.8,0.9,1)\big) =$
$= ((0.57,0.738,0.91)(0.8,0.9,1));$
$(1-\beta)ZQ_t(\sigma_3,\alpha_1) = 0.1 \cdot \big((0,0,0.1)(0.8,0.9,1)\big) =$
$= (0,0,0.01)(0.8,0.9,1);$
$((0,0,0.01)(0.8,0.9,1)) + ((0.57,0.738,0.91)(0.8,0.9,1)) =$
$= \big((0.57,0.738,0.92)(0.8,0.9,1)\big)$
$\boldsymbol{ZQ_C(\sigma_3,\alpha_1)} = \mathbf{((0.57,0.738,0.92)(0.8,0.9,1))}$
$ZV(\sigma_2) = \max\big(\mathrm{ZQ}(2,2), \mathrm{ZQ}(2,3)\big) = \max\,(((0,0,0.1)(0.8,0.9,1))$
$\big((0,0,0.1)(0.8,0.9,1)\big) = \big((0,0,0.1)(0.8,0.9,1)\big);$
$Zr_{t+1} + \gamma ZV(\sigma_2) = \big((0,0.1,0.2)(0.8,0.9,1)\big) + 0.9 \cdot$
$\cdot ((0,0,0.1)(0.8,0.9,1)) = \big((0,0,0.1)(0.8,0.9,1)\big) +$
$+((0,0,0.9)(0.8,0.9,1)) = \big((0,0,0.19)(0.8,0.9,1)\big);$
$\beta\big[\big(Zr_{t+1} + \gamma ZV(\sigma_2)\big) \wedge ZQ_C(\sigma_2,\alpha_2)\big] =$
$= 0.9 \cdot \big[\big((0,0,1.0.19)(0.8,0.9,1)\big) \wedge \big((0.9,1,1)(0.8,0.9,1)\big)\big] =$
$= 0.9 \cdot \big((0,0,0.19)(0.8,0.9,1)\big) = ((0,0,0.171)(0.8,0.9,1));$
$(1-\beta)ZQ_t(\sigma_2,\alpha_2) = 0.1 \cdot \big((0,0,0.1)(0.8,0.9,1)\big)$
$= (0,0,0.01)(0.8,0.9,1);$
$((0,0,0.01)(0.8,0.9,1)) + ((0,0,0.171)(0.8,0.9,1)) =$
$= \big((0,0,0.181)(0.8,0.9,1)\big)$
$\boldsymbol{ZQ_C(\sigma_2,\alpha_2)} = \mathbf{((0,0,0.181)(0.8,0.9,1));}$

⋮

VII EPİSODE

$\mathrm{ZV}(\sigma_4) = \max\,(\mathrm{ZQ}(4,4)) = ((0.9,1,1)\ (0.8,0.9,1));$
$Zr_{t+1} + \gamma ZV(\sigma_4) = \big((0,0.1,0.2)(0.8,0.9,1)\big) + 0.9 \cdot$
$\big((0.9,1,1)(0.8,0.9,1)\big) = \big((0,0.1,0.2)(0.8,0.9,1)\big) +$
$+((0.81,0.9,0.9)(0.8,0.9,1)) = \big((0.81,1.1)(0.8,0.9,1)\big);$
$\beta\big[\big(Zr_{t+1} + \gamma ZV(\sigma_4)\big) \wedge ZQ_C(\sigma_1,\alpha_1)\big] =$
$= 0.9 \cdot \big((0.81,1.1)(0.8,0.9,1)\big) \wedge \big((0.5,0.6,0.7)(0.8,0.9,1)\big) =$
$= 0.9 \cdot \big((0.5,0.6,0.7)(0.8,0.9,1)\big) = ((0.45,0.54,0.63)(0.8,0.9,1));$
$(1-\beta)ZQ_t(\sigma_1,\alpha_1) = 0.1 \cdot ((0.5,0.6,0.7)(0.8,0.9,1)) =$
$= ((0.05,0.06,0.07)(0.8,0.9,1));$

$((0.05,0.06,0.07)(0.8,0.9,1)) + ((0.45,0.54,0.63)(0.8,0.9,1)) =$
$= \big((0.5,0.6,0.7)(0.8,0.9,1)\big).$
$\boldsymbol{ZQ_C(\sigma_1, \alpha_1) = \big((0.5, 0.6, 0.7)(0.8, 0.9, 1)\big)};$
$ZV(\sigma_2) = \max\big(\mathrm{ZQ}(2,2), \mathrm{ZQ}(2,3)\big) =$
$= \max\big(((0.57,0.74,1)(0.8,0.9,1)\big),$
$\big((0.57,0.74,1)(0.8,0.9,1))\big) = \big((0.57,0.74,1)(0.8,0.9,1)\big);$
$Zr_{t+1} + \gamma ZV(\sigma_2) = \big((0,0,0.1)(0.8,0.9,1)\big) + 0.9 \cdot$
$\big((0.57,0.74,1)(0.8,0.9,1)\big) = \big((0,0,0.1)(0.8,0.9,1)\big) +$
$+((0.51,0.67,1)(0.8,0.9,1)) = \big((0.51,0.67,1)(0.8,0.9,1)\big);$
$\beta\big[\big(Zr_{t+1} + \gamma ZV(\sigma_2)\big) \wedge ZQ_C(\sigma_1, \alpha_2)\big]=$
$= 0.9 \cdot \big((0.51,0.67,1)(0.9,1,1)\big) \wedge \big((0.9,1,1)(0.8,0.9,1)\big) =$
$= 0.9 \cdot \big((0.51,0.67,1)(0.9,1,1)\big) = ((0.46,0.6,0.9)(0.8,0.9,1));$
$(1-\beta)ZQ_t(\sigma_1, \alpha_2) = 0.1 \cdot \big((0.51,0.67,1)(0.8,0.9,1)\big)=$
$= ((0.051,0.067,0.1)(0.8,0.9,1));$
$((0.46,0.6,0.9)(0.8,0.9,1)) + ((0.051,0.067,0.1)(0.8,0.9,1)) =$
$= \big((0.51,0.67,1)(0.8,0.9,1)\big);$
$ZQ_C(\sigma_1, \alpha_2) = \big((0.51,0.67,1)(0.9,1,1)\big);$
$ZV(\sigma_3) = \max\big(\mathrm{ZQ}(3,5), \mathrm{ZQ}(3,1)\big) =$
$= \max\big((0.46,0.59,1)(0.8,0.9,1)\big), \big((0.63,0.82,1.01)(0.8,0.9,1))\big)$
$= ((0,63,0.82,1.01)(0.8,0.9,1.01));$
$Zr_{t+1} + \gamma ZV(\sigma_3) = \big((0,0,0.1)(0.8,0.9,1)\big) + 0.9 \cdot$
$((0,63,0.82,1.01)(0.8,0.9,1)) = \big((0,0,0.1)(0.8,0.9,1)\big) +$
$+((0.57,0.74,0.9)(0.8,0.9,1)) = \big((0.57,0.74,1)(0.8,0.9,1)\big);$
$\beta\big[\big(Zr_{t+1} + \gamma ZV(\sigma_3)\big) \wedge ZQ_C(\sigma_2, \alpha_1)\big] = 0.9 \cdot$
$\cdot \big[\big((0.57,0.74,1)(0.8,0.9,1)\big) \wedge \big((0.7,0.8,0.9)(0.8,0.9,1)\big)\big] =$
$= 0.9 \cdot \big((0.57,0.74,1)(0.8,0.9,1)\big) = ((0.51,0.67,0.9)(0.8,0.9,1));$
$(1-\beta)ZQ_t(\sigma_2, \alpha_1) = 0.1 \cdot \big((0.57,0.74,1)(0.8,0.9,1)\big) =$
$= ((0.057,0.074,0.1)(0.8,0.9,1));$
$((0.057,0.074,0.1)(0.8,0.9,1)) + ((0.51,0.67,0.9)(0.8,0.9,1)) =$
$= \big((0.57,0.74,1)(0.8,0.9,1)\big)$

$\boldsymbol{ZQ_C(\sigma_2,\alpha_1)} = \big((0.57{,}0.74{,}1)(0.8{,}0.9{,}1)\big)$;
$ZV(\sigma_1) = \max\big(\mathrm{ZQ}(1{,}4), \mathrm{ZQ}(1{,}2)\big) =$
$= \max\Big(\big((0.51{,}0.67{,}1)(0.8{,}0.9{,}1)\big), \big((0.5{,}0.6{,}0.7)(0.8{,}0.9{,}1)\big)\Big) =$
$= ((0.51{,}0.67{,}1)(0.8{,}0.9{,}1))$;
$Zr_{t+1} + \gamma ZV(\sigma_1) = \big((0{,}0{,}0.1)(0.8{,}0.9{,}1)\big) + 0.9\,\cdot$
$\cdot\, ((0.51{,}0.67{,}1)(0.8{,}0.9{,}1)) = \big((0{,}0{,}0.1)(0.8{,}0.9{,}1)\big) +$
$+((0.46{,}0.6{,}0.9)(0.8{,}0.9{,}1)) = \big((0.46{,}0.6{,}1)(0.8{,}0.9{,}1)\big)$;
$\beta\big[\big(Zr_{t+1} + \gamma ZV(y\sigma_1)\big) \wedge ZQ_C(\sigma_3,\alpha_2)\big] =$
$= 0.9\cdot[\big((0.46{,}0.6.1)(0.8{,}0.9{,}1)\big) \wedge \big((0.6{,}0.7{,}0.8)(0.8{,}0.9{,}1)\big)] =$
$= 0.9\cdot\big((0.46{,}0.6{,}1)(0.8{,}0.9{,}1)\big) = ((0.41{,}0.54{,}0.9)(0.8{,}0.9{,}1))$;
$(1-\beta)ZQ_t(\sigma_3,\alpha_2) = 0.1\cdot\big((0.46{,}0.6{,}1)(0.8{,}0.9{,}1)\big) =$
$= ((0.046{,}0.06{,}0.1)(0.8{,}0.9{,}1))$;
$((0.41{,}0.54{,}0.9)(0.8{,}0.9{,}1)) + ((0.046{,}0.06{,}0.1)(0.8{,}0.9{,}1)) =$
$= ((0.46{,}0.6{,}1)(0.8{,}0.9{,}1))$;
$\boldsymbol{ZQ_C(\sigma_3,\alpha_2)} = ((\mathbf{0.46, 0.6, 1})(\mathbf{0.8, 0.9, 1}))$**;**
$ZV(\sigma_5) = \max\big(\mathrm{ZQ}(5{,}5)\big) = \big((0.7{,}0.8{,}0.9)(0.8{,}0.9{,}1)\big)$;
$Zr_{t+1} + \gamma ZV(\sigma_5) = \big((0{,}0.1{,}0.2)(0.8{,}0.9{,}1)\big) + 0.9\,\cdot$
$((0.7{,}0.8{,}0.9)(0.8{,}0.9{,}1)) = \big((0{,}0.1{,}0.2)(0.8{,}0.9{,}1)\big) +$
$+((0.63{,}0.72{,}0.81)(0.8{,}0.9{,}1)) = \big((0.63{,}0.82{,}1.01)(0.8{,}0.9{,}1)\big)$;
$\beta\big[\big(Zr_{t+1} + \gamma ZV(\sigma_5)\big) \wedge ZQ_C(\sigma_3,\alpha_1)\big]$
$= 0.9\cdot[\big((0.63{,}0.82{,}1.01)(0.8{,}0.9{,}1)\big) \wedge \big((0.9{,}1{,}1)(0.8{,}0.9{,}1)\big)] =$
$= 0.9\cdot\big((0.63{,}0.82{,}1.01)(0.8{,}0.9{,}1)\big) =$
$=((0.57{,}0.74{,}0.91)(0.8{,}0.9{,}1))$;
$(1-\beta)ZQ_t(\sigma_3,\alpha_1) = 0.1\cdot\big((0.63{,}0.82{,}1.01)(0.8{,}0.9{,}1)\big) =$
$= ((0.063{,}0.082{,}0.1)(0.8{,}0.9{,}1))$;
$((0.063{,}0.082{,}0.1)(0.8{,}0.9{,}1)) + ((0.57{,}0.74{,}0.91)(0.8{,}0.9{,}1)) =$
$= \big((0.63{,}0.82{,}1.01)(0.8{,}0.9{,}1)\big)$;
$\boldsymbol{ZQ_C(\sigma_3,\alpha_1)} = \big((\mathbf{0.63, 0.82, 1.01})(\mathbf{0.8, 0.9, 1})\big)$;
$ZV(\sigma_2) = \max\big(\mathrm{ZQ}(2{,}2), \mathrm{ZQ}(2{,}3)\big) =$
$= \max\,(\big((0.57{,}0.74{,}1)(0.8{,}0.9{,}1)\big),$

$\left((0.51,0.67,1)(0.8,0.9,1)\right)) = \left((0.57,0.74,1)(0.8,0.9,1)\right);$
$Zr_{t+1} + \gamma ZV(\sigma_2) = \left((0,0,0.1)(0.8,0.9,1)\right) + 0.9 \cdot$
$\cdot \left((0.57,0.74,1)(0.8,0.9,1)\right) = \left((0,0,0.1)(0.8,0.9,1)\right) +$
$+((0.51,0.67,0.9)(0.8,0.9,1)) = \left((0.51,0.67,1)(0.8,0.9,1)\right);$
$\beta\left[\left(Zr_{t+1} + \gamma ZV(\sigma_2)\right) \wedge ZQ_C(\sigma_2, \alpha_2)\right] = 0.9 \cdot$
$\cdot \left[\left((0.51,0.67,1)(0.8,0.9,1)\right) \wedge \left((0.9,1,1)(0.8,0.9,1)\right)\right] =$
$= 0.9 \cdot \left((0.51,0.67,1)(0.8,0.9,1)\right) = ((0.46,0.6,0.9)(0.8,0.9,1));$
$(1-\beta)ZQ_t(\sigma_2, \alpha_2) = 0.1 \cdot \left((0.51,0.65,1)(0.8,0.9,1)\right) =$
$= ((0.051,0.065,0.1)(0.8,0.9,1));$
$(0.051,0.065,0.1)(0.8,0.9,1)) + ((0.46,0.6,0.9)(0.8,0.9,1)) =$
$= \left((0.51,0.67,1)(0.8,0.9,1)\right);$
$\boldsymbol{ZQ_C(\sigma_2, \alpha_2)} = \left((\mathbf{0.51}, \mathbf{0.67}, \mathbf{1})(\mathbf{0.8}, \mathbf{0.9}, \mathbf{1})\right);$
The obtained results are shown below in sequential order.

I EPİSODE

$ZQ_C(\sigma_1, \alpha_1)$ = ((0.45,0.54,0.73)(0.8,0.9,1))
$ZQ_C(\sigma_1, \alpha_2)$ = ((0,0,0.181)(0.8,0.9,1))
$ZQ_C(\sigma_2, \alpha_1)$ = ((0,0,0.181)(0.8,0.9,1))
$ZQ_C(\sigma_3, \alpha_1)$ = ((0.57,0.738,0.92)(0.8,0.9,1))
$ZQ_C(\sigma_3, \alpha_2)$ = ((0,0,0.181)(0.8,0.9,1))
$ZQ_C(\sigma_2, \alpha_2)$ = ((0,0,0.181)(0.8,0.9,1))

II EPİSODE

$ZQ_C(\sigma_1, \alpha_1)$ = ((0.495,0.594,0.693)(0.8,0.9,1))
$ZQ_C(\sigma_1, \alpha_2)$ = ((0,0,0.231)(0.8,0.9,1))
$ZQ_C(\sigma_2, \alpha_1)$ = ((0.46,0.59,0.86)(0.8,0.9,1))
$ZQ_C(\sigma_3, \alpha_1)$ = ((0.63,0.81,1)(0.8,0.9,1))
$ZQ_C(\sigma_3, \alpha_2)$ = ((0.37,0.44,0.69)(0.8,0.9,1))
$ZQ_C(\sigma_2, \alpha_2)$ = ((0,0,0.255)(0.8,0.9,1))
⋮

VI EPİSODE

$ZQ_C(\sigma_1, \alpha_1)$ = ((0.5,0.6,0.7)(0.8,0.9,1))
$ZQ_C(\sigma_1, \alpha_2)$ = ((0.51,0.67,1)(0.8,0.9,1))
$ZQ_C(\sigma_2, \alpha_1)$ = ((0.57,0.74,1)(0.8,0.9,1))

$ZQ_C(\sigma_3, \alpha_1)$ = ((0.63,0.82,1.01)(0.8,0.9,1))
$ZQ_C(\sigma_3, \alpha_2)$ = ((0.46,0.6,1)(0.8,0.9,1))
$ZQ_C(\sigma_2, \alpha_2)$ = ((0.51,0.67,1)(0.8,0.9,1))

VII EPİSODE
$ZQ_C(\sigma_1, \alpha_1)$ = ((0.5,0.6,0.7)(0.8,0.9,1))
$ZQ_C(\sigma_1, \alpha_2)$ = ((0.51,0.67,1)(0.8,0.9,1))
$ZQ_C(\sigma_2, \alpha_1)$ = ((0.57,0.74,1)(0.8,0.9,1))
$ZQ_C(\sigma_3, \alpha_1)$ = ((0.63,0.82,1.01)(0.8,0.9,1))
$ZQ_C(\sigma_3, \alpha_2)$ = ((0.46,0.6,1)(0.8,0.9,1))
$ZQ_C(\sigma_2, \alpha_2)$ = ((0.51,0.67,1)(0.8,0.9,1))

Thus, the condition $D\big(ZQ_{t+1}(x,a), ZQ_t(x,a)\big) > \epsilon$ is satisfied. The solution is illustrated in Table 2.

Table 2. Solution of the problem

State	Action
σ_1	α_2
σ_2	α_1
σ_3	α_1

Chapter 6 covers the application of Z- and U-numbers to decision-making problems in business environments, as well as computer simulations.

Multicriteria decision making for investment problem under Z-number valued information

One of the main tasks faced by an investment company is to select the most appropriate investment option from among the available alternatives [19]. Let us assume that the company is limited to five

[19] Aliyev, Rafig R. Similarity based multi-attribute decision making under z-information / Proceedings 8th International Conference on Soft Computing, Computing with Words and Perceptions in System Analysis, Decision and Control. Antalya, Turkey. – 2015. - p. 33-38.

alternatives: a_1, a_2, a_3, a_4 and a_5. In order to evaluate these alternatives comparatively, a set of criteria is taken into account. For the evaluation process, four primary criteria have been selected:

C_1-level of risk,

C_2-growh potential,

C_3- service level,

C_4- environmental impact.

Thus, the investment company aims to assess each alternative according to these criteria and to select the most appropriate option.

The codebook of linguistic terms for Z-number-based attributes is provided below (*Level of Criteria– Linguistic Value*)[20]:

Linguistic terms for the A component of a Z-number:

Very Low (VL)	(0,0.1,0.2,0.3)
Low (L)	(0.2,0.3,0.4,0.5)
Medium (M)	(0.4,0.5,0.6,0.7)
High (H)	(0.6,0.7,0.8,0.9)
Very Low (VL)	(0.8,0.9,1,1)

Linguistic terms for the B component of a Z-number (reliability):

Low (L)	(0.05,0.25,0.45)
Medium (M)	(0.25,0.45,0.65,0.85)
High (H)	(0.65,0.85,1)

For the considered multi-criteria decision-making problem, the solution matrix $D_{n\times m}$ is given below:

	C_1	C_2	C_3	C_4
a_1	(VL,L)	(VH,M)	(M,H)	(L,M)
a_2	(VH,M)	(VH,L)	(L,L)	(H,M)
a_3	(H,M)	(VH,M)	(H,M)	(L,L)
a_4	(VH,H)	(H,M)	(M,L)	(L,M)
a_5	(VH,M)	(H,L)	(H,M)	(H,M)

Weights for the criteria:

[20] Jabbarova, K.I. Multi-attribute decision making for investment problem under Z-number valued information // Proceedings of the 9th World Conference on Intelligent Systems for Industrial Automation (WCIS-2016). Tashkent, Uzbekistan, – 2016. - p. 106-109.

$$w_1 = 0.35; \qquad w_2 = 0.3; \qquad w_3 = 0.2; \qquad w_4 = 0.15.$$

The solution of the considered problem is performed in the following sequence:

In Step 1, the positive ideal alternative is identified[21]:

$$a_p^{id} = \big((\mathrm{VH}, \mathrm{H}), (\mathrm{VH}, \mathrm{M}), (\mathrm{H}, \mathrm{M}), (\mathrm{H}, \mathrm{M})\big).$$

In Step 2, the distance between each component of the alternative and the ideal point is obtained[22]:

$$d(Z_{11}, Z_{p_1}^{id}) = 1.6; \qquad d(Z_{12}, Z_{p_2}^{id}) = 0;$$
$$d(Z_{13}, Z_{p_3}^{id}) = 0.6; \qquad d(Z_{14}, Z_{p_4}^{id}) = 0.4;$$
$$d(Z_{21}, Z_{p_1}^{id}) = 0.4; \qquad d(Z_{22}, Z_{p_2}^{id}) = 0.4;$$
$$d(Z_{23}, Z_{p_3}^{id}) = 0.8; \qquad d(Z_{24}, Z_{p_4}^{id}) = 0.$$
$$d(Z_{31}, Z_{p_1}^{id}) = 0.6; \qquad d(Z_{32}, Z_{p_2}^{id}) = 0;$$
$$d(Z_{33}, Z_{p_3}^{id}) = 0; \qquad d(Z_{34}, Z_{p_4}^{id}) = 0.8;$$
$$d(Z_{41}, Z_{p_1}^{id}) = 0; \qquad d(Z_{42}, Z_{p_2}^{id}) = 0.2;$$
$$d(Z_{43}, Z_{p_3}^{id}) = 0.6; \qquad d(Z_{44}, Z_{p_4}^{id}) = 0.4;$$
$$d(Z_{51}, Z_{p_1}^{id}) = 0.4; \qquad d(Z_{52}, Z_{p_2}^{id}) = 0.6;$$
$$d(Z_{53}, Z_{p_3}^{id}) = 0; \qquad d(Z_{54}, Z_{p_4}^{id}) = 0;$$

Next, for each alternative a_i $D(a_i, a_p^{id}) = \sum_{j=1}^{m} W_j D(Z_{ij}, Z_{p_j}^{id})$ is computed:

$$D(a_1, a_p^{id}) = 0.74; \qquad D(a_2, a_p^{id}) = 0.42;$$
$$D(a_3, a_p^{id}) = 0.33; \qquad D(a_4, a_p^{id}) = 0.24;$$

[21] Hwang C.L., Yoon K. Multiple attribute decision making methods and applications. Springer: Berlin Heidelberg, -1981.

[22] Aliev, R., Memmedova, K. Application of Z-Number Based Modeling in Psychological Research // Computational Intelligence and Neuroscience, - 2015. 2015(6), – p. 1-7.

$$D(a_5, a_p^{id}) = 0.32$$

In Step 3, it is obtained that a_4 is the optimal alternative.

Decision making with Z-numbers in evaluating job satisfaction levels

According to the Minnesota Satisfaction Questionnaire (MSQ), the factors used to assess job satisfaction are shown below [23]:

Activity (x_1), Independence (x_2), Variety (x_3), Social status (x_4), Supervision-human relations (x_5), Supervision–technical (x_6), Moral values (x_7), Security (x_8), Social service (x_9), Authority (x_{10}), Ability (x_{11}), Company policies and practices (x_{12}), Compensation (x_{13}), Advancement (x_{14}), Responsibility (x_{15}), Creativity (x_{16}), Working conditions (x_{17}), Co-workers (x_{18}), Recognition (x_{19}), Achievement (x_{20})[24][25].

The codebook of linguistic terms for the factors is provided below:

Linguistic terms for the A component of a Z-number:

Level of Satisfaction	*Linguistic Values*
Unsatisfied (U)	(1,1,1,2)
Less Satisfied (LS)	(1,2,2.5,3)
Quite Satisfied (QS)	(2.5,3,3.5,4)
Satisfied (S)	(3.5,4,4.5,5)
Very Satisfied (VS)	(4.5,5,5)

Linguistic terms for the B component of a Z-number (reliability):

Low (L)	(0,0,0.3,0.4)

[23]Weiss, D.J., Manual for the Minnesota Satisfaction Questionnaire / Dawis, R.V., England, G.W.Lofquist, L.H. // Minneapolis MN: The University of Minnesota Press, - 1967.

[24] Cabbarova, K.I. Z-informasiya şəraitində iş məmnunluğunun qiymətləndirilməsi / - Sumqayıt: Sumqayıt Dövlət Universitetinin Elmi Xəbərləri, -2018. №4, -s.78-83.

[25] Eyupoglu, S.Z., Jabbarova K.I., Aliyeva, K.R. The Identification of Job Satisfaction under Z-Information // Intelligent Automation & Soft Computing, - 2017. №24, - p. 1-5.

Medium (M)	(0.3,0.4,0.6,0.7)
High (H)	(0.6,0.7,1,1)

The linguistic terms for the factors are shown below:

Job Factors/Facets	**Linguistic label**
Activity	Very satisfied, Satisfied, Quite satisfied, Less satisfied, Unsatisfied
Independence	Satisfied, Quite satisfied, Less satisfied
Variety	Very satisfied, Satisfied, Quite satisfied, Less satisfied, Unsatisfied
Social Status	Very satisfied, Satisfied, Quite satisfied, Less satisfied
Supervision-human relations	Very satisfied, Satisfied, Quite satisfied, Unsatisfied
Supervision-technical	Very satisfied, Satisfied, Quite satisfied, Less satisfied, Unsatisfied
Moral Values	Very satisfied, Satisfied, Quite satisfied
Security	Very satisfied, Satisfied, Quite satisfied, Less satisfied, Unsatisfied
Social Service	Very satisfied, Satisfied, Less satisfied
Authority	Very satisfied, Satisfied, Quite satisfied
Ability	Very satisfied, Satisfied, Quite satisfied
Company Policies and Practices	Satisfied, Quite satisfied, Less satisfied, Unsatisfied
Compensation	Very satisfied, Satisfied, Quite satisfied, Less satisfied, Unsatisfied
Advancement	Very satisfied, Satisfied, Quite satisfied, Less satisfied, Unsatisfied
Responsibility	Very satisfied, Satisfied, Quite satisfied, Less satisfied
Creativity	Very satisfied, Satisfied
Working conditions	Satisfied, Quite satisfied, Less satisfied
Co-workers	Very satisfied, Satisfied, Quite satisfied

Recognition	Very satisfied, Satisfied, Less satisfied, Unsatisfied
Achievement	Very satisfied, Satisfied, Quite satisfied

The Z-rules are as follows:

Rule 1:
IF X_1 (VS,H) , X_2 (VS,H), X_3 (QS,H), X_4 (U,M), X_5 (QS,H),X_6 (QS,H), X_7 (VS,H), X_8 (S,H), X_9 (VS,H), X_{10} (QS,H), X_{11} (VS,H),X_{12} (S,H), X_{13} (VS,H), X_{14} (VS,H), X_{15} (VS,H), X_{16} (VS,H), X_{17} (VS,H),X_{18} (QS,H), X_{19} (S,H), X_{20} (VS,H) THEN Y (S,H);

Rule 2:
IF X_1 (VS,H) , X_2 (S,H), X_3 (VS,H), X_4 (VS,H), X_5 (VS,H), X_6 (S,H), X_7 (QS,H), X_8 (QS,H), X_9 (VS,H), X_{10} (S,H), X_{11} (VS,H),X_{12} (QS,H), X_{13} (S,H), X_{14} (VS,H), X_{15} (VS,H), X_{16} (VS,H), X_{17} (S,H),X_{18} (QS,H), X_{19} (VS,H), X_{20} (VS,H) THEN Y (S,H);

Rule 3:
IF X_1 (S,H) , X_2 (S,H), X_3 (S,H), X_4 (S,H), X_5 (QS,H),X_6 (QS,H), X_7 (QS,H), X_8 (QS,H), X_9 (S,H), X_{10} (S,H), X_{11} (S,H), X_{12} (LS,M), X_{13} (LS,M), X_{14} (QS,H), X_{15} (S,H), X_{16} (S,H), X_{17} (LS,M), X_{18} (S,H), X_{19} (QS,H), X_{20} (QS,H) THEN (QS,H);

Rule 4:
IF X_1 (LS,M) , X_2 (S,H), X_3 (LS,M), X_4 (QS,H), X_5 (VS,H), X_6 (S,H), X_7 (S,H), X_8 (S,H), X_9 (S,H), X_{10} (S,H), X_{11} (VS,H), X_{12} (S,H), X_{13} (LS,M), X_{14} (S,H), X_{15} (S,H), X_{16} (QS,M), X_{17} (S,H), X_{18} (S,H), X_{19} (S,H), X_{20} (LS,H) THEN Y (QS,H);

Rule 5:
IF X_1 (S,H) , X_2 (QS,H), X_3 (LS,M), X_4 (S,H), X_5 (S,H),X_6 (S,H), X_7 (LS,M), X_8 (LS,M), X_9 (S,H), X_{10} (S,H), X_{11} (S,H), X_{12} (S,H), X_{13} (LS,M), X_{14} (QS,H), X_{15} (S,H), X_{16} (S,H), X_{17} (QS,H),X_{18} (S,H), X_{19} (S,H), X_{20} (S,H) THEN Y (S,H);

Rule 6:
IF X_1 (QS,H) , X_2 (S,H), X_3 (S,H), X_4 (QS,H), X_5 (S,H), X_6 (M,Y), X_7 (S,H), X_8 (QS,M), X_9 (S,H), X_{10} (QS,H), X_{11} (S,H), X_{12} (QS,H), X_{13} (QS,H), X_{14} (QS,H), X_{15} (QS,H), X_{16} (S,H), X_{17} (QS,H),X_{18} (QS,H), X_{19} (S,H), X_{20} (QS,H) THEN Y (QS,H);

Rule 7:
IF X_1 (M,H) , X_2 (S,H), X_3 (LS,M), X_4 (S,H), X_5 (U,M),X_6 (U,M), X_7 (U,M), X_8 (S,H), X_9 (QS,H), X_{10} (U,M), X_{11} (LS,M),X_{12} (LS,M), X_{13} (LS,H), X_{14} (S,H), X_{15} (LS,M), X_{16} (LS,M), X_{17} (LS,H),X_{18} (S,H), X_{19} (VS,H), X_{20} (S,H) THEN Y (LS,M);

Rule 8:
IF X_1 (S,H) , X_2 (QS,H), X_3 (S,H), X_4 (QS,H), X_5 (S,H),X_6 (S,H), X_7 (QS,H), X_8 (QS,H), X_9 (QS,H), X_{10} (QS,H), X_{11} (S,H),X_{12} (QS,H), X_{13} (LS,M), X_{14} (QS,H), X_{15} (QS,H), X_{16} (S,H), X_{17} (LS,M),X_{18} (S,H), X_{19} (QS,H), X_{20} (S,H) THEN Y (S,H);

Rule 9
IF X_1 (S,H) , X_2 (S,H), X_3 (S,H), X_4 (QS,H), X_5 (S,H),X_6 (S,H), X_7 (QS,H), X_8 (QS,H), X_9 (QS,H), X_{10} (QS,H), X_{11} (S,H),X_{12} (QS,H), X_{13} (LS,M), X_{14} (LS,M), X_{15} (QS,H), X_{16} (QS,H), X_{17} (QS,H),X_{18} (S,H), X_{19} (QS,H), X_{20} (QS,H) THEN Y (QS,H);
Rule 9
IF X_1 (QS,H), X_2 (QS,H), X_3 (QS,H), X_4 (QS,H), X_5 (S,H), X_6 (S,H), X_7 (QS,H), X_8 (QS,H), X_9 (QS,H), X_{10} (QS,H), X_{11} (QS,H), X_{12} (LS,M), X_{13} (LS,M), X_{14} (LS,H), X_{15} (S,H), X_{16} (QS,H), X_{17} (LS,M),X_{18} (S,H), X_{19} (QS,H), X_{20} (QS,H) THEN Y (QS,H);

The current input values are as follows:
IF X_1 (S,H) , X_2 (U,M), X_3 (LS,H), X_4 (S,H), X_5 (U,M), X_6 (LS,M), X_7 (S,H), X_8 (VS,H), X_9 (U,M), X_{10} (VS,H), X_{11} (LS,M),X_{12} (U,M), X_{13} (S,H), X_{14} (S,H), X_{15} (VS,H), X_{16} (S,H),X_{17} (VS,H),X_{18} (QS,H), X_{19} (U,M), X_{20} (QS,H).

For these inputs, the overall Z-number-based job satisfaction index needs to be calculated. The problem is solved as follows:

In the first step, we calculate the activation degree λ_i of the *i-th* rule based on the similarity between the current Z-evaluation vector Z_j of the variables X_j, $j = 1, \dots, 20$ and the Z-number antecedent Z_{ij} of the corresponding rule:

$$\lambda_i = \min_{j=1,\dots,20} S(Z_j, Z_{ij}).$$

As a similarity measure, we suggest computing the function $S(Z_j, Z_{ij})$ using the Jaccard similarity index[26]:

$$S(Z_j, Z_{ij}) =$$

$$= \frac{1}{2} \frac{\sum_{k=1}^{K} \mu_{A_j}(x_k) \cdot \mu_{A_{ij}}(x_k)}{\sum_{k=1}^{K} (\mu_{A_i}(x_k))^2 + \sum_{k=1}^{K} (\mu_{A_{ij}}(x_k))^2 - \sum_{k=1}^{K} \mu_{A_i}(x_k) \cdot \mu_{A_{ij}}(x_k)}$$

$$+ \frac{1}{2} \frac{\sum_{k=1}^{K} \mu_{B_j}(x_k) \cdot \mu_{B_{ij}}(x_k)}{\sum_{k=1}^{K} (\mu_{B_j}(x_k))^2 + \sum_{k=1}^{K} (\mu_{B_{ij}}(x_k))^2 - \sum_{k=1}^{K} \mu_{B_j}(x_k) \cdot \mu_{B_{ij}}(x_k)}.$$

For example, the similarity $S(Z_j, Z_{ij})$ between the current input and the antecedent of Rule 3 is computed as follows: $S(Z_2, Z_{32}) = S((ÇAM, O), (M, Y)) = 0.54$, . $S(Z_8, Z_{38}) = S((ÇM, Y), (OM, Y)) =$ $= 0.5$. According to the given values of $S(Z_j, Z_{3j})$ the activation degree of Rule 3 is determined as $\lambda_3 = 0.014$.

In the second step, by aggregating the Z-number based evaluations Z_{Y_i} corresponding to the consequents Y_i, $i = 1, \dots, 10$, the overall Z-number based job satisfaction evaluation Z_{Y_i} is calculated as follows:

[26] Aliev, R. A. Approximate Reasoning on a Basis of Z-number valued If-Then Rules / R. A. Aliev, W. Pedrycz, O.H. Huseynov [et al. // IEEE Transactions on Fuzzy Systems, - 2017. 25(6), - p. 1589–1600.

$$Z_Y = \sum_{i=1}^{10} \lambda_i Z_{Y_i} \Big/ \sum_{i=1}^{10} \lambda_i$$

The computed Z-number representation of overall job satisfaction is illustrated in Figure 4.

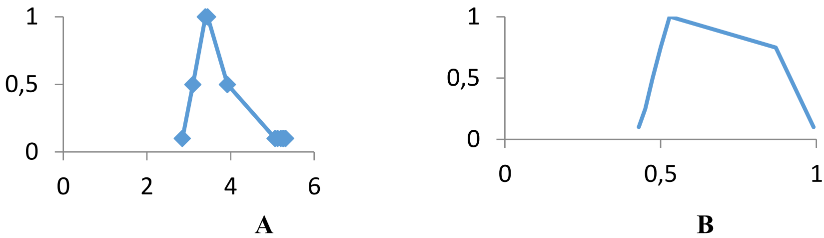

Figure 4. Z-number based value of overall job satisfaction

According to the obtained results, the output of the current rule (Z_y) is (Satisfied, High).

U-number based decision making in production planning

One of the critical issues frequently encountered in the business environment is the process of making sound decisions. To illustrate this issue, let us consider the case of a company engaged in the production of garden sheds [27][28]. The company specializes in manufacturing sheds intended for outdoor storage, and one of the primary questions facing management is the extent to which expanding production is advisable.

[27] Jabbarova, K.I. Decision making on planning construction of plants under u-number-valued information // Proceedings of the 10th World Conference on Intelligent Systems for Industrial Automation (WCIS-2018). Tashkent, Uzbekistan, - 2018, - p. 159-162.
[28] Cabbarova K.İ. U-informasiya şəraitində tikintinin planlaşdırılması məsələsinin həlli // - Sumqayıt: Sumqayıt Dövlət Universitetinin Elmi Xəbərləri, - 2024. №1, s.78-81.

The construction firm's manager aims to make an optimal decision by evaluating various alternative options. The alternatives under consideration are as follows:

a_1-construction of a large factory for shed production;

a_2- construction of a small factory for shed production;

a_3- maintaining the current situation with no changes.

The manager's decision will largely depend on market conditions.

Here, two possible states of nature are distinguished:

S_1 - favorable market conditions;

S_2.- unfavorable market conditions.

Thus, the decision-making process involves the manager choosing among alternatives. The alternatives and probabilities of the states of nature are represented using U-numbers.

	States of Nature	
Alternatives	Favorable Market, s_1	Unfavorable Market, s_2
	$P(s_1)$=(VL,U)	$P(s_2)$=(NVL,U)
a_1	(VH,O)	(VL,U)
a_2	(H,O)	(M,U)
a_3	0	0

The problem is to find the best alternative as an alternative with the maximal U-number valued EU:

Find a^* such that malik a^* tapmaq. $EU(a^*) = \max_{i=1,\dots,4} EU\ (a_i)$.

Solution of the problem

The codebook of linguistic terms for the alternatives and probabilities is presented below:

The codebook of linguistic terms for the A-part of U-number valued alternatives *(Linguistic value- Fuzzy value)*:

Very Low (VL)	(-267, -233, -167, -133)
Low (L)	(-167, -133, -67, -33)
Medium (M)	(-67, -33, 33, 67)
High (H)	(33, 67, 133, 167)
Very High (VH)	(133, 167, 233, 267)

Linguistic terms for the A-part of U-number based probabilities:

Unlikely (U)	(0.15,0.15,0.15,0.25)
Not very likely (NVL)	(0.15,0.25,0.35,0.45)
Likely (L)	(0.35,0.45,0.55,0.65)
Very likely (VL)	(0.55,0.65,0.75,0.85)
Extremely likely (EL)	(0.75,0.85,1,,1)

The codebook of the linguistic terms for the B-part of U-valued alternatives and probabilities (*(Linguistic value- Fuzzy value)*:

Rarely (R)	(0, 0, 0.1, 0.5)
Seldom (S)	(0, 0, 0.3, 0.9)
Often (O)	(0.1, 0.4, 0.6, 0.9)
Usually (U)	(0.1, 0.7, 1, 1)

We will apply the expected utility (EU) model to the problem under consideration:

$$\mathrm{EU}(a_i) = x_{i1}p(s_1) + x_{i2}p(s_2)$$

where $x_{i1} = a_i(s_1), x_{i2} = a_i(s_2), i = 1,2,$ are the U-valued outcomes of alternatives at states of nature, the probabilities $p(s_1)$ and $p(s_2)$ are U-valued.

The computed EU values of the alternatives are as follows:

The U-valued EU of alternative a_1 is:

(1) $EU(a_1)$=(VH,O)·(VL,U)+(VL,U) (NVL,U)=
=((133,167,233,267)(0.1,0.4,0.6,0.9)) ·
((0.55,0.65,0.75,0.85)(0.1,0.7,1,1))+
+((-267,-233,-167,- 133)(0.1,0.7,1,1)
·((0.15,0.25,0.35,0.45)(0.1,0.7,1,1))=
=(-47,27,133,207)(0.0004,0.995,0.995,0.995);

The U-valued EU of alternative a_2 is:

$EU(a_2)$=(H,O) (VL,U)+(M,U) (NVL,U)=

$$=((33,67,133,167)(0.1,0.4,0.6,0.9)) \cdot$$
$$\cdot((0.55,0.65,0.75,0.85)(0.1,0.7,1,1))+$$
$$+((-67,-33,33,67)(0.1,0.7,1,1)\cdot$$
$$\cdot((0.15,0.25,0.35,0.45)(0.1,0.7,1,1))=$$
$$=((-12,32,111,172.1)(0.0003,0.994,0.994,0.994));$$

(2) Obviously, for alternative a_3 we have the following result in terms of U-numbers:

$$EU(a_3)=((0,0,0,0),(1,1,1,1)).$$

This implies that no gain and no loss will certainly be obtained.

To compare the obtained EU results, we will use an approach based on the FPO principle (Definition 3). The comparison results are as follows:

$do(a_1)=0.016$, $do(a_2)=1$,
$do(a_1)=1$, $do(a_3)=0$,
$do(a_2)=1$, $do(a_3)=0$.

The comparison results indicate that a_2 is the best alternative.

Application of expected utility to business decision making under U-number valued information

One of the main challenges faced in the operation of a manufacturing enterprise is making proper business decisions. Let us consider the decision-making process of the enterprise over the course of one year[29]. Suppose that the existing production capacity is insufficient to meet the current market demand, and therefore the production rates are significantly below the demand. This situation

[29] Jabbarova K.I. Application of expected utility to business decision making under U-number valued information // Advances in Intelligent Systems and Computing, - 2018. 896, - p. 716-723.

forces management to evaluate various alternatives for increasing production capacity.

The management of the enterprise is considering the following possible decision alternatives[30]:

Alternative 1 – a relatively low increase in production capacity;
Alternative 2 –a moderate increase in production capacity through the establishment of a new plant;
Alternative 3 – a high increase in production capacity by creating a new product line and implementing additional technologies;
Alternative 4 – maintaining the current situation, i.e., taking no additional measures.

At the same time, the outcomes that the enterprise will achieve depend on market conditions – in other words, on the “states of nature”. Four possible scenarios are distinguished here:

S_1 – low demand;
S_2 – medium demand;
S_3 – high demand;
S_4 – very high demand..

The following states of nature (market conditions) are considered: low demand, medium demand, high demand, and very high demand. Due to the uncertainty of future demand, information on the potential profit of the alternatives is characterized by imprecision and partial reliability. In this regard, the decision-making problem is represented in the form of the following U-valued payoff matrix:

	S_1	S_2	S_3	S_4
a_1	(A,U)	(AA,U)	(HA,U)	(EA,S)
a_2	(L,U)	(A,O)	(EA,O)	(VH,O)
a_3	(EL,S)	(BA,U)	(AA,U)	(EH,U)
a_4	(A,U)	(A,U)	(A,U)	(A,U)

[30] Cabbarova K.İ., Hüseynov, O.H. U-ədədlər nəzəriyyəsinin biznes qərar qəbuletmə məsələsinə tətbiqi.// - Sumqayıt: Sumqayıt Dövlət Universitetinin Elmi Xəbərləri, - 2024. №3, s.75-78.

The problem is to compute U-number valued EUs of the alternatives U(a_i) and find the best alternative as an alternative with the maximal U:

Find a^* such that, $do\big(EU(a^*)\big) = \max_{i=1,\dots,4} do\,\big(\mathrm{EU}(a_i)\big)$.

Solution of the problem

The codebook of linguistic terms for the U-numbers used is as follows (*Linguistic value – Fuzzy value*):

Linguistic terms for the A component of the U-number

Extremely Low (EL)	(-92,-83,-67,-58)
Very Low (VL)	(-67,-58,-42,-33)
Low (L)	(-42,-33,-17,-8)
Low Average (LA)	(-17,-8,8,17)
Below Average (BA)	(8,17,33,42)
Average (A)	(33,42,58,67)
Above Average (AA)	(58,67,83,92)
High Average (HA)	(83,92,108,117)
Extremely Average (EA)	(108,117,133,142)
High (H)	(133,142,158,167)
Very High (VH)	(158,167,183,192)
Extremely High (EH)	(183,192,200)

Linguistic terms for the B component of the U-number

Rarely (R)	(0,0,0.1,0.5)
Seldom (S)	(0,0,0.3,0.9)
Often (O)	(0.1,0.4,0.6,0.9)
Usually (U)	(0.1,0.7,1,1)

Information on the probabilities of the states of nature is also characterized by imprecision and partial reliability. However, for the sake of simplicity in calculations, we will not consider U-valued probabilities of the states of nature. We will examine the cases of numerical and fuzzy probabilities:

Case I. In this case, let the probabilities be taken as:$P(S_1)=0.3$, $P(S_2)=0.25$, $P(S_3)=0.35$, $P(S_4)=0.1$ Based on these probabilities, the expected utilities (EU) are calculated using the utility values U of the alternatives EU results are obtained using the corresponding values for each alternative:

$EU(a_1)=P(S_1)\cdot(A,U)+P(S_2)\cdot(AA,U)+P(S_3)\cdot(HA,U)+P(S_4)\cdot(EA,S)=$
$=0.3\cdot((33,42,58,67)(0.1,0.7,1,1))+0.25\cdot$
$\cdot(58,67,83,92)\ (58,67,83,92))+\ 0.35\cdot((83,92,108,117)\ (0.1,0.7,1,1))+$
$+0.1\cdot((108,117,133,142)(0,0,0.3,0.9))=$
$=((62.4,69.6,82.9,90.1)(0,0.0007,0.299,0.897));$

$EU(a_2)=P(S_1)\cdot(L,U)+P(S_2)\cdot(A,O)+P(S_3)\cdot(EA,O)+P(S_4)\cdot(VH,O)=$
$=0.3\cdot((-42,-33,-17,-8)\ (0.1,0.7,1,1))+\ 0.35\cdot$
$\cdot((33,42,58,67)\ (0.1,0.4,0.6,0.9))+0.35$
$\cdot((108,117,133,142)\ (0.1,0.4,0.6,0.9))+0.1\cdot$
$\cdot((158,167,183,192)(0.1,0.4,0.6,0.9))=$
$=((48.95,57.55,72.95,81.55)(0.0004,0.089,0.286,0.74));$

$EU(a_3)=P(S_1)\cdot(EL,S)+P(S_2)\cdot(BA,U)+P(S_3)\cdot(AA,U)+P(S_4)\ (EH,U)=$
$0.3\cdot((-92,-83,-67,-58)(\ (0,0,0.3,0.9))+0.25\cdot$
$\cdot((8,17,33,42)\ (0.1,0.7,1,1))+0.35\ ((58,67,83,92)\ (0.1,0.7,1,1))+$
$+0.1\cdot((183,192,200)\ (0.1,0.7,1,1))=$
$=\ ((12.25,20.25,33.95,41.05)(0.0002,0.226,0.599,0.897));$

$EU(a_4)=P(S_1)\cdot(A,U)+P(S_2)\cdot(A,U)+P(S_3)(A,U)+P(S_4)\cdot(A,U)=$
$=0.3\cdot((33,42,58,67)\ (0.1,0.7,1,1))+\ 0.25\cdot$
$\cdot((33,42,58,67)(0.1,0.7,1,1))+0.35\cdot((33,42,58,67)(0.1,0.7,1,1)+0.1\cdot$
$\cdot((33,42,58,67)(0.1,0.7,1,1))=$
$=((33,42,58,67)(0.0004,0.994,0.994,0.994)).$

Subsequently, in order to determine the most optimal alternative, the U-values will be compared based on the Fuzzy Pareto Optimality principle (Definition 3). The results are presented below:

$do(EU(a_1)) = 1, do(EU(a_2)) = 0.59,\ do(EU(a_4)) = 1,$
$do(EU(a_3)) = 0,$
$do(EU(a_1)) = 1, do(EU(a_3)) = 0.49,\ do(EU(a_4)) = 1,$
$do(EU(a_2)) = 0.67,$

$do(EU(a_1)) = 0.9, do(EU(a_4)) = 1,\ do(EU(a_2)) = 1,$
$do(EU(a_3)) = 0.5.$

As one can see, the best alternative is a_4.

Case II. Now let us consider the probabilities as fuzzy. It is known that fuzzy probabilities can only be assigned to n-1 states of nature, while the remaining ones must be computed[31].

Suppose that $P(S_1)$=(0.285,0.3,0.315), $P(S_2)$=(0.2375,0.25, 0.2625), $P(S_3)$=(0.3325,0.35,0.3675) and $P(S_4)$ is unknown. We obtain $P(S_4)$=(0.055,0.1,0.145).

Then, the U-valued Expected Utilities (EU) of the alternatives are calculated:

EU(a_1)=$P(S_1)$·(A,U)+$P(S_2)$·(AA,U)+$P(S_3)$·(HA,U)+$P(S_4)$ (EA,S)=
=(0.285,0.3,0.315) ((33,42,58,67)(0.1,0.7,1,1))+
+(0.2375,0.25,0.2625)·((83,92,108,117)(0.1,0.7,1,1))+
+(0.3325,0.35,0.3675) ((83,92,108,117) (0.1,0.7,1,1))+
+(0.055,0.1,0.145)· ((108,117,133,142) (0,0,0.3,0.9))=
=((55.28,69.6,82.9,99.4)(0,0.0008,0.299,0.895));

EU(a_2)=$P(S_1)$ (L,U)+$P(S_2)$ (A,O)+$P(S_3)$ (EA,O)+$P(S_4)$ (VH,O)=
=(0.285,0.3,0.315)· ((-42,-33,-17,-8)(0.1,0.7,1,1))+
+(0.2375,0.25,0.2625) ((33,42,58,67) (0.1,0.4,0.6,0.9))+
+(0.3325,0.35,0.3675) ((108,117,133,142) (0.1,0.4,0.6,0.9))+
+(0.055,0.1,0.145) ((158,167,183,192) (0.1,0.4,0.6,0.9))=
= ((40.3,57.55,72.95,92.63)(0.0004,0.089,0.286,0.59));

EU(a_3)=$P(S_1)$ (EL,S)+$P(S_2)$ (BA,U)+$P(S_3)$ (AA,U)+$P(S_4)$ (EH,U)=
=(0.285,0.3,0.315) ((-92,-83,-67,- 58)(0,0,0.3,0.9))+
+(0.2375,0.25,0.2625) ((8,17,33,42)(0.1,0.7,1,1))+
+(0.3325,0.35,0.3675) ((83,92,108,117) (0.1,0.7,1,1))+
+(0.055,0.1,0.145) ((183,192,200) (0.1,0.7,1,1))=
=((4.32,20.25,33.95,51.1)(0.0001,0.0002,0.299,0.89));

EU(a_4)=$P(S_1)$ (A,U)+$P(S_2)$ (A,U)+$P(S_3)$ (A,U)+$P(S_4)$ (A,U)=
=(0.285,0.3,0.315) (33,42,58,67)(0.1,0.7,1,1))+(0.2375,0.25,0.2625) ((33,42,58,67) (0.1,0.7,1,1))+(0.3325,0.35,0.3675)

[31] Zadeh, L.A., Generalized theory of uncertainty (GTU) – principal concepts and ideas // Computational Statistics & Data Analysis, - 2006, 51, - p. 15-46.

((33,42,58,67) (0.1,0.7,1,1))+(0.055,0.1,0.145)
((33,42,58,67) (0.1,0.7,1,1))=
=((30.03,42,58,73)(0.0004,0.32,0.99,0.99)).

Based on the obtained utility values, the comparison results of the alternatives will be as follows (Definition 3).

$do(EU(a_1)) = 1, do(EU(a_2)) = 0.6,$
$do(EU(a_4)) = 1, do(EU(a_3)) = 0,$
$do(EU(a_1)) = 1, do(EU(a_3)) = 0,$
$do(EU(a_4)) = 1, do(EU(a_2)) = 0.6,$
$do(EU(a_1)) = 0.9, do(EU(a_4)) = 1,$
$do(EU(a_2)) = 1, do(EU(a_3)) = 0.$

Let us analyze the sensitivity of the decision obtained in Case II with respect to the application of the outcomes of the alternatives presented in the solution table below. To investigate what occurs in situations with a low level of usuality, we conduct two experiments.

1) The level of utility of the outcomes decreases from "usually" to "seldom". Pay-off table with the reduced level of usuality of the ouctomes is as follows:

	S_1	S_2	S_3	S_4
a_1	(A,S)	(AA,S)	(HA,S)	(EA,S)
a_2	(L,S)	(A,O)	(EA,O)	(VH,O)
a_3	(EL,S)	(BA,S)	(AA,S)	(EH,S)
a_4	(A,S)	(A,S)	(A,S)	(A,S)

The calculated values of the EU are as follows:

EU(a_1)=P(S_1)· (A,S)+P(S_2)·(AA,S)+P(S_3)·(HA,S)+P(S_4)
·(EA,S)=(0.285,0.3,0.315)·((33,42,58,67)(0,0,0.3,0.9))+
+(0.2375,0.25,0.2625)·((58,67,83,92) (0,0,0.3,0.9))+
+(0.3325,0.35,0.3675)·((58,67,83,92)(0,0,0.3,0.9))+
+(0.055,0.1,0.145)·(108,117,133,142)(0,0,0.3,0.9)=
=((55.28,69.6,82.9,99.4)(0,0,0.02,0.7));

EU(a_2)=P(S_1)(L,S)+P(S_2)· (A,O)+P(S_3)· (EA,O)+P(S_4)· (VH,O)=
=(0.285,0.3,0.315) ((-42,-33,-17,-8) (0,0,0.3,0.9))+
+(0.2375,0.25,0.2625)((33,42,58,67) (0.1,0.4,0.6,0.9))+

+(0.3325,0.35,0.3675) ((83,92,108,117) (0.1,0.4,0.6,0.9))+
+(0.055,0.1,0.145) ((158,167,183,192) (0.1,0.4,0.6,0.9))=
= ((40.3,57.55,72.95,92.63)(0.0002,0.001,0.115,0.678));
EU(a_3)=P(S1) (EL,S)+P(S2) (BA,S)+P(S3) (AA,S)+P(S4) (EH,S)=
=(0.285,0.3,0.315) ((-92,-83,-67,-58)(0,0,0.3,0.9))+
+(0.2375,0.25,0.2625) ((8,17,33,42)(0,0,0.3,0.9))+
+(0.3325,0.35,0.3675) ((58,67,83,92)(0,0,0.3,0.9))+
+(0.055,0.1,0.145) ((183,192,200)(0,0,0.3,0.9))=
=((4.32,20.25,33.95,51.1)(0,0,0.02,0.68));
EU(a_4)=P(S1) (A,S)+P(S2) (A,S)+P(S3) (A,S)+P(S4) (A,S)=
=(0.285,0.3,0.315) ((33,42,58,67) (0,0,0.3,0.9))+
+(0.2375,0.25,0.2625) ((33,42,58,67) (0,0,0.3,0.9))+
+(0.3325,0.35,0.3675) ((33,42,58,67) (0,0,0.3,0.9))+
+(0.055,0.1,0.145) (33,42,58,67) (0,0,0.3,0.9))=
=((30.03,42,58,73)(0,0,0.02,0.68)).

The results of the comparison of the alternatives based on their EU values are as follows (Definition 3):

$do\big(EU(a_1)\big) = 0.5, do\big(EU(a_2)\big) = 1,$
$do(EU(a_4)) = 1, do(EU(a_3)) = 0,$
$do(EU(a_1)) = 1, do(EU(a_3)) = 0,$
$do(EU(a_4)) = 0, do(EU(a_2)) = 1,$
$do\big(EU(a_1)\big) = 1, do\big(EU(a_4)\big) = 0,$
$do(EU(a_2)) = 1, do(EU(a_3)) = 0,$

As can be seen, the results are sensitive to changes in the level of usuality, so that a_2 becomes the best alternative instead of a_4. his is due to the fact that this alternative is characterized by a relatively high degree of reliability of the outcomes.

2) The level of usuality of outcomes is decreased from “often” to “seldom”. The computed values of expected utility:
EU(a_1)=P(S1) (A,U)+P(S2) (AA,U)+P(S3) (HA,U)+P(S4) (EA,S)=
=(0.285,0.3,0.315) (33,42,58,67)(0.1,0.7,1,1))+(0.2375,0.25,0.2625)
((58,67,83,92) (0.1,0.7,1,1))+(0.3325,0.35,0.3675)
((83,92,108,117) (0.1,0.7,1,1))+(0.055,0.1,0.145)

((83,92,108,117) (0,0,0.3,0.9))=
=(55.28,69.6,82.9,99.4)(0,0.0008,0.299,0.895));
EU(a_2)=P(S_1) (L,U)+P(S_2) (A,S)+P(S_3) (EA,S)+P(S_4) (VH,S)=
=(0.285,0.3,0.315) ((-42,-33,-17,-8)(0.1,0.7,1,1))+
+(0.2375,0.25,0.2625) ((33,42,58,67)(0.1,0.4,0.6,0.9))+
+((0.3325,0.35,0.3675) ((108,117,133,142) (0,0,0.3,0.9))+
+(0.055,0.1,0.145) ((158,167,183,192) (0,0,0.3,0.9))=
=((40.3,57.55,72.95,92.63)(0,0,0.05,0.74));
EU(a_3)=P(S_1) (EL,S)+P(S_2) (BA,U)+P(S_3) (AA,U)+P(S_4) (EH,U)=
=(0.285,0.3,0.315) ((-92,-83,-67,-58)(0,0,0.3,0.9))+
+(0.2375,0.25,0.2625) · ((8,17,33,42) (0.1,0.7,1,1))+
(0.3325,0.35,0.3675) · ((58,67,83,92) (0.1,0.7,1,1))+
+(0.055,0.1,0.145) · ((183,192,200) (0.1,0.7,1,1))=
=((4.32,20.25,33.95,51.1)(0.0001,0.0002,0.299, 0.89));
EU(a_4)=P(S_1) · (A,U)+P(S_2) · (A,U)+P(S_3) · (A,U)+P(S_4) · (A,U)=
=(0.285,0.3,0.315) ((33,42,58,67) (0.1,0.7,1,1))+
+(0.2375,0.25,0.2625) ((33,42,58,67) (0.1,0.7,1,1))+
(0.3325,0.35,0.3675) · ((33,42,58,67) (0.1,0.7,1,1))+
+((0.3325,0.35,0.3675) ((33,42,58,67) (0.1,0.7,1,1))=
=((30.0342,58,73)(0.0004,0.32,0.99,0.99)).

The results of the comparison of the alternatives based on the obtained values are given below (Definition 3):

$do(EU(a_1)) = 1, do(EU(a_2)) = 0,$
$do(EU(a_4)) = 1, do(EU(a_3)) = 0,$
$do(EU(a_1)) = 1, do(EU(a_3)) = 0,$
$do(EU(a_4)) = 1, do(EU(a_2)) = 0.4,$
$do\big(EU(a_1)\big) = 0.9, do\big(EU(a_4)\big) = 1,$
$do(EU(a_2)) = 1, do(EU(a_3)) = 0.5,$

As can be seen, a_4 remains the best alternative. However, its optimality decreases due to a reduction in the utility level of its outcomes.

It can be concluded that the considered solution is sensitive to changes in the utility levels of the outcomes. If the reliability of the initially selected alternative decreases, the decision-maker would choose an alternative with higher outcome reliability.

Solving U- information based project selection problem

Let us now consider the project selection problem based on U-information. Three projects P1, P2, P3 are evaluated by four criteria: net present value (C1), quality (C2), contractor's technology (C3) and contractor's economic status (C4)[32][33]. The values of the criteria and their importance weights are characterized by U-numbers in the form U = (A, B).

The evaluations of the alternatives with respect to the U-number–valued criteria:

	C_1	C_2
f_1	(0.55,0.61,0.67)(0.5,0.7,0.9)	(0.44,0.48,0.53)(0.3,0.5,0.7)
f_2	(0.39,0.43,0.48)(0.5,0.7,0.9)	(0.53,0.59,0.65)(0.3,0.5,0.7)
f_3	(0.6,0.66,0.73)(0.5,0.7,0.9)	(0.58,0.65,0.71)(0.3,0.5,0.7)

	C_3	C_4
f_1	(0.52,0.57,0.63)(0.5,0.7,0.9)	(0.50,0.56,0.61)(0.5,0.7,0.9)
f_2	(0.44,0.49,0.54)(0.5,0.7,0.9)	(0.46,0.51,0.56)(0.5,0.7,0.9)
f_3	(0.6,0.66,0.72)(0.5,0.7,0.9)	(0.59,0.65,0.72)(0.5,0.7,0.9)

The importance weights of the criteria:

w_1	(0.16,0.25,0.38)(0.5,0.7,0.9)
w_2	(0.16,0.24,0.36)(0.5,0.7,0.9)
w_3	(0.15,0.23,0.35)(0.5,0.7,0.9)
w_4	(0.18,0.28,0.42)(0.5,0.7,0.9)

Let us solve this problem using the PROMETHEE algorithm based on U-number–valued information [34].

1) A weighted normalized decision matrix is constructed by multiplying the value of each criterion by its corresponding importance weight:

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[32]Rouyendegh, B.D., Erol, S. Selecting the Best Project Using the Fuzzy ELECTRE Method // Mathematical Problems in Engineering, - 2012. Article ID 790142,-12 p.
[33] Jabbarova, K.I. Project selection under U-number-valued information // Lecture Notes in Networks and Systems, -2022, 362, - p. 286-293.
[34]Jabbarova, K. I., Jabbarova, A. I. Z-information Based PROMETHEE Method // Advances in Intelligent Systems and Computing, -2021. 1323, - p. 287-293.

	C_1	C_2
f_1	((0.09,0.15,0.25)(0.3,0.52,0.82))	((0.07,0.12,0.19)(0.19,0.39,0.64))
f_2	((0.06,0.11,0.18)(0.3,0.53,0.82))	((0.08,0.14,0.23)(0.19,0.39,0.64))
f_3	((0.1,0.17,0.23)(0.3,0.53,0.82))	((0.09,0.16,0.26)(0.19,0.39,0.64))

	C_3	C_4
f_1	((0.08,0.13,0.22)(0.3,0.53,0.82))	((0.09,0.16,0.26)(0.3,0.53,0.82))
f_2	((0.07,0.11,0.19)(0.3,0.53,0.82))	((0.08,0.14,0.24)(0.3,0.53,0.82))
f_3	((0.09,0.15,0.25)(0.3,0.53,0.82))	((0.11,0.18,0.3)(0.3,0.53,0.82))

2) For comparing two alternatives g and f on each criterion, distances values

$$D\left(U_{gj}(A,B),((0.9,1,1)(0.9,1,1)\right),$$

$$D\left(U_{fj}(A,B),((0.9,1,1)(0.9,1,1))\right),$$

$D\left(U_{gj}(A,B),Z_{fj}(A,B)\right)$ are computed.

Here, ((0.9,1,1),(0.9,1,1))((0.9,1,1),(0.9,1,1)) represents the ideal solution.

IF $D(U_{gj}(A,B),((0.9,1,1)(0.9,1,1)) \leq$
$D(U_{fj}(A,B),((0.9,1,1)(0.9,1,1))$ THEN
$U_{gj}(A,B) \geq U_{fj}(A,B)$ olar.

Then, the preference function is defined as follows:

$$P\left(U_{gj}(A,B),U_{fj}(A,B)\right) = \\ = \begin{cases} 0, U_{gj}(A,B) \leq U_{fj}(A,B) \\ D\left(U_{gj}(A,B),U_{fj}(A,B)\right), U_{gj}(A,B) > U_{fj}(A,B). \end{cases} \quad (21)$$

The obtained results are presented below:

	C_1	C_2	C_3	C_4
$P(1,2)$	0.08	0	0.03	0.03
$P(1,3)$	0	0	0	0
$P(2,1)$	0	0.04	0	0
$P(2,3)$	0	0	0	0
$P(3,1)$	0.03	0.07	0.03	0.04

3) U-number valued preference index is computed to define the value of the outranking relation $(j = 1,2,\dots,n)$:

$$\pi(g,f) = \sum_{j=1}^{n}\left[w_j P_j(g,f)\right]. \qquad (22)$$

The obtained results will be as follows:

	f_1	f_2	f_3
f_1	-	(0.023,0.035,0.054) (0.22,0.43,0.76)	-
f_2	(0.006,0.01,0.014), (0.5,0.7,0.9)	-	-
f_3	(0.0282,0.043,0.0634), (0.15,0.35,0.71)	(0.042,0.066,0.099), (0.18,0.37,71)	-

4) The leaving and entering flows are computed for ranking of alternatives:

$$\phi^{+}(g) = \sum_{\substack{f \neq g \\ f=1}}^{m} \pi(g,f), \quad \phi^{-}(g) = \sum_{\substack{f \neq g \\ f=1}}^{m} \pi(f,g) \qquad (23)$$

The results obtained based on (23) are presented below:

	Leaving flows	Entering flows
f_1	(0.0115,0.0175,0.027) (0.22,0.43,0.76)	(0.017,0.0265,0.0385) (0.15,0.35,0.73)
f_2	(0.003,0.005,0.007) (0.5,0.7,0.9)(	(0.0325,0.0505,0.0765) (0.18,0.37,0.73)
f_3	(0.035,0.055,0.08) (0.15,0.35,0.73)	-

5) Net flow is computed as follows:

$$\phi(g) = \phi^{+}(g) - \phi^{-}(g) \qquad (24)$$

The results derived from (24) are shown below:

	$\phi(g)$
f_1	(-0.006,-0.009,-0.012)(0.19,0.35,0.62)
f_2	(-0.03,-0.05,-0.07)(0.21,0.4,0.62)
f_3	(0.04,0.06,0.08),(0.15,0.35,0.73)

Finally, the alternatives are ranked using Definition 3

$f_3 \succ f_2 \succ f_1$.

Consequently, alternative f_3 demonstrates the highest preference.

Chapter 7 is devoted to the application of the proposed method under Z-information conditions to hierarchical decision-making problems to optimal port selection and the evaluation of technical systems. These two problems, which have a hierarchical (two-level) structure, are solved using both Z-number theory and interval arithmetic based on the Relative Distance Measure (RDM), and a comparative analysis of the obtained results has been conducted.

Decision-making in port selection using Z-number theory

Port selection is one of the well-known hierarchical (two-level) multi-criteria decision-making problems that attracts both theoretical and practical interest. Numerous key factors influencing port selection research have been reflected in the existing literature. The identification of the key factors affecting port selection has been more clearly demonstrated in the literature[35][36][37]. Thus, each port has criteria $C_j, j = 1, \ldots, 7$: C_1-: hınterland condıtıon, C_2 - port servıces, C_3 - logıstıcs cost, C_4 - connectıvıty, C_5 - convenıence, C_6 - avaılabılıty, C_7 - regıonal center. The evaluation for each criterion is the aggregate of the related assessments across its sub-criteria. Each criterion $C_j, j = 1, \ldots, 7$ has sub-criteria $C_{ji}, i = 1, \ldots, n_j$. For example, criterion C_1: "hınterland condıtıon" has 3 sub-criteria: C_{11} - professionals and skilled labors in port operation, C_{12} - size and activity of FTZ in port hinterland, C_{13}-volume of total container cargo. After reviewing the relevant literature in this field, selected

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experts identify the key criteria for port selection. Each criterion and sub-criterion is characterized by its corresponding importance weight.

The criteria and sub-criteria are presented below:

C1: HINTERLAND CONDITION:

1. Professionals and skilled labors in port operation
2. Size and activity of FTZ in port interland
3. Volume of total container cargo

C2: PORT SERVICES

1. Prompt response
2. 24hours/7 a week service
3. Zero waiting time

C3: LOGISTICS COST

1. Inland transportation cost
2. Cost related vessel and cargo entering
3. Free dwell time on the terminal

C4:CONNECTIVITY

1. Land distance and connectivity to major supplier
2. Efficient inland transport network

C5:CONVENIENCE

1. Wather depth in approach channel and at berth
2. Sophistication level of port information & its application scope
3. Stability of Port's labour

C6: AVAILABILITY

1. Availability of vessel berth on arrival in port
2. Port Congestion

C7:REGIONAL CENTER

1. Port Accessibility
2. Deviation from main trunk routes

The decision maker must select the optimal port using the criteria and sub-criteria presented above. The alternatives under consideration are as follows: port of Busan, port of Tokyo, port of Hong Kong, port of Qingdao, port of Shanghai, port of Kaohsiung, port of Shenzhen[38]. The available information is characterized by uncertainty and partial reliability. Taking all this into account, the criterion values and their importance weights are represented using Z-numbers.

The codebook for the A and B fuzzy components of the Z-numbers is as follows[39] (*Level - Linguistic value*):

Linguistic terms for the A component of the Z-number

Very Low (VL) (1,1,2)

Low (L) (1,2,3)

Medium (M) (2,3,4)

High (H) (3, 4, 5)

Very High (VH) (4,5,5)

Linguistic terms for the B component of the Z-number

Unlikely (U) (0.05, 0.05, 0.25)

Not very likely (NVL) (0.05, 0.25, 0.5)

Likely (L) (0.25, 0.5, 0.75)

Very Likely (VL) (0.5, 0.75, 1)

Extremely likely (EL) (0.75, 1, 1)

Z-number–based results:

Hong Kong:-(0.89,3.9,9.5)(0.85,0.98,0.99)

Busan -(0.8,3.5,10.7)(0.5,0.7,0.8)

Tokyo-(0.71,3.54,13.6)(0.4,0.6,0.7)

Shanghai-(0.5,3.32,12.9)(0.48,0.81,0.82)

Shenzen-(0.71,3.54,13.6)(0.4,0.6,0.7)

Kaohsiung-(0.62,2.98,10.3)(0.48,0.7,0.72)

Qingdao-(0.7,3.5,13.4)(0.66,0.96,0.97)

Z-number–based results are compared and ranked according to Definition 3.

Hong-Kong-1

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Qingdao -2
Shanghai-3
Shenzen-4
Busan-6
Tokyo-5
Kaohsiung-7

MAIN SCIENTIFIC RESULTS

1. For the first time, a Z-number-based Linear Programming (Z-LP) problem has been formulated in this study, and a solution method has been proposed. Unlike existing linear programming problems with uncertain parameters, the key distinction of the proposed method is that it accounts for the reliability of the obtained solutions. In contrast to the gradient methods typically used for such problems, a Differential Evolution algorithm has been employed for optimization, which enables to find the global solution.

2. The decision-making problem has been formalized using the concept of Z-number as a comprehensive extension of classical fuzzy logic, and the interpolation method used for classical cases has been extended to decision analysis under Z-information. This approach has been developed for problems in which decision-making is based on expert knowledge characterized by partial reliability and expressed in the form of "If...Then..." rules. To account for such knowledge, a computational mechanism based on "If...Then..." rules, where antecedents and consequents are represented by Z-numbers, has been proposed.

3. The concept of U-numbers, a special case of Z-numbers, has been applied to recurring situations in decision-making. The results obtained facilitate simpler and more efficient decision-making in such situations.

4. Hierarchical decision analysis under Z-information conditions has been considered. In the proposed solution method, the aggregation of Z-number-based evaluations of criteria and subcriteria relies on computational operations performed on Z-numbers.

5. A learning method based on confirmation under Z-information conditions has been proposed. In this method, the partial reliability of information related to the objectives and constraints of decision-making is represented using Z-numbers. The proposed method is based on computational operations performed on Z-numbers.

6. Theoretical results proposed in this dissertation were applied in port selction, identification of technical systems, multicriteria decision making for car choice and other problems. The obtained results illustrate validity of the proposed theoretical findings.

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Author's personal role in co-published works:

[1]- Computer simulation, and analysis of results;
[2]- Problem solution, computer simulation, and analysis of results;
[3]- Author of the idea, problem statement, problem solution, and analysis of results;
[4]- Problem solution, computer simulation, and analysis of results;
[8]- Problem solution, computer simulation, and analysis of results;
[9]- Problem solution, computer simulation, and analysis of results;
[11]- Author of the idea, problem statement, computer simulation, and analysis of results;
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[15]- Author of the idea, problem statement, and analysis of results;
[18]- Author of the idea, problem statement, computer simulation, and analysis of results;
[21]- Author of the idea, problem solution, computer simulation;
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[27]- Author of the idea, problem statement, computer simulation;
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Address: Baku, Azadlig avenue 20, AZ 1010
e-mail: info@asoiu.edu.az

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