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ABSTRACT

of the dissertation for the degree of Doctor of Science

OPERATION ON Z-NUMBERS AND THEIR APPLICATION ON DECISION MAKING

Specialty: 3338.01 – System analysis, control and information processing (control and decision making)

Field of science: Technical sciences

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İMZANI TƏSDİQ EDİRƏM
ADNSU-nun Elmi katibi
dosent N.Əliyeva

GENERAL CHARACTERISTICS OF THE WORK

Relevance of the topic and degree of elaboration. In the modern world, the problem of decision-making is a particularly relevant issue, in many areas of human activity. The growing complication of various technical, economic and social structures requires finding and applying new methods, as well as improving existing methods for decision-making by involving decision-maker.

As an object of research, complex decision-making systems are used in various fields of science, for example, management of various technological processes, system analysis, etc. Currently, in many modern production and work processes, specialists have to work in the environment characterized by non-stationary production processes, price fluctuations, changes in working conditions, etc. In these areas, the theory of fuzzy sets proposed by Professor Lotfi Zadeh plays an important role, which allows to describe linguistic information in the form of mathematical expressions and creates conditions for the creation and application of new scientific methods.

Starting from von Neumann and Morgenstern's theory of expected utility, most of the existing decision-making theories are based on a rigorous mathematical foundation and provide satisfactory results. In addition, it is designed to make decisions in special situations. Current decision-making theories have several important shortcomings: technics based on perfect information is used instead of real imperfect information; well-formed structured knowledge of future objective conditions is required; the classical probability measure is used, in reality the probabilities are imprecise; a measure of uncertainty based on binary logic is used when there may be real advantages; the fact that a person draws conclusions from the linguistic representation of information is not taken into account; interaction of decision-makers' behavior determinants is not taken into account; partial reliability of relevant information is not taken into account for making a true decision. Most of the work is devoted to decision-making under first-order uncertainty. Thus, there is a need to develop a theory of decision making that is free from the above limitations. The dissertation proposes the foundations of a new decision theory based on a comprehensive review of relevant imperfect information under two types of decision-making uncertainty. The above mentioned shows the importance of the scientific problem posed in the dissertation. Validity and application of the proposed theory, models, and decision-making methods are demonstrated by using benchmark standard and real decision-making problems. From this point of view, the relevance of the research conducted in the dissertation is clear.

The goals and objectives of the dissertation work. The aim of the dissertation is to develop the theory, models and decision-making methods under conditions of uncertainty expressed by Z-numbers. To achieve this goal, the following tasks were set and relevant problems were solved: – Comparative analysis of existing decision-making methods; – Analysis of information in decision-making processes; – Analysis of information based on Z numbers; – Development of operations on Z-numbers; – Building a decision-making model based on Z-information; – Building a linear programming model based on Z-information; – Creating a decision-making method based on the comparison of Z-numbers; – Verification of the effectiveness of the proposed methods and models based on Z-information.

Research methods. Fuzzy set theory, possibility theory, soft computing technologies, utility theory, system analysis methods, decision-making methods based on partially reliable information, Z-number theory, uncertainty theory, fuzzy inference are used to solve these problems in the dissertation work. Experimental research methods were used together with mathematical and simulation modeling methods to confirm the obtained theoretical results. The computer simulation was carried out in the ZLab software package created according to the MatLab software package and shows the effectiveness of the results.

Scientific novelty of the research. The scientific novelty of the work results is as follows: - A methodology for solving decision-making problems based on fuzzy information was proposed; - Theoretical basics of decision-making in conditions of uncertainty expressed through Z-numbers were proposed; - Arithmetic operations on discrete Z-numbers were developed for the first time; - A decision-making method was proposed in the conditions of Z-information expressed directly by Z-numbers; - A model of decision-making in Z-

information conditions was developed without using utility function;
- A method for building a multi-criteria decision-making model in Z-information conditions is proposed.

Theoretical and practical significance of research. The proposed models and decision-making methods differ from existing classical methods by using a more detailed description of uncertainty, taking into account psychological and individual determinants of decision-making, as well as consideration of interval, fuzzy and Z-based information. The obvious advantages of the proposed models and decision-making methods from a practical point of view have been proven by solving various practical decision-making problems. The theoretical basics proposed in the dissertation have been widely applied in the Zlab software package, including operations on Z-numbers, Z-linear programming, Pareto optimality, etc. It is included in the Zlab package and is widely used in different countries around the world. The results of the dissertation are general in nature, the proposed models and decision-making methods can be applied in various fields of economics, psychology, sociology, technical fields, etc.

Approval and application. The main scientific and practical results of the dissertation were discussed in Research laboratory "Intelligent control and decision-making systems in industry and economics" of Azerbaijan State Oil and Industry University as well as in scientific seminars organized at international conferences:

– Second International Conference on Application of Fuzzy Systems and Soft Computing, Siegen, Germany, June 25-27, 1996.
– Third International Conference on Application of Fuzzy Systems and Soft Computing, Wiesbaden, Germany, October 5-7, 1998.
– Fifth International Conference on Soft Computing with Words and Perceptions in System Analysis, Decision and Control. Famaqusta, North Cyprus, 2-4 September, 2009.
– Ninth International Conference on Application of Fuzzy Systems and Soft Computing. Prague, Czech Republic, August, 2010.
– Sixth World Conference on Intelligent Systems for Industrial Automation Tashkent, Uzbekistan, November 25-27, 2010.

– Sixth International Conference on Soft Computing with Words and Perceptions in System Analysis, Decision and Control. Antalya, Turkey, August 26-27, 2011.
– Tenth International Conference on Application of Fuzzy Systems and Soft Computing. Lisbon, Portugal. August 29-30, 2012.
– Seventh International Conference on Soft Computing with Words and Perceptions in System Analysis, Decision and Control. İzmir, Turkey. August 29-30, 2013.
– Seventh World Conference on Intelligent Systems for Industrial Automation Tashkent, Uzbekistan, November 25-27, 2012
– Eleventh International Conference on Application of Fuzzy Systems and Soft Computing. Paris, France. September 2-3, 2014.
– Eighth International Conference on Soft Computing, Computing with Words and Perceptions in System Analysis, Decision and Control. Antalya, Turkey. September 3-4, 2015.
– ICAFS-2016, 12th International Conference on Application of Fuzzy Systems and Soft Computing, ICAFS 2016, 29-30 August 2016, Vienna, Austria;
– ICSCCW-2017, 9th International Conference on Theory and Application of Soft Computing, Computing with Words and Perception - ICSCCW-2017, 22-23 August 2017, Budapest, Hungary;
– ICAFS-2018, 13th International Conference on Theory and Application of Fuzzy Systems and Soft Computing - ICAFS-2018, 26-27 August 2018, Warsaw, Poland;
– ICAFS-2020, 14th International Conference on Theory and Application of Fuzzy Systems and Soft Computing - ICAFS-2020, 27-28 August 2020, Budva, Montenegro;
– WCIS-2020, 11th World Conference on Intelligent systems for industrial automation - WCIS-2020, 26-28 November, Tashkent, Uzbekistan.
– ICSCCW-2021, 11th International Conference on Theory and Application of Soft Computing, Computing with Words and Perceptions and Artificial Intelligence - ICSCCW-2021;

– ICAFS-2022, 15th International Conference on Applications of Fuzzy Systems, Soft Computing and Artificial Intelligence Tools – ICAFS-2022;
– WCIS-2022, 12th World Conference "Intelligent System for Industrial Automation", (WCIS-2022).

The name of the institution where the dissertation work was performed. Azerbaijan State Oil and Industry University, Research laboratory "Intelligent control and decision-making systems in industry and economics".

The structure of the dissertation. The dissertation consists of an introduction, 6 chapters, a conclusion, a list of used literature and an appendix.

Publications. In total, 61 articles were published. Out of 32 published research articles, 13 were included in Web of Science, 7 in SCOPUS, and 12 in Conference Proceedings Citation Index databases.

MAIN CONTENTS OF THE WORK

In the introduction, the relevance of the subject area, the goals and objectives of the research, the main propositions defended, research methods, and the theoretical and practical importance of the research are mentioned.

The first chapter ("The state of research of Z-numbers theory) analysis of its application to decision-making" explained the information expressed by Z-numbers and reviewed the existing works related to decision-making based on Z-numbers. The shortcomings of the existing works have been investigated. A verbal formulation of the research problems was given. Decision making is based on information. For information to be useful, it must be reliable. Basically, the concept of Z-number is related to the reliability of information. Z-number has two components $Z = (A, B)$. The first component A is a constraint on the value of the real-valued uncertain variable X. The second component B is a measure of certainty (probability) about the value of the first component. A and B are often expressed through natural language. The concept of Z-numbers has many applications, particularly in economics, decision analysis, risk assessment, forecasting, , and characterization of relationships and imprecise functions based on production rules. In the real world, uncertainty is a widespread phenomenon. Most of the information on which decision-making is based is uncertain. One of the remarkable abilities of humans is the ability to make rational decisions based on uncertain, imprecise, and incomplete information. At least some formalization of this capability is a rare approach. It is this approach that forms the basis of the main concepts and ideas of the dissertation work.

The constraint $R(X)$: X is A understood as a possibility constraint, where A serves as the probability distribution of X . More specifically, can be written as $R(X)$: X is $A \rightarrow Poss(X = u) = \mu_A(u)$. Here μ_A is the membership function of A and u is the general value of X. $\mu_A(u)$ can be viewed as a constraint conditioned by $R(X)$. $\mu_A(u)$ means the degree to which u satisfies the constraint condition[1].

When X is a random variable, the probability distribution of X acts as a probability constraint on X . The probabilistic restriction is expressed as follows: $R(X)$: $X\ isp\ p$, where p is the density function of the probability distribution of X. In this case

$$R(X):\ X\ isp\ p \rightarrow Prob(u \leq X \leq u + du) = p(u)du.$$

A Z-valuation is expressed as an ordered triple (X, A, B). Z-valuation is equivalent to the X is (A, B) operator. If A is not a singleton, then X is an uncertain variable. Accordingly, uncertainty calculus is a system of calculations in which the objects of calculation are not the values of the variables, but the constraints on the values of the variables. X is assumed to be a random variable unless otherwise specified in the thesis. For simplicity, even when we say that the value of X is A, we do not mean that the value of X is equal to A, but that the values that X can take are constrained to A. The second component represents certainty. As concepts closely related to certainty, B is

[1] Rafik A. Aliev, Rashad R. Aliyev, Oleg H. Huseynov, Akif V. Alizadeh, The Arithmetic of Z-Numbers, Theory and Applications, Word Scientific Publishing, 2015 İSBN 978981-4675-28-4, 2015, 316, https://www.worldcat.org/title/arithmetic-of-z-numbers-theory-and-applications/oclc/907652071

certainty, reliability, degree of confidence, probability, possibility, etc. It can be shown. When X is a random variable, certainty is equated with probability.

If X is a random variable, then X is A fuzzy event on the real number axis. The probability P of this event can be expressed as:

$$P = \int_R \mu_A(u) p_X(u) du,$$

where the probability density function p_X of the random variable X is implicitly given. In fact, the Z-valuation (X, A, B) may be viewed as a restriction on X defined by

Prob(X is A) is B.

It should be emphasized that the hidden probability distribution p_X in the representation of the Z-number (A, B) in the form of an ordered pair is unknown. What is known is the constraint on p_X. This restriction is expressed as follows:

$$\int_R \mu_A(u) p_X(u) du \ \ is \ \ B.$$

In the second chapter ("Operations on Z-numbers and Z-sets") preliminary information on operations on Z-numbers, operations on random variables, operations on continuous Z-numbers by L.Zade's method and difficulties of this method are shown. Operations on discrete Z-numbers, the importance of operations on discrete Z-numbers is justified when performing operations on Z-numbers.

Arithmetic operations on continuous Z-numbers[2,3]. Assume that $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ are the Z-numbers describing imperfect information about real-valued random variables X_1 and X_2. Let's look at the calculation of the sum of $Z_{12} = Z_1 + Z_2$. As mentioned, performing this operation on Z-numbers starts with performing the corresponding operation on the corresponding

[2] Rafik A. Aliev, Rashad R. Aliyev, Oleg H. Huseynov, A.V. Alizadeh, The Arithmetic of Z-Numbers, Theory and Applications, Word Scientific Publishing, 2015 İSBN 978981-4675-28-4, 2015

[3] A.V. Alizadeh, Rashad R. Aliev, Rafig R. Aliyev. Operational approach to z-information-based decision making, ICAFS-2012, Tenth International Conference on Application of Fuzzy Systems and Soft Computing, Lisbon, Portugal, August 29-30, 2012, 269-277.

Z^+ numbers. The result of the sum $Z_{12}^+ = Z_1^+ + Z_2^+$ expressed in Z^+number is determined as follows: $Z_1^+ + Z_2^+ = (A_1 + A_2, R_1 + R_2)$.

In general, we consider broad family of probability distributionsR_1 and R_2. For simplicity, let's consider normal distributions:

$$p_1(x_1) = \frac{1}{\sqrt{2\pi\sigma_1}} e^{-\frac{(x_1-\mu_1^2)}{2\sigma_1^2}}, \quad (1)$$

$$p_2(x_2) = \frac{1}{\sqrt{2\pi\sigma_2}} e^{-\frac{(x_2-\mu_2^2)}{2\sigma_2^2}}. \quad (2)$$

The sum of fuzzy numbers $A_1 + A_2$ is calculated using the formulas of fuzzy arithmetic. And $R_1 + R_2$ is determined according to (3) as the convolution of continuous probability density functions $p_{12} = p_1 \circ_+ p_2$. As a result, the following expression is used[4]:

$$p_{12}(x_{12}) = \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} \exp\left[-\frac{(x_{12}-(\mu_1+\mu_2))^2}{2(\sigma_1^2+\sigma_2^2)}\right]. \quad (3)$$

Taking these into account, we get Z_{12}^+ as $Z_{12}^+ = (A_1 + A_2, p_{12})$, which is the corresponding Z^+ number –it is the first step in addition of Z -numbers.

In the next step, we take into account that given the Z -numbers $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ the "real" probability distributions p_1 and p_2 are not known exactly. Instead, the available information is represented as fuzzy constraints:

$$\int_{\mathcal{R}} \mu_{A_1}(x_1) p_{R_1}(x_1)\ dx_1\ is\ B_1, \quad (4)$$

$$\int_{\mathcal{R}} \mu_{A_2}(x_2) p_{R_2}(x_2) dx_2\ \ is\ B_2, \quad (5)$$

and is expressed through membership functions as follows:

$$\mu_{B_1}\left(\int_{\mathcal{R}} \mu_{A_1}(x_1) p_{R_1}(x_1) dx_1\right), \quad (6)$$

$$\mu_{B_2}\left(\int_{\mathcal{R}} \mu_{A_2}(x_2) p_{R_2}(x_2) dx_2\right). \quad (7)$$

These constraints indicate the degree to which the probability distributions p_1 and p_2 belong to the corresponding fuzzy sets and are expressed as follows:

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[4] http://thirteen-01.stat.iastate.edu/wiki/stat430/files?filename=Ch-2.2-trivedi.pdf

$$\mu_{p_{R_1}}(p_{R_1}) = \mu_{B_1}\left(\int_{\mathcal{R}} \mu_{A_1}(x_1) p_{R_1}(x_1) dx_1\right) \tag{6a}$$

$$\mu_{p_{R_2}}(p_{R_2}) = \mu_{B_2}\left(\int_{\mathcal{R}} \mu_{A_2}(x_2) p_{R_2}(x_2) dx_2\right). \tag{7a}$$

Thus, $B_j, j = 1,2$ is a fuzzy number, which plays the role of a soft constraint on a value of a probability measure of A_j. Here we will use discretized version of (6)-(7). In this case basic values $b_{jl} \in \text{supp}(B_j), j = 1,2;\ l = 1,\ldots,m$ of a discretized fuzzy number $B_j, j = 1,2$ are values of a probability measure of A_j, $b_{jl} = P(A_j)$. Thus, given b_{jl}, we have to find such probability distribution p_{jl} which satisfies:

$$b_{jl} = \mu_{A_j}(x_{j1}) p_{jl}(x_{j1}) + \mu_{A_j}(x_{j2}) p_{jl}(x_{j2}) + \ldots + \mu_{A_j}(x_{jn_j}) p_{jl}(x_{jn_j}).$$

At the same time, we know that p_{jl} has to satisfy:

$$\sum_{k=1}^{n_j} p_{jl}(x_{jk}) = 1, p_{jl}(x_{jk}) \geq 0.$$

Therefore, the following goal programming problem should be solved to findp_j:

$$\mu_{A_j}(x_{j1}) p_{jl}(x_{j1}) + \mu_{A_j}(x_{j1}) p_{jl}(x_{j1}) + \ldots + \mu_{A_j}(x_{jn_j}) p_{jl}(x_{jn_j}) \to b_{jl} \tag{8}$$

subject to

$$\left.\begin{array}{l} p_{jl}(x_{j1}) + p_{jl}(x_{j2}) + \ldots + p_{jl}(x_{jn}) = 1 \\ p_{jl}(x_{j1}), p_{jl}(x_{j2}), \ldots, p_{jl}(x_{jn}) \geq 0 \end{array}\right\}. \tag{9}$$

Now, denote $c_k = \mu_{A_j}(x_{jk})$ and $v_k = p_j(x_{jk}), k = 1,..,n$. As c_k and b_{jl} are known numbers, and v_k are unknown decision variables, we see that the problem (8)-(9) is nothing but the following goal linear programming problem:

$$c_1 v_1 + c_2 v_2 + \ldots + c_n v_n \to b_{jl} \tag{8a}$$

subject to

$$\left.\begin{array}{l} v_1 + v_2 + \ldots + v_n = 1 \\ v_1, v_2, \ldots, v_n \geq 0 \end{array}\right\}. \tag{9a}$$

Having obtained the solution $v_k, k = 1,..,n$ of problem (8a)-(9a) for each $l = 1,\ldots,m$, we recall that $v_k = p_{jl}(x_{jk}),\ k = 1,..,n$.

Therefore, the probability distribution p_{jl} is obtained. Next, as p_{jl} is obtained given b_{jl}, the desired degree is $\mu_{p_{jl}}(p_{jl}) = \mu_{B_j}\big(b_{jl}\big), j = 1,2,$ that is

$$\mu_{p_{jl}}(p_{jl}) = \mu_{B_j}\left(\sum_{k=1}^{n_j} \mu_{A_j}(x_{jk}) p_{jl}(x_{jk})\right).$$

Thus, to construct a fuzzy set of probability distributionsp_{jl}, it is needed to solve n simple goal linear programming problems (8a)-(9a). For normal random variables, taking into account compatibility conditions, this problem is reduced to optimization problem with one parameter σ. This problem may be solved by a simple optimization method.

The fuzzy sets of probability distributions p_1 and p_2 obtained from approximation of calculated $p_{jl}(x_{jk})$ by a normal distribution, induce the fuzzy set of convolutions p_{12s}, $s = 1,\ldots,m^2$, with the membership function defined as

$$\mu_{p_{12}}(p_{12}) = \max_{p_1,p_2}[\mu_{p_1}(p_1) \wedge \mu_{p_2}(p_2)] \quad (10)$$

$$\text{subject to } p_{12} = p_1 \circ_+ p_2, \quad (11)$$

where $\wedge$ is *min* operation.

At the next step we should compute probability measure of $A_{12} = A_1 + A_2$ given p_{12}, that is, to compute probability measure $P(A_{12})$ of the fuzzy event $X \ \ is \ \ A_{12}$:

$$P(A_{12}) = \int_{\mathcal{R}} \mu_{A_{12}}(u) p_{12}(u) du \quad (12)$$

Thus, when p_{12} is known, $P(A_{12})$ is a number $P(A_{12}) = b_{12}$. However, what is only known is a fuzzy restriction on p_{12} described by the membership function $\mu_{p_{12}}$. Therefore, $P(A_{12})$ will be a fuzzy set B_{12} with the membership function $\mu_{B_{12}}$ defined as follows:

$$\mu_{B_{12}}(b_{12s}) = \sup(\mu_{p_{12s}}(p_{12s})) \quad (13)$$

subject to

$$b_{12s} = \int_{\mathcal{R}} \mu_{A_{12}}(x) p_{12s}(x_k) dx \quad (14)$$

As a result, $Z_{12} = Z_1 + Z_2$ is obtained as $Z_{12} = (A_{12}, B_{12})$.

Other operations on continuous Z-numbers are performed analogously to the addition operation.

Operations on discrete Z-numbers[5].

Addition of Discrete Z-numbers. Let $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ be discrete Z-numbers describing imperfect information about values of real-valued uncertain variables X_1andX_2. Consider the problem of computation of addition $Z_{12} = Z_1 + Z_2$. Computation with discrete Z-numbers, as that with continuous Z-numbers, starts with the computation over the corresponding discrete Z^+ -numbers. The discrete Z^+-number $Z_{12}^+ = Z_1^+ + Z_2^+$ is determined as follows:

$Z_1^+ + Z_2^+ = (A_1 + A_2, R_1 + R_2)$,

where R_1 and R_2 are represented by discrete probability distributions:

$p_1 = p_1(x_{11})\backslash x_{11} + p_1(x_{12})\backslash x_{12} + \ldots + p_1(x_{1n})\backslash x_{1n}$,

$p_2 = p_2(x_{21})\backslash x_{21} + p_2(x_{22})\backslash x_{22} + \ldots + p_2(x_{2n})\backslash x_{2n}$,

for which one necessarily has

$$\sum_{k=1}^{n} p_1(x_{1k}) = 1, \qquad (15)$$

$$\sum_{k=1}^{n} p_2(x_{2k}) = 1. \qquad (16)$$

As the operands in $A_1 + A_2$ and in $R_1 + R_2$ are represented by different types of restrictions, then the meanings of $+$ are also different[6] . The addition $A_1 + A_2$ of discrete fuzzy numbers is defined in accordance with addition over fuzzy numbers and $R_1 + R_2$ is a convolution $p_{12} = p_1 \circ p_2$ of discrete probability distributions:

$p_{12}(x) = \sum_{x = x_1 + x_2} p_1(x_1) p_2(x_2)$.

So, we will have Z_{12}^+ as $Z_{12}^+ = (A_1 + A_2, p_{12})$, which is the result of computation with discrete Z^+-numbers being the first step of computation with Z-numbers.

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[5] Rafik A. Aliev, A.V. Alizadeh, Oleg H. Huseynov. The arithmetic of discrete Z-numbers. Information Sciences https://doi.org/10.1016/j.ins.2014.08.024, January 2015, Volume 290, 1, 134-155

[6] Zadeh, L. A. (2010). A note on Z-numbers, *Inform. Sciences,* 181, pp. 2923–2932.

At the next stage we realize that in Z-numbers $Z_1=(A_1,B_1)$ and $Z_2=(A_2,B_2)$, the 'true' probability distributions p_1 and p_2 are not exactly known. In contrast, the information available is represented by the fuzzy restrictions:

$$\sum_{k=1}^{n_1}\mu_{A_1}(x_{1k})p_1(x_{1k})\ \ is\ \ B_1,\ \ \sum_{k=1}^{n_2}\mu_{A_2}(x_{2k})p_2(x_{2k})\ \ is\ \ B_2,$$

which are represented in terms of the membership functions as

$$\mu_{B_1}\left(\sum_{k=1}^{n_1}\mu_{A_1}(x_{1k})p_1(x_{1k})\right),\ \ \mu_{B_2}\left(\sum_{k=1}^{n_2}\mu_{A_2}(x_{2k})p_2(x_{2k})\right).$$

These restrictions imply that one has the fuzzy sets of probability distributions of p_1 and p_2 with the membership functions defined as

$$\mu_{p_1}(p_1)=\mu_{B_1}\left(\sum_{k=1}^{n_1}\mu_{A_1}(x_{1k})p_1(x_{1k})\right),$$

$$\mu_{p_2}(p_2)=\mu_{B_2}\left(\sum_{k=1}^{n_2}\mu_{A_2}(x_{2k})p_2(x_{2k})\right).$$

Thus, $B_j, j=1{,}2$ is a discrete fuzzy number, which plays the role of a softconstraint on a value of a probability measure of A_j. Therefore, basic values $b_{jl}\in\mathrm{supp}(B_j),\ j=1{,}2;\ l=1,\ldots,m$ of a discrete fuzzy number $B_j, j=1{,}2$ are values of a probability measure of A_j, $b_{jl}=P(A_j)$. Thus, given b_{jl}, we have to find such probability distribution p_{jl} which satisfies:

$$b_{jl}=\mu_{A_j}(x_{j1})p_{jl}(x_{j1})+(x_{j2})p_{jl}(x_{j2})+\ldots+\mu_{A_j}(x_{jn_j})p_{jl}(x_{jn_j}).$$

At the same time, we know that p_{jl} has to satisfy:

$$\sum_{k=1}^{n_j}p_{jl}(x_{jk})=1, p_{jl}(x_{jk})\geq 0.$$

Therefore, the following goal programming problem should be solved to find p_j:

$$\mu_{A_j}(x_{j1})p_{jl}(x_{j1})+\mu_{A_j}(x_{j1})p_{jl}(x_{j1})+\ldots+\mu_{A_j}(x_{jn_j})p_{jl}(x_{jn_j})\to b_{jl} \quad (17)$$

subject to

$$\left.\begin{array}{l} p_{jl}(x_{j1})+p_{jl}(x_{j2})+\ldots+p_{jl}(x_{jn})=1 \\ p_{jl}(x_{j1}),p_{jl}(x_{j2}),\ldots,p_{jl}(x_{jn})\geq 0 \end{array}\right\}. \quad (18)$$

Now, denote $c_k=\mu_{A_j}(x_{jk})$ and $v_k=p_j(x_{jk}), k=1,..,n$. As c_k and b_{jl} are known numbers and v_k are unknown decision variables, we see that problem (17)-(18) is nothing but the following goal linear programming problem:

$$c_1v_1 + c_2v_2 + \ldots + c_nv_n \rightarrow b_{jl} \quad (17a)$$

subject to

$$\left.\begin{matrix} v_1 + v_2 + \ldots + v_n = 1 \\ v_1, v_2, \ldots, v_n \geq 0 \end{matrix}\right\}. \quad (18a)$$

Having obtained the solution $v_k,\ k = 1,..,n$ of problem (17a)-(18a) for each $l = 1,\ldots,m$, recall that $v_k = p_{jl}(x_{jk}), k = 1,..,n$. As a result, $p_{jl}(x_{jk}), k = 1,..,n$ is found and, therefore, the probability distribution p_{jl} is obtained. Next, as p_{jl} is obtained given b_{jl}, then the desired membership degree is $\mu_{p_{jl}}(p_{jl}) = \mu_{B_j}\left(b_{jl}\right), j = 1,2$, that is $\mu_{p_{jl}}(p_{jl}) = \mu_{B_j}\left(\sum_{k=1}^{n_j}\mu_{A_j}(x_{jk})p_{jl}(x_{jk})\right)$. Thus, to construct a fuzzy set of probability distributions p_{jl}, we need to solve n simple goal linear programming problems (17a)-(18a).

The fuzzy sets of probability distributions p_{1l} and p_{2l} induce the fuzzy set of convolutions $p_{12s},\ s = 1,\ldots,m^2$, with the membership function defined as

$$\mu_{p_{12}}(p_{12}) = \max_{p_1,p_2}[\mu_{p_1}(p_1) \wedge \mu_{p_2}(p_2)] \quad (19)$$

subject to

$$p_{12} = p_1 \circ_+ p_2, \quad (20)$$

where $\wedge$ is min operation.

At the next step we should compute probability measure of $A_{12} = A_1 + A_2$ given p_{12}, that is, to compute probability of the fuzzy event $X\ is\ A_{12}$.

Thus, when p_{12} is known, $P(A_{12})$ is a number $P(A_{12}) = b_{12}$. However, what is only known is a fuzzy restriction on p_{12} described by the membership function $\mu_{p_{12}}$. Therefore, $P(A_{12})$ will be a fuzzy set B_{12} with the membership function $\mu_{B_{12}}$ defined as follows:

$$\mu_{B_{12}}(b_{12s}) = \sup(\mu_{p_{12s}}(p_{12s})) \quad (21)$$

subject to

$$b_{12s} = \sum_k p_{12s}(x_k)\mu_{A_{12}}(x_k) \quad (22)$$

As a result, $Z_{12} = Z_1 + Z_2$ is obtained as $Z_{12} = (A_{12}, B_{12})$.

An example 1. Let us consider computation of an addition $Z_{12} = Z_1 + Z_2$ of two discrete Z-numbers $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$, given:

$$A_1 = 0/1 + 0.3/2 + 0.5/3 + 0.6/4 + 0.7/5 + 0.8/6 + 0.9/7 + 1/8 + \\ + 0.8/9 + 0.6/10 + 0/11,$$

$$B_1 = 0/0 + 0.5/0.1 + 0.8/0.2 + 1/0.3 + 0.8/0.4 + 0.7/0.5 + 0.6/0.6 + \\ + 0.4/0.7 + 0.2/0.8 + 0.1/0.6 + 0/1;$$

$$A_2 = 0/1 + 0.5/2 + 0.8/3 + 1/4 + 0.8/5 + 0.7/6 + 0.6/7 + 0.4/8 + \\ + 0.2/9 + 0.1/10 + 0/11,$$

$$B_2 = 0/0 + 0.3/0.1 + 0.5/0.2 + 0.6/0.3 + 0.7/0.4 + 0.8/0.5 + \\ + 0.9/0.6 + 1/0.7 + 0.9/0.8 + 0.8/0.6 + 0/1.$$

At the first step of computation of Z_{12} we proceed to the discrete Z^+-numbers. Let us consider $Z_1^+ = (A_1, R_1)$ and $Z_2^+ = (A_2, R_2)$ where R_1 and R_2 are the following discrete probability distributions R_1 andR_2:

$$p_1 = 0.27\backslash 1 + 0\backslash 2 + 0\backslash 3 + 0.0027\backslash 4 + 0.04\backslash 5 + 0.075\backslash 6 + \\ +0.11\backslash 7 + 0.15\backslash 8 + 0.075\backslash 9 + 0.0027\backslash 10 + 0.27\backslash 11,$$

$$p_2 = 0.09\backslash 1 + 0\backslash 2 + 0.18\backslash 3 + 0.32\backslash 4 + 0.18\backslash 5 + 0.1\backslash 6 + \\ +0.036\backslash 7 + 0\backslash 8 + 0\backslash 9 + 0\backslash 10 + 0.09\backslash 11.$$

As one can verify, the constraints (15)-(16) are satisfied.

At the second step we should determine the discrete Z^+-number $Z_{12}^+ = (A_1 + A_2, R_1 + R_2)$. Here we first compute $A_{12} = A_1 + A_2$, we have:

$$A_{12} = \bigcup_{\alpha \in [0,1]} \alpha A_{12}^{\alpha},$$

$A_{12}^{\alpha} = \{x \in \{\text{supp}\,(A_1) + \text{supp}\,(A_2)\} | \min\{A_1^{\alpha} + A_2^{\alpha}\} \le x \le \max\{A_1^{\alpha} + A_2^{\alpha}\}\}$. We will use $\alpha = 0, 0.1, \ldots, 1$. The resulting A_{12} is found as follows.

$$A_{12} = 0/1 + 0/2 + 0.19/3 + 0.36/4 + 0.5/5 + 0.58/6 + 0.65/7 + 0.73/8 + \\ + 0.8/9 + 0.87/10 + 0.93/11 + 1/12 + 0.9/13 + 0.8/14 + 0.73/15 + \\ + 0.7/16 + 0.6/17 + 0.45/18 + 0.3/19 + 0.17/20 + 0.086/21.$$

Next, we compute $R_1 + R_2$ as a convolution $p_{12} = p_1 \circ_+ p_2$ of the considered p_1 and p_2.

For example, compute $p_{12}(x)$ for $x = 4$. The latter can be $x = x_{11} + x_{23} = 1 + 3 = 4$, $x = x_{13} + x_{21} = 3 + 1 = 4$ or $x = x_{12} + x_{22} = 2 + 2 = 4$. Then

$$p_{12}(4) = p_1(1)p_2(3) + p_1(3)p_2(1) + p_1(2)p_2(2) =$$
$$= 0.27 \cdot 0.18 + 0 \cdot 0.09 + 0 \cdot 0 = 0.0486.$$

The p_{12} obtained in accordance with (2.36) is given below:

$$p_{12} = 0\backslash 1 + 0.0243\backslash 2 + 0\backslash 3 + 0.0486\backslash 4 + \ldots + 0.007\backslash 19 + $$
$$+ 0.0002\backslash 20 + 0.0243\backslash 21.$$

Thus, $Z_{12}^{+} = (A_1 + A_2, R_1 + R_2) = (A_1 + A_2, p_{12})$ is obtained.

At the third step we realize, that 'true' probability distributions p_1 and p_2 are not exactly known, but only fuzzy restrictions μ_{p_1} and μ_{p_2} for p_1 and p_2 are available which are induced by B_1 and B_2 respectively. We compute the membership degrees $\mu_{p_j}(p_j)$, $j = 1,2$, of the fuzzy restrictions given the solutions of the goal linear programming problems (17a)-(18a). Let us consider determination of the membership degrees $\mu_{p_1}(p_1)$ and $\mu_{p_2}(p_2)$ for distributions p_1 and p_2 considered above. It is known that $\mu_{p_1}(p_1) = \mu_{B_1}\left(\sum_{k=1}^{n_1} \mu_{A_1}(x_{1k}) p_1(x_{1k})\right)$, and as for p_1 considered above we have

$$\sum_{k=1}^{n_1} \mu_{A_1}(x_{1k}) p_1(x_{1k}) = 0 \cdot 0.27 + 0.3 \cdot 0 + 0.5 \cdot 0 + 0.6 \cdot 0.003 +$$
$$+ 0.7 \cdot 0.04 + 0.8 \cdot 0.075 + 0.9 \cdot 0.11 + 1 \cdot 0.15 + 0.8 \cdot 0.075 +$$
$$+ 0.6 \cdot 0.002 + 0 \cdot 0.27 = 0.4,$$

then $\mu_{p_1}(p_1) = \mu_{B_1}(0.4) = 0.8$. Analogously, we find that $\mu_{p_2}(p_2) = 1$ for p_2 considered above. Finally, we compute the membership degrees for all the considered p_1 and p_2.

At the fourth step, we should determine the fuzzy restriction $\mu_{p_{12}}$ over all the convolutions p_{12} obtained on the base of (19)-(20) from all the considered p_1 and p_2. It is clear that the fuzzy restriction $\mu_{p_{12}}$ is induced by fuzzy restrictions μ_{p_1} and μ_{p_2}. For example, the membership degree of this fuzzy restriction for the convolution p_{12} obtained above is

$$\mu_{p_{12}}(p_{12}) = \mu_{p_1}(p_1) \wedge \mu_{p_2}(p_2) = 0.8 \wedge 1 = 0.8.$$

Analogously, we computed the degrees for all the considered p_{12}.

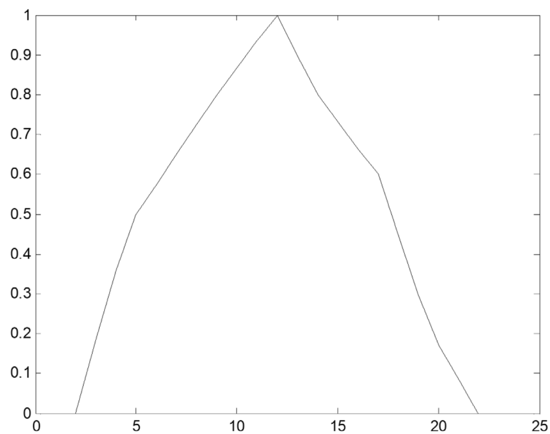
At the fifth step, we should proceed to construction of B_{12} as a soft constraint on a probability measure $P(A_{12})$ based on (21)-(22). First, we compute values of probability measure $P(A_{12})$ by using the obtained convolutions p_{12}. For example, $P(A_{12})$ computed with respect to p_{12} considered above is

$$P(A_{12}) = \sum_{k=1}^{n_1} \mu_{A_{12}}(x_{12k}) p_{12}(x_{12k}) = 0 \cdot 0 + 0 \cdot 0.243 + 0.19 \cdot 0 + $$
$$+0.36 \cdot 0.0486 + 0.087 \cdot 0.5 + \ldots + 0.086 \cdot 0.243 = 0.63.$$

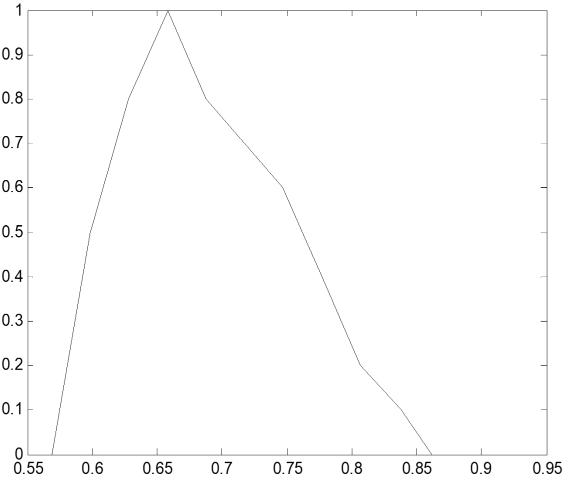
As the computed $P(A_{12})$ is one possible value of probability measure within the fuzzy restriction B_{12} to be constructed, we can say that one basic value of B_{12} is found as $b_{12} = 0.63$. Now we recall that $\mu_{B_{12}}\left(b_{12} = \sum_k \mu_{A_{12}}(x_{12k}) p_{12}(x_{12k})\right) = \mu_{p_{12}}(p_{12})$. Then, given $\mu_{p_{12}}(p_{12}) = 0.8$, we obtain $\mu_{B_{12}}(b_{12} = 0.63) = 0.8$ for $b_{12} = \sum_k \mu_{A_{12}}(x_{12k}) p_{12}(x_{12k})$. By carrying out analogous computations, we constructed B_{12}as follows:

$$B_{12} = 0/0.56 + 0.5/0.60 + 0.8/0.63 + 1/0.66 + 0.8/0.69 + 0.7/0.72 + $$
$$+ 0.6/0.75 + 0.4/0.78 + 0.2/0.81 + 0.1/0.84 + 0/0.86 + 0/1.$$

Thus, the result of addition $Z_{12} = (A_{12}, B_{12})$ is obtained, where A_{12}, B_{12} are shown in Fig. 1.



(a)



(b)

Fig. 1. The results of addition of the discrete Z-numbers: (a) A_{12}, (b) B_{12}

Multiplication of discrete Z-numbers [7]**.** Let us consider multiplication $Z_{12} = Z_1 \cdot Z_2$ of $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$. First, $Z_{12}^+ = Z_1^+ \cdot Z_2^+$ should be determined:

$$Z_1^+ \cdot Z_2^+ = (A_1 \cdot A_2, R_1 \cdot R_2),$$

where R_1 and R_2 are represented by discrete probability distributions:

$$p_1 = p_1(x_{11})\backslash x_{11} + p_1(x_{12})\backslash x_{12} + \ldots + p_1(x_{1n})\backslash x_{1n},$$

$$p_2 = p_2(x_{21})\backslash x_{21} + p_2(x_{22})\backslash x_{22} + \ldots + p_2(x_{2n})\backslash x_{2n},$$

for which (15)-(16) are satisfied. The product $A_1 \cdot A_2$ of discrete fuzzy numbers is defined and $R_1 \cdot R_2$ is a convolution $p_{12} = p_1 \circ_* p_2$ of discrete probability distributions defined:

$$p_{12}(x) = \sum_{x = x_1 \cdot x_2} p_1(x_1) p_2(x_{2j}).$$

[7] A.V. Alizadeh, Rashad R. Aliev, Oleg H.Huseynov. Numerical computations with discrete z-numbers, ICSCCW-2013, Seventh International Conference on Soft Computing, Computing with Words and Perceptions in System Analysis, Decision and Control, Izmir, Turkey, September 2-3, 2013, 71-82

Thus, we will have $Z_{12}^{+} = (A_1 \cdot A_2, p_{12})$. Next, analogously to the procedure described for addition, we construct the fuzzy sets $\mu_{p_{jl}}(p_{jl})$, $l = 1,\ldots,m$, and the fuzzy set of convolutions $p_{12s},\ s = 1,\ldots,m^2$, with the membership function defined by (19)-(20) and a convolution defined.

At the next step probability measure of $A_{12} = A_1 \cdot A_2$ is computed. Finally, a fuzzy set B_{12} is constructed according to (21)-(22). As a result, $Z_{12} = Z_1 \cdot Z_2$ is obtained as $Z_{12} = (A_{12}, B_{12})$.

An example 2. Let us consider multiplication of the Z-numbers considered in Section 4.1.3. Again, first we proceed to the discrete Z^{+}-numbers. Second, we should calculate $Z_{12}^{+} = (A_{12}, R_{12}) = (A_1 \cdot A_2, R_1 \cdot R_2)$. In accordance with the approach described above, we compute $A_{12} = A_1 \cdot A_2$ and $R_1 \cdot R_2$ as a convolution of p_1 and p_2 (taken the same as in the case of addition). The results (obtained analogously to the procedures used previously for addition and subtraction) are shown below:

$$A_{12} = 0/1 + 0.16/2 + \ldots + 1/32 + \ldots + 0.17/100 + 0/121.$$

$$p_{12} = 0.243\backslash 1 + 0\backslash 2 + \ldots + 0\backslash 100 + 0.243\backslash 121.$$

As a result, $Z_{12}^{+} = (A_1 \cdot A_2, p_{12})$ is obtained.

Third, we compute membership degrees $\mu_{p_1}(p_1)$ and $\mu_{p_2}(p_2)$. Fourth, the membership degrees of the convolutions p_{12} are obtained on the basis of $\mu_{p_1}(p_1)$ and $\mu_{p_2}(p_2)$ analogously to the cases of addition.

Fifth, we compute B_{12}. For this purpose, we compute values of probability measure $P(A_{12})$ with respect to the obtained convolutions p_{12}. For example, $P(A_{12})$ computed for p_{12} considered above is

$$P(A_{12}) = b_{12} = 0.67.$$

At the final stage, we construct B_{12} based on (21)-(22). For example, $\mu_{B_{12}}(b_{12} = 0.67) = 0.8$. The constructed B_{12} is given below:

$$B_{12} = 0/0.54 + 0.5/0.57 + 0.8/0.61 + 1/0.63 + 0.8/0.67 +$$
$$+ 0.7/0.7 + 0.6/0.73 + 0.4/0.76 + 0.2/0.79 + 0.1/0.819 + 0/0.82.$$

Thus, $Z_{12} = (A_{12}, B_{12})$ as the result of multiplication is obtained and A_{12}, B_{12} are shown in Fig. 2.

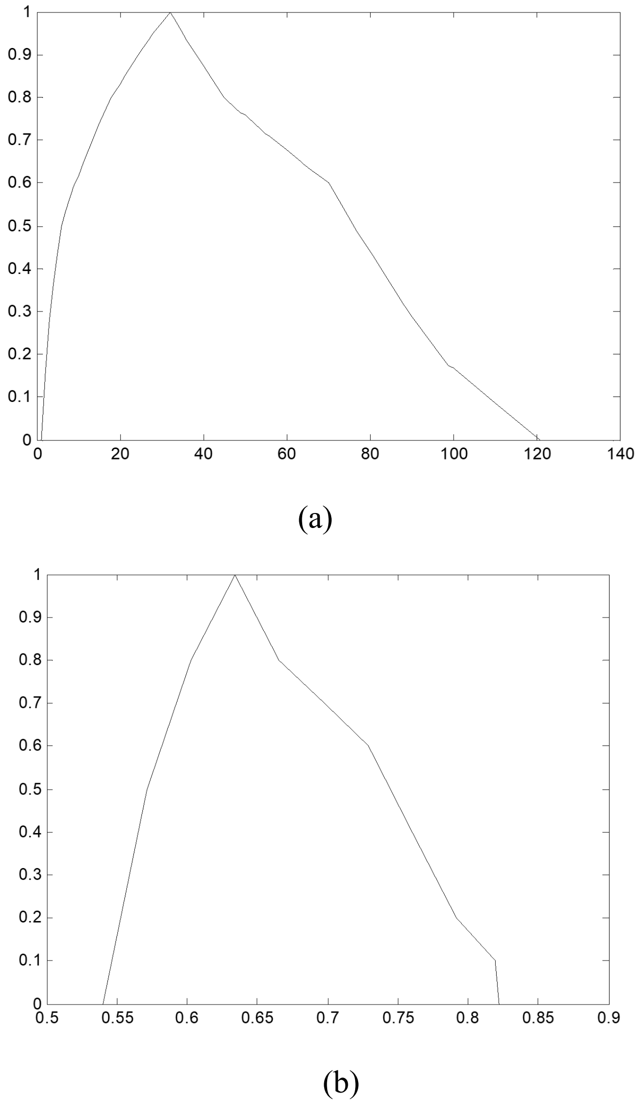


Fig. 2. The results of multiplication of the discrete Z-numbers:(a) A_{12}, (b) B_{12}.

Square Root of a Discrete Z-number.

Let us consider computation of $Z_Y = \sqrt{Z_X}$. Let Z_X^+ and Z_X be the same as those considered in the same as in the previous example. Then the discrete Z^+-number Z_Y^+ is determined as follows:

$Z_Y^+ = (A_Y, R_Y)$,

where $A_Y = \sqrt{A_X}$, $\sqrt{A_X}$ is determined and R_Y is represented by a discrete probability distribution

$$p_{R_Y} = p_{R_Y}(y_1)\backslash y_1 + p_{R_Y}(y_2)\backslash y_2 + \ldots + p_{R_Y}(y_n)\backslash y_n,)$$

such that

$$y_k = \sqrt{x_k} \text{ and } p_{R_Y}(y_k) = p_{R_X}(x_k),$$

Then we construct $\mu_{p_X}(p_{X,l}) = \mu_{B_X}\left(\sum_{k=1}^{n} \mu_{A_X}(x_k) p_{X,l}(x_k)\right)$ and recall that

$$\mu_{p_Y}(p_{Y,l}) = \mu_{p_X}(p_{X,l}).$$

Next, we compute probability measure of A_Y and, given the membership function μ_{p_Y}, we construct a fuzzy set B_Y analogously to that we did in multiplication. As a result, $\sqrt{Z}$ is obtained as $\sqrt{Z} = (A_Y, B_Y)$. Let us mention that analogously to the case of the square of a discrete Z-number with non-negative first component, it is not needed to carry out computation of B_Y. One can easily verify that for the case of the square root of a discrete Z-number, $B_Y = B_X$ holds.

An example 3. Let us consider computation of the square root $Z_3 = \sqrt{Z_2}$ of $Z_2 = (A_2, B_2)$ considered in Section 4.1.3. Given the Z^+-number $Z_2^+ = (A_2, R_2)$ used in the example above, we computed the corresponding Z^+-number $Z_3^+ = (A_3, R_3)$, where $A_3 = \sqrt{A_2}$ computed and R_3 computed are given below:

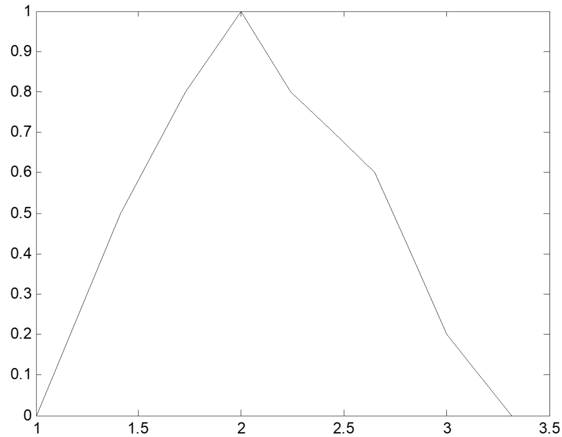
$A_3 = 0/1 + 0.5/1.4 + 0.8/1.7 + 1/2 + 0.8/2.2 + 0.7/2.4 + 0.6/2.6 +$
$+ 0.4/2.8 + 0.2/3 + 0.1/3.2 + 0/3.3$,

$p_3 = 0.09\backslash 1 + 0\backslash 1.4 + 0\backslash 1.7 + 0.32\backslash 2 + 0.18\backslash 2.2 + 0.1\backslash 2.4 +$
$+0.036\backslash 2.6 + 0\backslash 2.8 + 0.2\backslash 3 + 0.1\backslash 3.2 + 0.09\backslash 3.3.$

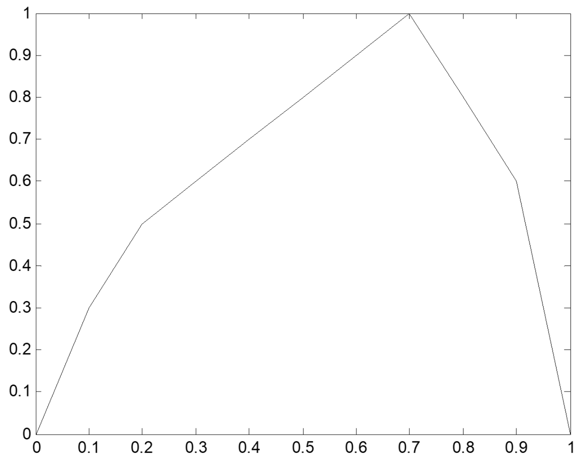
As it was shown above, $B_3 = B_2$. However, we computed B_3 givenA_3, the membership degree

$$\mu_{p_2}(p_2) = \mu_{B_2}\left(\sum_{k=1}^{n} \mu_{A_2}(x_k)p_2(x_k)\right)$$

and taking into account the fact that $\mu_{p_3}(p_3) = \mu_{p_2}(p_2)$. Thus, Z-number $Z_3 = (A_3, B_3)$ as the square root of Z_2 is obtained and A_3, B_3 are shown in Fig. 3.



(a)



(b)

Fig. 3. The square root of the discrete Z-number: (a) A_3, (b) B_3

As one can see, we have $B_3 = B_2$.

Minimum and maximum of discrete Z-numbers [8].

Let $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ be discrete Z-numbers describing imperfect information about values of real-valued random variables X_1 and X_2. Consider the problem of computation of minimum $Z_{12} = MIN(Z_1, Z_2)$. Computation of the maximum, $Z_{12} = MAX(Z_1, Z_2)$ is treated analogously.

Computation with discrete Z-numbers, as that with continuous Z-numbers, starts with the computation over the corresponding discrete Z^+-numbers. The discrete Z^+-number $Z_{12}^+ = \min(Z_1^+, Z_2^+)$ is determined as follows:

$$\min(Z_1^+, Z_2^+) = (MIN(A_1, A_2), \min(R_1, R_2))$$

whereR_1 and $R_2$are represented by discrete probability distributions for which one necessarily has satisfied:

$$\sum_{k=1}^{n} p_1(x_{1k}) = 1, \sum_{k=1}^{n} p_2(x_{2k}) = 1.$$

As the operands in $MIN(A_1, A_2)$ and in $\min(R_1, R_2)$ are represented by different types of restrictions, then the meanings of MIN and min are also different[9]. The minimum $MIN(A_1, A_2)$ of DFNs is defined. $\min(R_1, R_2)$is a convolution $p_{12} = p_1 \circ_{min} p_2$ of discrete probability distributions and is defined.

For the case of the maximum of Z_1, Z_2, $Z_{12} = MAX(Z_1, Z_2)$, instead of the MIN operation over discrete fuzzy numbers and min operation over probability distributions, MAX operation defined and max operation defined are used respectively.

So, we will have Z_{12}^+ as $Z_{12}^+ = (MIN(A_1, A_2), p_{12})$, which is the result of computation with discrete Z^+-numbers, being the first step of computation with Z-numbers.

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[8] A.V. Alizadeh, Oleg H. Huseynov Minimum and maximum of discrete z-numbers, Eleventh International Conference on Application of Fuzzy Systems and Soft Computing, ICAFS – 2014, Paris, France, September 2-3, 2014, 205-218.

[9] Zadeh, L. A. (2010). A note on Z-numbers, *Inform. Sciences,* 181, pp. 2923–2932.

Next, we realize that the 'true' probability distributions p_1 and p_2 are not exactly known, but the fuzzy restrictions which may be represented in terms of membership functions are only available:

$$\mu_{B_1}\left(\sum_{k=1}^{n}\mu_{A_1}(x_{1k})p_1(x_{1k})\right),\mu_{B_2}\left(\sum_{k=1}^{n}\mu_{A_2}(x_{2k})p_2(x_{2k})\right).$$

These restrictions induce fuzzy sets of probability distributions of p_1 and p_2:

$$\mu_{p_1}(p_1)=\mu_{B_1}\left(\sum_{k=1}^{n}\mu_{A_1}(x_{1k})p_1(x_{1k})\right),$$

$$\mu_{p_2}(p_2)=\mu_{B_2}\left(\sum_{k=1}^{n}\mu_{A_2}(x_{2k})p_2(x_{2k})\right).$$

Next, to construct B_1 and B_2, one needs to compute the values of $\mu_{B_j}(b_{jl})$, $b_{jl}\in \mathrm{supp}B_j, j=1,2;\ l=1,\dots,n$, by solving a series of n goal linear programming problems.

The fuzzy sets of probability distributions p_{1l} and p_{2l} induce the fuzzy set of convolutions $p_{12s},\ s=1,\dots,m^2$, with the membership function defined as

$$\mu_{p_{12}}(p_{12})=\max_{p_1,p_2}[\mu_{p_1}(p_1)\wedge\mu_{p_2}(p_2)]$$

subject to $p_{12}=p_1\circ p_2$,

where $\wedge$ is min operation.

At the next step, we should compute probability measure of

$A_{12}=MIN(A_1,A_2)$:

$$P(A_{12})=\sum_{x_k}p_{12}(x_k)\mu_{A_{12}}(x_k).$$

However, as a fuzzy restriction on p_{12} described by the membership function $\mu_{p_{12}}$ is only known, $P(A_{12})$ will be defined as a fuzzy set B_{12} with the membership function $\mu_{B_{12}}$ defined as follows:

$$\mu_{B_{12}}(b_{12s})=\sup(\mu_{p_{12s}}(p_{12s}))$$

subject to

$$b_{12s}=\sum_k p_{12s}(x_k)\mu_{A_{12}}(x_k).$$

As a result, $Z_{12}=MIN(Z_1,Z_2)$ is obtained as $Z_{12}=(A_{12},B_{12})$.

Z-sets and operations on them.

Join and meet operations on Z-sets, set-theoretic operations on Z-sets: complement, intersection, union operations are defined and calculation rules are given. These rules by which these operations are implemented are proved and explained on examples.

Definition 1. A Z-set is described as a triple $Z = (A, B, G)$, where A is a fuzzy set, and B is a fuzzy constraint on a probability measure $P(A)$ constructed through a set of G-probability distributions, such that,

$G = \{p_Z(x): \int_{-\infty}^{+\infty} p_Z(x)dx =$

$1, \int_{-\infty}^{+\infty} p_Z(x)\mu_A(x) \ \ is \ \ B, \int_{-\infty}^{+\infty} xp_Z(x) = \int_{-\infty}^{+\infty} x\mu_A(x)dx \, /$

$\int_{-\infty}^{+\infty} \mu_A(x)dx\}.$

$Z_i = (A_i, B_i, G_i)$ We consider three basic operations on the Z-set.

Complement. Assume that $Z = (A, B, G)$ is a given Z-set, such that the elements of G is set with probability distributions: $G = \{p_Z(x): \int_{-\infty}^{+\infty} p_Z(x)dx = 1, \int_{-\infty}^{+\infty} p_Z(x)\mu_A(x) \ \ is \ \ B, \int_{-\infty}^{+\infty} xp_Z(x) = \int_{-\infty}^{+\infty} x\mu_A(x)dx \, / \int_{-\infty}^{+\infty} \mu_A(x)dx\}$. The complement $\bar{Z}$ of the Z-set $Z = (A, B, G)$ is defined as follows.

The set of probability distributions $\bar{G}$ for the complement $\bar{Z}$ is defined as follows:

$\overline{G} = \{p_Z(x): \int_{-\infty}^{+\infty} p_Z(x)dx =$

$1, \int_{-\infty}^{+\infty} p_Z(x)\mu_A(x) \ \ is \ \ B, \int_{-\infty}^{+\infty} xp_Z(x) = \int_{-\infty}^{+\infty} x\mu_A(x)dx \, /$

$\int_{-\infty}^{+\infty} \mu_A(x)dx\}$. Then $\bar{Z} = (\overline{A}, 1 - B, G)$, so that $\mu_{\overline{A}}(x) = 1 - \mu_A(x)$ and $1 - B$ is understood as a difference.

Join of Z-sets. Join of Z-sets $Z_i = (A_i, B_i, G_i), i = 1{,}2$is defined as follows: $Z_{12} = Z_1 \sqcup Z_2 = \max(Z_1, Z_2) = (A_{12}, B_{12})$. Let us at first consider the case of continuous Z-sets. The maximum of two given fuzzy numbers $A_{12} = MAX(A_1, A_2)$ is defined as follows: $A_{12}(z) = MAX(A_1, A_2)(z) = \sup_{z=\max(x,y)} \min[A_1(x), A_2(y)]$ [10].

The G_{12} set of p_{12} probability distributions of the join is defined as: $G_{12} = G_1 \sqcup G_2$, here

$G_i = \{p_{Z_i}(x): \int_{-\infty}^{+\infty} p_{Z_i}(x)dx = 1,\ \int_{-\infty}^{+\infty} p_{Z_i}(x)\mu_{A_i}(x)\ \ is\ B_i,$

$\int_{-\infty}^{+\infty} xp_{Z_i}(x) = \int_{-\infty}^{+\infty} x\mu_{A_i}(x)dx \,/\, \int_{-\infty}^{+\infty} \mu_{A_i}(x))dx\}$.

Probability distributions $p_{12} = p_1 \circ_{\max} p_2$ are defined as follows: $p_{12}(x) = p_1(x)F_1(x) + p_2(x)F_2(x)$ where F_1 and F_2 are cumulative distribution functions: $F_1(x) = \int_{-\infty}^{x} p_1(x)dx$, $F_2(x) = \int_{-\infty}^{x} p_2(x)dx$.

Then $Z_{12} = (A_{12}, B_{12}, G_{12})$ so $G_{12} = \{p_{12}(x): p_{12}(x) = p_1(x)F_1(x) + p_2(x)F_2(x), p_i(x) \in G_i\}$, $B_{12} = \{(\mu_{p_{12}}(p_{12}), \mu_{A_{12}} \cdot p_{12}): p_{12} \in G_{12}\}$.

Meet of Z-sets. The meet of Z-sets $Z_i = (A_i, B_i, G_i)$, $i = 1,2$ is defined as follows: $Z_{12} = Z_1 \sqcap Z_2 = \min(Z_1, Z_2) = (A_{12}, B_{12}, G_{12})$.

The minimum of fuzzy numbers $MIN(A_1, A_2)$ is defined as follows:

$$A_{12}(z) = MIN(A_1, A_2)(z) = \sup_{z=\min(x,y)} \min[A_1(x), A_2(y)].$$

The G_{12} set of probability distributions of the result p_{12} is defined as follows: $G_{12} = G_1 \sqcap G_2$,, here

$G_i = \{p_{Z_i}(x): \int_{-\infty}^{+\infty} p_{Z_i}(x)dx = 1,\ \int_{-\infty}^{+\infty} p_{Z_i}(x)\mu_{A_i}(x)\ \ is\ B_i,$

$\int_{-\infty}^{+\infty} xp_{Z_i}(x) = \int_{-\infty}^{+\infty} x\mu_{A_i}(x)dx \,/\, \int_{-\infty}^{+\infty} \mu_{A_i}(x))dx\}$.

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[10] A.V. Alizadeh, Properties of Join and Meet Operations over Z-numbers, 14th International Conference on Theory and Application of Fuzzy Systems and Soft Computing – ICAFS-2020, Montenegro, Advances in Intelligent Systems and Computing, Springer, Cham. Online ISBN978-3-030-64058-3, Print ISBN978-3-030-64057-6, eBook Packages Intelligent Technologies and RoboticsIntelligent Technologies and Robotics (R0), 2020, vol 1306, 580-589, https://www.springer.com/gp/book/9783030640576, https://doi.org/10.1007/978-3-030-64058-3_72

The convolution $p_{12} = p_1 \circ_{\min} p_2$ of the given probability distributions is defined as follows:

$$p_{12}(x) = p_1(x) + p_2(x) - p_1(x)F_1(x) - p_2(x)F_2(x),$$

where F_1 and F_2 are the corresponding cumulative probability distribution functions:

$$F_1(x) = \int_{-\infty}^{x} p_1(x)dx, \; F_2(x) = \int_{-\infty}^{x} p_2(x)dx.$$

Then

$$G_{12} = \{p_{12}(x): p_{12}(x) = p_1(x) + p_2(x) - p_1(x)F_1(x) - p_2(x)F_2(x), p_i(x) \in G_i\}.$$

Thus,

$$B_{12} = \{(\mu_{p_{12}}(p_{12}), \mu_{A_{12}} \cdot p_{12}): p_{12} \in G_{12}\}.$$

The union of Z-sets[11]. The union of Z-sets (A_i, B_i, G_i), $i = 1{,}2$ is denoted as $Z_{12} = Z_1 \cup Z_2 = Zor(Z_1, Z_2) = (A_{12}, B_{12}, G_{12})$ and defined as follows: The union $A_{12} = A_1 \cup A_2$ of fuzzy sets A_1 and A_2 is defined as:

$$A_{12}(x) = (A_1 \cup A_2)(x) = \max(A_1(x), A_2(x)).$$

Denote the set G_{12} of probability distributions p_{12} by $G_{12} = G_1 \cup G_2$, where for $i = 1{,}2$ is defined as

$$G_i = \{p_{Z_i}(x): \int_X p_{Z_i}(x)dx = 1, \int_X p_{Z_i}(x)A_i(x)dx \;\; is \;\; B_i, x \in X\}$$

Convolution of probability distributions $p_{12} = p_{Z_1 \cup Z_2}(x) = p_{Z_1}(x) \circ_{\cup} p_{Z_2}(x) = p_1 \circ_{\cup} p_2$ is defined as

$$p_{12}(x) = p_{Z_1 \cup Z_2}(x) = \frac{(p_{Z_1}(x) + p_{Z_2}(x) - p_{Z_1}(x)p_{Z_2}(x))p_X(x)}{\int_X (p_{Z_1}(x) + p_{Z_2}(x) - p_{Z_1}(x)p_{Z_2}(x))p_X(x)dx}.$$

Here $p_X(x) = \frac{1}{card(X)}$.

Then $Z_{12} = (A_{12}, B_{12}, G_{12})$, so G_{12} and B_{12} are defined as:

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[11] A.V. Alizadeh, Properties of Set-Theoretical Operations over Z-Sets, Advances in Intelligent Systems and Computing Springer, Cham. https://doi.org/10.1007/978-3-030-35249-3_84, 2019, vol 1095, 654-661, https://link.springer.com/chapter/10.1007%2F978-3-030-35249-3_84

G_{12}={$p_{12}(x)$: $p_{12}(x) = p_{Z_1 \cup Z_2}(x) =$
$\frac{(p_{Z_1}(x)+p_{Z_2}(x)-p_{Z_1}(x)p_{Z_2}(x))p_X(x)}{\int_X (p_{Z_1}(x)+p_{Z_2}(x)-p_{Z_1}(x)p_{Z_2}(x))p_X(x)dx}, p_i(x) \in G_i, x \in X\}$.

Thus,
$B_{12} = \{(\mu_{p_{12}}(p_{12}), \mu_{A_{12}} \cdot p_{12}): p_{12} \in G_{12}\}$.

Intersection of Z-sets. The intersection of Z-sets $= (A_i, B_i, G_i)$, $i = 1{,}2$ is denoted as
$Z_{12} = Z_1 \cap Z_2 = Zand(Z_1, Z_2) = (A_{12}, B_{12}, G_{12})$
and defined as follows:

The intersection $A_1 \cap A_2$ of the fuzzy sets A_1 and A_2 is defined as follows: $A_{12}(x) = (A_1 \cap A_2)(x) = \min(A_1(x), A_2(x))$.

The probability distributions p_{12} corresponding to the intersection of the given Z-sets G_{12} is denoted as $G_{12} = G_1 \cap G_2$,.

Here the sets G_i are defined as in union.

Convolution $p_{12} = p_{Z_1 \cap Z_2}(x) = p_{Z_1}(x) \circ_\cap p_{Z_2}(x) = p_1 \circ_\cap p_2$ of probability distributions is defined as follows:

$p_{12}(x) = p_{Z_1 \cap Z_2}(x) = \frac{p_{Z_1}(x)p_{Z_2}(x)p_X(x)}{\int_X p_{Z_1}(x)p_{Z_2}(x)p_X(x)dx}$,

where $p_X(x) = \frac{1}{card(X)}$.

Then $Z_{12} = (A_{12}, B_{12}, G_{12})$, so

$G_{12} = \Big\{p_{12}(x): p_{12}(x) = p_{Z_1 \cap Z_2}(x) =$

$\frac{p_{Z_1}(x)p_{Z_2}(x)p_X(x)}{\int_X p_{Z_1}(x)p_{Z_2}(x)p_X(x)dx}, p_i(x) \in G_i,\ \ x \in X\Big\}$.

Thus,
$B_{12} = \{(\mu_{p_{12}}(p_{12}), \mu_{A_{12}} \cdot p_{12}): p_{12} \in G_{12}\}$.

The third chapter ("Application of Z-Linear Programming in Decision Making") covers the solution of the Z-valued linear programming problem. Linear programming is a method of operations research often used in the fields of science, economics, business, management, and engineering. Unfortunately, despite the research and application of various types of linear programming methods based on interval, fuzzy, generalized fuzzy, random numbers with different levels of generalization of information about the parameters of the model by engineers and researchers for more than sixty years, unfortunately, in linear programming, there is no method that takes into account the degree of reliability of information. With Z-valued decision variables and Z-valued parameters, we will look at the process of solving a linear programming problem based on Z-scoring The general formulation of the problem of linear programming based on Z-information is expressed as follows[12]:

$$Z_f(Z_{x_1}, Z_{x_2}, \dots, Z_{x_n}) = Z_{c_1}Z_{x_1} + Z_{c_2}Z_{x_2} + \dots + Z_{c_n}Z_{x_n} \rightarrow \max \quad (23)$$

subject to

$$\begin{aligned} &Z_{a_{11}}Z_{x_1} + Z_{a_{12}}Z_{x_2} + \dots + Z_{a_{1n}}Z_{x_n} \preceq Z_{b_1}, \\ &Z_{a_{21}}Z_{x_1} + Z_{a_{22}}Z_{x_2} + \dots + Z_{a_{2n}}Z_{x_n} \preceq Z_{b_2}, \\ &\dots \\ &Z_{a_{m1}}Z_{x_1} + Z_{a_{m2}}Z_{x_2} + \dots + Z_{a_{mn}}Z_{x_n} \preceq Z_{b_m}, \end{aligned} \quad (24)$$

$$Z_{x_1}, Z_{x_2}, \dots, Z_{x_n} \succeq Z_0 \quad (25)$$

With Z-valued decision variables and Z-valued parameters, we will look at the process of solving a linear programming problem based on Z-valuation. The general statement of the problem of linear programming based on Z-information is expressed as follows:

$$Z_f(Z_{x_1}, Z_{x_2}, \dots, Z_{x_n}) = Z_{c_1}Z_{x_1} + Z_{c_2}Z_{x_2} + \dots + Z_{c_n}Z_{x_n} \rightarrow \min \quad (26)$$

$$\begin{aligned} &Z_{a_{11}}Z_{x_1} + Z_{a_{12}}Z_{x_2} + \dots + Z_{a_{1n}}Z_{x_n} \preceq Z_{b_1}, \\ &Z_{a_{21}}Z_{x_1} + Z_{a_{22}}Z_{x_2} + \dots + Z_{a_{2n}}Z_{x_n} \preceq Z_{b_2}, \\ &\dots \\ &Z_{a_{m1}}Z_{x_1} + Z_{a_{m2}}Z_{x_2} + \dots + Z_{a_{mn}}Z_{x_n} \preceq Z_{b_m}, \end{aligned} \quad (27)$$

$$Z_{x_1}, Z_{x_2}, \dots, Z_{x_n} \succeq 0^Z. \quad (28)$$

According to expressions (26)-(27), Z-inequalities can be transformed into the following problem:

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[12] Aliev, R. A., Alizadeh, A. V., Huseynov, O. H., Jabbarova, K.I. Z-number based Linear Programming. *Int. J. Intell. Syst.*

$$Z_f(Z_{x_1}, Z_{x_2}, \ldots, Z_{x_n}) = Z_{c_1}Z_{x_1} + Z_{c_2}Z_{x_2} + \ldots + Z_{c_n}Z_{x_n} \to \min \quad (29)$$

Subject to

$$\begin{aligned} &Z_{a_{11}}Z_{x_1} + Z_{a_{12}}Z_{x_2} + \ldots + Z_{a_{1n}}Z_{x_n} \le Z_{b_1}, \\ &Z_{a_{21}}Z_{x_1} + Z_{a_{22}}Z_{x_2} + \ldots + Z_{a_{2n}}Z_{x_n} \le Z_{b_2}, \\ &\ldots \\ &Z_{a_{m1}}Z_{x_1} + Z_{a_{m2}}Z_{x_2} + \ldots + Z_{a_{mn}}Z_{x_n} \le Z_{b_m}, \end{aligned} \quad (30)$$

$$Z_{x_1}, Z_{x_2}, \ldots, Z_{x_n} \ge 0^Z. \quad (31)$$

Here, the decision variables and parameters are expressed in Z-numbers:

$$\begin{aligned} &Z_{x_i} = (\tilde{A}_{x_i}, \tilde{B}_{x_i}), \\ &Z_{c_i} = (\tilde{A}_{c_i}, \tilde{B}_{c_i}), \\ &Z_{a_{ij}} = (\tilde{A}_{a_{ij}}, \tilde{B}_{a_{ij}}), \\ &Z_{b_j} = (\tilde{A}_{b_j}, \tilde{B}_{b_j}), \\ &i = 1, \ldots, n, j = 1, \ldots, m. \end{aligned}$$

Solution of the problem. To essentially understand the problem of linear programming based on Z-valuation, we need to clarify the meaning of the maximum Z_f and inequalities expressed by Z-numbers. A method that can determine the optimal (max or min) value of Z_f is not found in the scientific literature. Therefore, we use a direct search method, called Differential Evolutionary Optimization (DEO) method, to solve the Z-LP problem (23)-(25).

Definition 2. A Z-valued slack variable. Assume that in the linear programming problem based on the ith constraint Z- valuation

$$\sum_{j=1}^{n} Z_{a_{ij}}Z_{x_j} \le Z_{b_i},$$

one uses

$$\sum_{j=1}^{n} Z_{a_{ij}}Z_{x_j} + Z_{x_{n+i}} = Z_{b_i},$$

if $Z_{x_{n+i}} \ge 0^Z$, then the variable $Z_{x_{n+i}}$ based on Z- valuation is called a Z-valued slack variable.

Definition 3. A Z-valued surplus variable. Assume that in the linear programming problem based on the ith Z-valued constraint

$\sum_{j=1}^{n} Z_{a_{ij}} Z_{x_j} \geq Z_{b_i}$

$\sum_{j=1}^{n} Z_{a_{ij}} Z_{x_j} - Z_{x_{n+i}} = Z_{b_i}, \quad Z_{x_{n+i}} \geq 0^Z,$

The variable $Z_{x_{n+i}}$ based on Z-estimation is called Z-valued surplus variable.

Definition 4. A Z-valued feasible solution. If any Z_x in (21) satisfies conditions (22)-(23), it is called a Z-valued feasible solution of (21)-(23).

Definition 5. The Z-valued optimal solution. Assume that Z_s is the set of possible solutions of (23)-(25) based on Z-values. If $Z_f(Z_{X_0}) \leq^Z Z_f(Z_X)$ satisfies the conditions (24)-(25), then Z-valued feasible solution $Z_{X_0} \in Z_s$ is called Z-valued optimal solution of (23)-(25).

First, we add the Z-valued slack variables.

$$Z_f(Z_{x_1}, Z_{x_2}, \ldots, Z_{x_n}) = Z_{c_1} Z_{x_1} + Z_{c_2} Z_{x_2} + \ldots + Z_{c_n} Z_{x_n} \rightarrow \min$$

subject to

$$\begin{aligned}
&Z_{a_{11}} Z_{x_1} + Z_{a_{12}} Z_{x_2} + \ldots + Z_{a_{1n}} Z_{x_n} + Z_{x_{n+1}} = Z_{b_1},\\
&Z_{a_{21}} Z_{x_1} + Z_{a_{22}} Z_{x_2} + \ldots + Z_{a_{2n}} Z_{x_n} + Z_{x_{n+2}} = Z_{b_2},\\
&\ldots\\
&Z_{a_{m1}} Z_{x_1} + Z_{a_{m2}} Z_{x_2} + \ldots + Z_{a_{mn}} Z_{x_n} + Z_{x_{n+m}} = Z_{b_m},\\
&\quad Z_{x_1}, Z_{x_2}, \ldots, Z_{x_n}, Z_{x_{n+1}}, Z_{x_{n+2}}, \ldots, Z_{x_{n+m}} \geq 0^Z.
\end{aligned}$$

Then we rewrite the problem under consideration in the appropriate equivalent form:

$$\begin{aligned}
&Z_g(Z_{x_1}, Z_{x_2}, \ldots, Z_{x_n}, Z_{x_{n+1}}, Z_{x_{n+2}}, \ldots, Z_{x_{n+m}}) = Z_{c_1} Z_{x_1} + Z_{c_2} Z_{x_2} + \ldots + Z_{c_n} Z_{x_n} +\\
&+(Z_{b_1} - (Z_{a_{11}} Z_{x_1} + Z_{a_{12}} Z_{x_2} + \ldots + Z_{a_{1n}} Z_{x_n} + Z_{x_{n+1}})) +\\
&+(Z_{b_2} - (Z_{a_{21}} Z_{x_1} + Z_{a_{22}} Z_{x_2} + \ldots + Z_{a_{2n}} Z_{x_n} + Z_{x_{n+2}})) +\\
&\ldots\\
&+(Z_{b_m} - (Z_{a_{m1}} Z_{x_1} + Z_{a_{m2}} Z_{x_2} + \ldots + Z_{a_{mn}} Z_{x_n} + Z_{x_{n+m}})) \rightarrow \min
\end{aligned}$$

subject to

$$Z_{x_1}, Z_{x_2}, \ldots, Z_{x_n}, Z_{x_{n+1}}, Z_{x_{n+2}}, \ldots, Z_{x_{n+m}} \geq 0^Z$$

To solve this optimization problem, we use the Differential Evolutionary Optimization algorithm.

Solution method. To solve the optimization problem, first all decision variables Z_X are initialized by generating random values in the interval [-1, 1]. To start the optimization, we first define the parameters of Differential Evolutionary Optimization (DEO), Z_f as the objective function of Differential Evolutionary Optimization (DEO) $Z_g\left(Z_{x_1}, Z_{x_2}, \dots, Z_{x_n}, Z_{x_{n+1}}, Z_{x_{n+2}}, \dots, Z_{x_{n+m}}\right)$ and choose the population size. (typically, 10 times the optimization parameters, i.e. 10*Npar). Then the Differential Evolutionary Optimization process begins.

First, we define the template parameters (Z_{xs}) of the measurement (Npar) to know the decision variables (Z_x). Then we need to define the parameters of the algorithm: mutation rate (F), crossover rate (CR) and population size (PN).

We calculate the fitness function as an objective function. We randomly generate PN parameter vectors (for example, from the appropriate parameter space [-1, 1]) and create a population:

$$P=\{Z_{X_1}, Z_{X_2}, \dots, Z_{X_{ps}}\}.$$

In the end, if the result (either the specified number of generations should be obtained or the required error level should be obtained) is not as expected, a new set of parameters should be generated. We choose the next vector:

$$Z_{X_i} \text{ (i=1,...,PopSize).}$$

Then we take 3 different test vectors from P: $Z_{X_{r1}}, Z_{X_{r2}}, Z_{X_{r3}}$ so that each of them is different from the current vector Z_{X_i}. We generate the test vector:

$$Z_{X_t} = Z_{X_{r1}} + F \cdot (Z_{X_{r2}} - Z_{X_{r3}}).$$

We generate a new vector from the test vector Z_{X_t}. The individual vector parameters of Z_{X_t} are transformed into a new vector Z_{X_i} together with the probability of the crossover norm. If the cost function of $Z_{X_{new}}$ is better (or lower) than the cost function of Z_{X_i}, the current Z_{X_i}, function is replaced by the population P of $Z_{X_{new}}$. Then, from the population P, we select the parameter vector with the best value function (objective function) $Z_{X_{best}}$. Then we extract the vectors of decision variables from $Z_{X_{best}}$.

Then we generate a new vector from the test vector S_t . The individual vector parameters of S_t are inherited together with the probability of the calama norm and assigned to the vector S_new. If the value function of S_{new} is better (or lower) than the value function of S_i, the current S_i is replaced by the population P of S_{new}. Then from the population P we have the highest value function (Z_f). S_{best}. we select the parameter vector (the best decision variables). Now we can extract all the decision variables from S_{best}.

In the fourth chapter ("Decision-making in the Z-information environment without using the utility function") methods of decision-making in the Z-information environment without using the utility function were given. A common approach in the field of decision-making methods is the application of utility theories. The main shortcoming of utility theories is that they are based on the evaluation of vector-valued alternatives by means of a scalar-valued quantity. This transformation always leads to loss of information and contradicts intuition. In real life, one does not switch from vector values to scalar values when comparing attributes for thinking or decision making. Although there are approaches based on a utility function described by a vector, there is no fundamental axiomatic theory. On the other hand, preferences such as human judgment are often vague and cannot be described with precise numerical values. However, existing works on vector-valued utility function-based approaches are devoted to situations characterized by perfect decision-related information, which is rarely encountered in real-life decision-making. However, utility-based approaches rely on restrictive assumptions such as independence or its various relaxed conditions, completeness, transitivity, regularity, and alike. A useful utility model based on very limited assumptions is simple but inadequate; models based on less restrictive assumptions make the utility model more complex though more adequate. There are also cases where utility functions cannot be applied (for example, lexicographic order).

The above cases, on the one hand, necessitate the development of new decision-making approaches based on direct pairwise comparison of vector-valued alternatives. On the other hand, new approaches must be based on linguistic comparison of alternatives to work with uncertain vector alternatives, since real-life alternatives are almost always significant to some degree. Linguistic modeling of preferences allows to reduce the set of Pareto optimal alternatives of alternatives or helps to obtain a narrowed subset of optimal alternatives when all relevant information is explained in natural language (NL). For this purpose, the concept of fuzzy optimality [13, 14] can be used as a means of redefining existing scientific concepts based on Computation with Words (CW).

Definition 6. (Pareto dominance). For any two points $f_i, f_k \in A$ (candidate solutions), the alternative f_i is considered Pareto superior (P-dominated) to f_k if and only if the following conditions are satisfied:

$f_i(s_j) \geq f_k(s_j)$ for each $j \in \{1,2,\ldots,M\}$,

$f_i(s_{j'}) > f_k(s_{j'})$ for at least one $j' \in \{1,2,\ldots,M$.

Definition 7. (Pareto Optimality). $f^* \in A$ is considered Pareto Optimal if it is impossible to find $f_i \in A$ such that the alternative f_i is P-dominant with respect to f^*.

Definition 8. (Pareto set and Pareto Front). We call the set of Pareto optimal solutions in the project area and the goal area S_P Pareto optimal set and F_P Pareto front, respectively.

Statement of Problem. Suppose that $S = \{\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_M\} \subset \mathcal{E}^n$ is a set of fuzzy natural states, and $X \subset \mathcal{E}^n$ is a set of fuzzy results. Fuzzy in nature is used for fuzzy granulation of objective conditions when clean division of the latter is not possible due to the inaccuracy of the relevant information described in natural language. A set of

[13] Zadeh, L. A. (2006). Generalized theory of uncertainty (GTU)—principal concepts and ideas. *Computational Statistics and Data Analysis*, *51*, 15–46.
[14] Farina,M.,&Amato, P. (2004). A fuzzy definition of ”optimality” for many-criteria optimization problems. *IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans*, *34*(3), 315–326.

alternatives A is understood as a set of fuzzy functions $\tilde{f}$ affecting S to X [15], [16]. Linguistic information about the probabilities $\tilde{P}^l$ of states of nature is expressed through fuzzy probabilities $\tilde{P}_j$ of states $\tilde{s}_j$:

$$\tilde{P}^l = \tilde{P}_1/\tilde{s}_1 + \tilde{P}_2/\tilde{s}_2 + \ldots + \tilde{P}_M/\tilde{s}_M,$$

here $\tilde{P}_j \in \mathcal{E}^1_{[0,1]}$.

Ambiguous preferences over ambiguous alternatives are modeled through a linguistic preference relation over A. For this, it is sufficient to define the set of terms $T = (T_1, \ldots, T_K)$ for the linguistic variable "precedence degree" [17, 18]. The conditions can be labeled as "equality", "low preference", "high preference" and each can be described by a fuzzy number set on a certain scale, for example in the interval [0,1] or [0,10]. The linguistic superiority of $\tilde{f}_i$ over $\tilde{f}_k$ is written as $\tilde{f}_i \gtrsim_l \tilde{f}_k$. This means that one can find $T_i \in T$ a linguistic degree $Deg(\tilde{f}_i \gtrsim_l \tilde{f}_k)$ such that the preference degree of alternative $\tilde{f}_i$ over $\tilde{f}_k$ is expressed as $Deg(\tilde{f}_i \gtrsim_l \tilde{f}_k) \approx T_i$.

Thus, fuzzy decision-making with imperfect information is described as a quadruple $\left(S, \tilde{P}^l, X, A\right)$. The decision-making problem consists of determining $\gtrsim_l$. This is described by the degrees of optimality of alternatives. The degree of optimality of the alternative $\tilde{f}_i$ is denoted $do(\tilde{f}_i)$ and is the overall degree to which $\tilde{f}_i$ is preferred over all other alternatives. The decision-making problem consists in

[15] Zadeh, L.A., Aliev, R.A., Fazlollahi, B.,Alizadeh, A.V., Guirimov,B.G.,&Huseynov, O.H. (2009).Decision Theory with Imprecise Probabilities. Contract on "Application of Fuzzy Logic and Soft Computing to communications, planning and management of uncertainty". Technical report, Berkeley, Baku, 95 p. http://www.raliev.com/report.pdf

[16] Aliev, R. A., Alizadeh, A. V., Guirimov, B. G., & Huseynov, O. H. (2010). Precisiated information-based approach to decision making with imperfect information. In *Proceedings of the ninth international conference on application of fuzzy systems and soft computing, 2010*, *ICAFS-2010* (pp. 91–103). Prague, Czech Republic.

[17] Borisov, A. N., Alekseyev, A. V., Merkuryeva, G. V., Slyadz, N. N., & Gluschkov, V. I. (1989). Fuzzy information processing in decision making systems. Moscow: Radio i Svyaz (in Russian).

[18] Liu, W. J., & Zeng, L. (2008). A new TOPSIS method for fuzzy multiple attribute group decision making problem. *Journal of Guilin University of Electronic Technology*, *28*(1), 59–62.

determining the optimal alternative $\tilde{f}^* \in A$ with the degree of optimality $do(\tilde{f}^*) = \max_{\tilde{f}_i \in A} do(\tilde{f}_i)$.

Solution method. The fuzzy Pareto optimality (FPO) formalism suggested in [19] is developed for a perfect information structure, i.e. when all the decision relevant information is represented by precise numerical evaluations. From the other side, this approach is developed for MADM. We will extend the FPO formalism for the considered framework of decision making with imperfect information. The method of solution is described below.

The solution to the considered problem consists in determining the degree of linguistic superiority of $\tilde{f}_i$ over $\tilde{f}_k$ for all $\tilde{f}_i, \tilde{f}_k \in A$ for direct comparison of $\tilde{f}_i$ and $\tilde{f}_k$ alternatives described by the vector.

At the first stage, the value of fuzzy probabilities $\tilde{P}_j$ should be determined for each fuzzy state $\tilde{s}_j$ of nature. However, partial information expressed by fuzzy probabilities can be given for all but one of the fuzzy states. An unknown fuzzy probability cannot be assigned, but must be calculated based on known fuzzy probabilities. Since the computation of the unknown fuzzy probability requires the construction of a membership function, it is an optimization problem. The problem of calculating the unknown fuzzy probability $\tilde{P}(\tilde{S}_j) = \tilde{P}_j$ is formulated as follows according to the rule proposed in [20]

$$\mu_{\tilde{P}_j}(p_j) = \sup_\rho \min_{j\prime=\{1,\dots,j-1,j+1,\dots,n\}} (\mu_{\tilde{P}_{j\prime}}(\int_S \mu_{\tilde{s}_{j\prime}}(s)\rho(s)ds))$$

$$\int_S \mu_{\tilde{s}_j}(s)\rho(s)ds = p_j, \int_S \rho(s)ds = 1.$$

Here, $\mu_{\tilde{s}_j}(s)$ is the membership function of the fuzzy state $\tilde{s}_j$. Thus, the unknown probability $\tilde{P}_j$ for the state $\tilde{s}_j$ is built not only on the basis of probabilities given for other states of nature, but also on the basis

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[19] Farina,M.,&Amato, P. (2004). A fuzzy definition of "optimality" for many-criteria optimization problems. *IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans*, *34*(3), 315–326.

[20] Zadeh, L. A. (2006). Generalized theory of uncertainty (GTU)-principal concepts and ideas. *Computational Statistics and Data Analysis*, *51*, 15–46.

of imperfect information about the state $\tilde{s}_j$ itself. After $\tilde{P}_j$ is found, the linguistic probability distribution $\tilde{P}^l$ for all states $\tilde{s}_j$ is determined:

$$\tilde{P}^l = \tilde{P}_1/\tilde{s}_1 + \tilde{P}_2/\tilde{s}_2 + \ldots + \tilde{P}_M/\tilde{s}_M.$$

An important problem that arises when calculating $\tilde{P}^l$ is testing for goodness-of-fit, completeness, and redundancy. In the second stage, taking into account the distribution of adaptive, complete and non-redundant fuzzy probabilities over all $\tilde{s}_j$ fuzzy states of nature, the superiority, equivalence of $\tilde{f}_i$ over $\tilde{f}_k$ for all situations, taking into account the fuzzy probability of each fuzzy and it is necessary to determine the general degrees of weakness.

The corresponding overall nbF, neF, and nwF degrees of superiority, equivalence, and weakness of $\tilde{f}_i$ to $\tilde{f}_k$ are determined based on the differences between the fuzzy results of $\tilde{f}_i$ and $\tilde{f}_k$ for each fuzzy state of nature as follows:

$$nbF(\tilde{f}_i, \tilde{f}_k) = \sum_{j=1}^{M} \mu_b^j \left(gmv((\tilde{f}_i(\tilde{s}_j) - \tilde{f}_k(\tilde{s}_j)) \cdot \tilde{P}_j)\right),$$

$$neF(\tilde{f}_i, \tilde{f}_k) = \sum_{j=1}^{M} \mu_e^j \left(gmv((\tilde{f}_i(\tilde{s}_j) - \tilde{f}_k(\tilde{s}_j)) \cdot \tilde{P}_j)\right),$$

$$nwF(\tilde{f}_i, \tilde{f}_k) = \sum_{j=1}^{M} \mu_w^j \left(gmv((\tilde{f}_i(\tilde{s}_j) - \tilde{f}_k(\tilde{s}_j)) \cdot \tilde{P}_j)\right).$$

Here, μ_b^j, μ_e^j, μ_w^j are the membership functions for “better”, “equivalent” and “worse” evaluations, respectively [21].

μ_b^j, μ_e^j, μ_w^j corresponding to state j are determined so that the Ruspini condition is satisfied. This condition is expressed as follows:

$$nbF\left(\tilde{f}_i, \tilde{f}_k\right) + neF\left(\tilde{f}_i, \tilde{f}_k\right) + nwF\left(\tilde{f}_i, \tilde{f}_k\right) =$$

$$= \sum_{j=1}^{M} (\mu_b^j + \mu_e^j + \mu_w^j) = M.$$

Based on $nbF(\tilde{f}_i, \tilde{f}_k)$, $neF(\tilde{f}_i, \tilde{f}_k)$, and $nwF(\tilde{f}_i, \tilde{f}_k)$, the $(1 - kF)$-dominance degree is determined. These concepts express the

[21] Farina,M.,&Amato, P. (2004). A fuzzy definition of ”optimality” for many-criteria optimization problems. *IEEE Transactions on Systems, Man and Cybernetics, Part A: Systems and Humans*, *34*(3), 315–326.

necessary and sufficient condition for the $(1-kF)$-dominance of $\tilde{f}_i$ with respect to $\tilde{f}_k$:

$$neF(\tilde{f}_i,\tilde{f}_k) < M,\ nbF(\tilde{f}_i,\tilde{f}_k) \geq \frac{M-neF(\tilde{f}_i,\tilde{f}_k)}{kF+1},$$

where $kF \in [0,1]$.

To determine the kF corresponding to the largest value of the $(1-kF)$-dominance of $\tilde{f}_i$ with respect to $\tilde{f}_k$, the function d is defined as follows[22]:

$$d(\tilde{f}_i,\tilde{f}_k) =$$

$$= \begin{cases} 0, & if\ \ nbF(\tilde{f}_i,\tilde{f}_k) \leq \frac{M-neF(\tilde{f}_i,\tilde{f}_k)}{2} \\ \frac{2\cdot nbF(\tilde{f}_i,\tilde{f}_k)+neF(\tilde{f}_i,\tilde{f}_k)-M}{nbF(\tilde{f}_i,\tilde{f}_k)}, & otherwise \end{cases}.$$

For a given d, the desired largest rate kF is found as $1-d(\tilde{f}_i,\tilde{f}_k)$.

The statement $d(\tilde{f}_i,\tilde{f}_k)=1$ means that $\tilde{f}_i$ is Pareto superior to $\tilde{f}_k$, and $d(\tilde{f}_i,\tilde{f}_k)=0$ means that $\tilde{f}_i$ is not Pareto superior to $\tilde{f}_k$.

In the fuzzy optimality formalism, instead of $\tilde{f}^*$ Pareto optimality concept, $\tilde{f}^*$ Pareto optimality concept with degree kF is introduced. $\tilde{f}^*$ is considered kF optimal if and only if $\tilde{f}_i \in A$ cannot be found such that the alternative $\tilde{f}_i$ is $(1-kF)$-dominant with respect to $\tilde{f}^*$.

The basic idea of the concept of fuzzy optimality is that the optimality degree $do(\tilde{f}^*)$ of the Pareto optimality of $\tilde{f}^*$ is defined as:

$$do(\tilde{f}^*) = 1 - \max_{\tilde{f}_i \in A} d(\tilde{f}_i,\tilde{f}^*).$$

Thus, the degree $do(\tilde{f}^*)$ is a degree resulting from considering the preference degrees of $\tilde{f}^*$ over all alternatives.

The function do can be considered as the membership function of a fuzzy set describing the concept of kF-optimality.

[22] Rafik A. Aliev, Witold Pedrycz, A.V. Alizadeh, Oleg H. Huseynov. Fuzzy optimality based decision making under imperfect information without utility, Journal Fuzzy Optimization and Decision Making, Kluwer Academic Publishers Hingham, MA, USA, In: FO & DM, https://doi.org/10.1007/s10700-013-9160-2, 2013, Volume 12 Issue 4

We call the set of kF -Pareto optimal solutions in the project area and the goal area as S_{kF} Pareto optimal set and kF -optimal $\mathcal{F}_{kF}$ Pareto front, respectively.

Assume that $\mu_D(\tilde{f}_i, \tilde{f}_k)$ is the membership function defined as:

$$\mu_D(\tilde{f}_i, \tilde{f}_k) = \varphi_{\mu_D}(nbF(\tilde{f}_i, \tilde{f}_k), neF(\tilde{f}_i, \tilde{f}_k), nwF(\tilde{f}_i, \tilde{f}_k)).$$

Then $\mu_D(\tilde{f}_i, \tilde{f}_k)$ is a fuzzy preference relation if for each $\alpha \in [0,1]$ $\mu_D(\tilde{f}_i, \tilde{f}_k) > \alpha$ from the expression $\tilde{f}_i$ to $\tilde{f}_k$ $(1 - kF)$-dominance.

In the special case, φ_{μ_D} is defined as:

$$\varphi_{\mu_D} = \frac{2 \cdot nbF(\tilde{f}_i, \tilde{f}_k) + neF(\tilde{f}_i, \tilde{f}_k)}{2M}.$$

A membership function $\mu_D(\tilde{f}_i, \tilde{f}_k)$ represents the fuzzy optimality relation if for any $0 \leq kF \leq 1$. $\tilde{f}^*$ belongs to the kF -cut of μ_D if and only if there is no $\tilde{f}_i \in A$ such that

$$\mu_D(\tilde{f}_i, \tilde{f}^*) > kF.$$

At the third stage, on the base of values of $nbF(\tilde{f}_i, \tilde{f}_k)$, $neF(\tilde{f}_i, \tilde{f}_k)$, and $nwF(\tilde{f}_i, \tilde{f}_k)$, the value of degree of optimality $do(\tilde{f}_i)$ as a degree of membership to a fuzzy Pareto optimal set, is determined by using formulas (17)–(21) for each $\tilde{f}_i \in A$. The obtained $do()$ allows for justified determination of linguistic preference relation $\_\gtrsim_l$ over A.

At the fourth stage, the degree $Deg(\tilde{f}_i \gtrsim_l \tilde{f}_k)$ of preference of $\tilde{f}_i$ to $\tilde{f}_k$ for any $\tilde{f}_i, \tilde{f}_k \in A$ should be determined based on $do()$. For simplicity, one can calculate $Deg(\tilde{f}_i \gtrsim_l \tilde{f}_k)$ as follows:

$$Deg(\tilde{f}_i \gtrsim_l \tilde{f}_k) = do(\tilde{f}_i) - do(\tilde{f}_k).$$

The ranking of alternatives is discussed in detail in Chapter 5.

In the fifth chapter ("The problem of ranking decisions in the Z-information environment based on the Pareto optimality principle"), the solution of decision ranking issues in the Z-information environment based on the Pareto optimality principle is considered. Decision-making based on the comparison of discrete Z-numbers, including decision-making that does not include the utility function, a method of comparison of Z-numbers based on the principle of fuzzy optimality is given, examples of comparison of Z-numbers are shown, a model of decision-making in the conditions of Z-information is proposed, Practical methods of decision-making based on Z-information are proposed, a multi-criteria decision-making method based on linear mathematical programming for Z-quantities is explained.

Comparison of Z-numbers based on fuzzy optimality principle. Ranking of discrete Z-numbers is a necessary operation in arithmetic of Z-numbers and is a challenging practical issue. In this section we suggest an approach to ranking of discrete Z-numbers.

In contrast to real numbers, Z-numbers are ordered pairs, for ranking of which there can be no unique approach.

For purpose of comparison, we suggest to consider a Z-number as a pair of values of two attributes – one attribute measures value of a variable, the other one measures the associated reliability. Then it will be adequate to compare Z-numbers as multiattribute alternatives. Basic principle of comparison of multiattribute alternatives is the Pareto optimality principle which is based on a counterintuitive assumption that all alternatives within a Pareto optimal set are considered equally optimal. The fuzzy Pareto optimality (FPO) concept[23] fits very well multiattribute decision making problems. This concept is an implementation of the ideas of CW-based redefinitions of the existing scientific concepts [24]. In this approach, by directly comparing alternatives, one arrives at total degrees to which one alternative is better than, is equivalent to and is worse than another one. These degrees are determined as graded sums of differences between attribute values for considered alternatives[25,26,27]. Such

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[23] Farina, M., and Amato, P. (2004). A fuzzy definition of "optimality" for many-criteria optimization problems, *IEEE T. Syst. Man Cy. A: Systems and Humans*, 34(3), pp. 315-326.
[24] Zadeh, L. A. (2006). Generalized theory of uncertainty (GTU) – principal concepts and ideas, *Comput. Stat. DataAn.,* 51, pp. 15-46
[25] Aliev, R. A. (2013) *Fundamentals of the Fuzzy Logic-Based Generalized Theory of Decisions*. (Springer, NewYork, Berlin).
[26] Aliev, R. A., Pedrycz, W., Alizadeh, A. V. and Huseynov, O. H. (2013). Fuzzy optimality based decision making under imperfect information without utility, *Fuzzy Optim. Decis. Ma.*, vol. 12, issue 4, pp. 357-372
[27] Farina, M., and Amato, P. (2004). A fuzzy definition of "optimality" for many-criteria optimization problems, *IEEE T. Syst. Man Cy. A: Systems and Humans*, 34(3), pp. 315-326.

comparison is closer to the way humans compare alternatives by confronting their attribute values.

We suggest to consider comparison of Z-numbers on the base of FPO principle as follows. Let Z-numbers $Z_1 = (A_1, B_1)$ and $Z_2 = (A_2, B_2)$ be given. It is needed to compare the corresponding components of these Z-numbers. For this purpose, it is needed to calculate the functions n_b, n_e, n_w which evaluate how much one of the Z-numbers is better, equivalent and worse than the other one with respect to the first and the second components A and B. The total degree n_b measures the number of components with respect to which $Z_1 = (A_1, B_1)$ dominates $Z_2 = (A_2, B_2)$ (minimum is 0, maximum is 2). The total degree n_w measures the number of components with respect to which $Z_1 = (A_1, B_1)$ is dominated by $Z_2 = (A_2, B_2)$ (minimum is 0, maximum is 2). The total degree n_e measures the number of components with respect to which $Z_1 = (A_1, B_1)$ is equivalent to $Z_2 = (A_2, B_2)$ (minimum is 0, maximum is 2).

The functions n_b, n_e, n_w are defined as follows:

$$n_b(Z_i, Z_j) = P_b(\delta_A^{i,j}) + P_b(\delta_B^{i,j}),$$

$$n_e(Z_i, Z_j) = P_e(\delta_A^{i,j}) + P_e(\delta_B^{i,j}),$$

$$n_w(Z_i, Z_j) = P_w(\delta_A^{i,j}) + P_w(\delta_B^{i,j}),$$

where $\delta_A^{i,j} = A_i - A_j, \delta_B^{i,j} = B_i - B_j$, $i, j = 1,2,\ i \neq j$. The meaning of these functions is as follows. As superiority, equivalence and inferiority of one Z-number with respect to the other is actually a matter of a degree for human intuition, $\delta_A^{i,j} = A_i - A_j$ and $\delta_B^{i,j} = B_i - B_j$ may be evaluated by using the function:

$$P_l(\delta_A^{i,j}) = \frac{Poss(\delta_A^{i,j}|n_l)}{\sum_{t\in\{b,e,w\}} Poss(\delta_A^{i,j}|n_t)},\ P_l(\delta_B^{i,j}) = \frac{Poss(\delta_B^{i,j}|n_l)}{\sum_{t\in\{b,e,w\}} Poss(\delta_B^{i,j}|n_t)},$$

where $Poss$ is a possibility measure[28].

[28] Aliev, R. A. (2013) *Fundamentals of the Fuzzy Logic-Based Generalized Theory of Decisions*. (Springer, NewYork, Berlin).

The function $P_l()$ is therefore used as a weighted possibility measure. As $\sum_{t\in\{b,e,w\}} P_l(\delta_k^{i,j}) = 1$ will always hold, one will always have $n_b(Z_i, Z_j) + n_e(Z_i, Z_j) + n_w(Z_i, Z_j) = N$, where N is the number of components of a Z-number, i.e. $N = 2$.

Next, on the base of n_b, n_e, n_w , the $(1-k)$ -dominance is determined as dominance in the terms of its degree. This concept suggests that Z_1 $(1-k)$-dominates Z_2 iff

$$n_e(Z_i, Z_j) < 2, n_b(Z_i, Z_j) \geq \frac{2-n_e(Z_i, Z_j)}{k+1},$$

with $k \in [0,1]$.

Next it is needed to determine the greatest k such that Z_i Pareto dominates Z_j to the degree$(1-k)$. For this purpose, a function d is introduced [29]:

$$d(Z_i, Z_j) = \\ = \begin{cases} 0, & if \;\; n_b(Z_i, Z_j) \leq \dfrac{2-n_e(Z_i, Z_j)}{2} \\ \dfrac{2 \cdot n_b(Z_i, Z_j) + n_e(Z_i, Z_j) - 2}{n_b(Z_i, Z_j)}, & otherwise \end{cases}$$

Given a value of d, the desired greatest k is found as $k = 1 - d(Z_i, Z_j)$, and then $(1-k) = d(Z_i, Z_j)$. $d(Z_i, Z_j) = 1$ implies Pareto dominance of Z_i over Z_j, whereas $d(Z_i, Z_j) = 0$ implies no Pareto dominance of Z_i over Z_j . The degree of optimality $do(Z_i)$ is determined as follows:

$$do(Z_i) = 1 - d(Z_j, Z_i).$$

Thus, we can consider $do(Z_i)$ as the degree to which one Z-number is higher than the other one. Then

$Z_i > Z_j$ iff $do(Z_i) > do(Z_j)$,

$Z_i < Z_j$ iff $do(Z_i) < do(Z_j)$,

[29] A.V. Alizadeh, Application of the Fuzzy Optimality Concept to Decision Making, Advances in Intelligent Systems and Computing Springer, Cham. https://doi.org/10.1007/978-3-030-35249-3_69, 2019, 542-54, https://link.springer.com/chapter/10.1007%2F978-3-030-35249-3_69

$Z_i = Z_j$ otherwise.

Recall that comparison of fuzzy numbers is a matter of a degree due to related vagueness. For Z-numbers, which are more complex constructs characterized by possibilistic-probabilistic uncertainty, degree-based comparison is even more desirable.

The suggested approach may be considered as basis of a human-oriented ranking of Z-numbers. In this viewpoint, we suggest to take into account degree of pessimism $\beta \in [0,1]$ as a mental factor which influences a choice of a preferred Z-number. The degree of pessimism is submitted by a human observer who wishes to compare the considered Z-numbers but does not completely rely on the results obtained by the above mentioned FPO approach. In this viewpoint, given $do(Z_j) \leq do(Z_i)$, we define for two Z-numbers Z_1 and Z_2:

$$r(Z_i, Z_j) = \beta do(Z_j) + (1 - \beta) do(Z_i).$$

Then

$$\left.\begin{array}{l} Z_i > Z_j \text{ iff } r(Z_i, Z_j) > \frac{1}{2}(do(Z_i) + do(Z_j)) \\ Z_i < Z_j \text{ iff } r(Z_i, Z_j) < \frac{1}{2}(do(Z_i) + do(Z_j)) \\ \text{and} \\ Z_i = Z_j \text{ otherwise} \end{array}\right\}$$

The degree of pessimism β is submitted by a human being and adjust ranking of Z-numbers to reflect human attitude to the do degree-based comparison. This attitude may result from the different importance of A and B components for a human being and other issues.

The sixth chapter ("Application of the proposed decision-making methods") is devoted to the application and simulation of decision-making methods in the Z-information environment. The results of the application of the proposed methods to the solution of standard decision-making problems such as decision-making for the supply problem, decision-making for the multi-criteria marketing problem, optimal planning of the company's production based on Z-line programming.

6.1. Decision making for the supply issue.

Consider the issue of decision-making based on imprecise information as a problem of supplier selection taking into account different economic conditions. The set of alternatives is represented by the set of five suppliers: $A = \{\tilde{f}_1, \tilde{f}_2, \tilde{f}_3, \tilde{f}_4, \tilde{f}_5\}$. The set of natural states is described by five possible economic conditions: $S = \{\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_5\}$. Each economic situation $\tilde{s}_j$ is characterized by demands on aspects of supplier profitability, relationship closeness, technological capabilities, compatibility quality, and conflict resolution. Fuzzy evaluations of the consequences of alternatives in economic conditions are given in the following table (Table 1):

Table 1. Table of fuzzy results

	s_1	s_2	s_3	s_4	s_5
f_1	(5.0,7.0,9.0)	(7.0,9.0,10.0)	(3.0,5.0,7.0)	(9.0,10.0,10.0)	(5.0,7.0,9.0)
f_2	(1.0,3.0,5.0)	(3.0,5.0,7.0)	(5.0,7.0,9.0)	(7.0,9.0,10.0)	(1.0,3.0,5.0)
f_3	(3.0,5.0,7.0)	(5.0,7.0,9.0)	(7.0,9.0,10.0)	(5.0,7.0,9.0)	(3.0,5.0,7.0)
f_4	(0.0,1.0,3.0)	(1.0,3.0,5.0)	(0.0,1.0,3.0)	(1.0,3.0,5.0)	(7.0,9.0,10.0)
f_5	(7.0,9.0,10.0)	(0.0,1.0,3.0)	(1.0,3.0,5.0)	(3.0,5.0,7.0)	(0.0,1.0,3.0)

Linguistic information about the probabilities of economic conditions is described as follows:

$$\tilde{P}^l = (0.2, 0.3, 0.4)/s_1 + (0.1, 0.2, 0.3)/s_2 + (0.0, 0.1, 0.2)/s_3 + \\ + (0.3, 0.3, 0.5)/s_4 + (0.0, 0.1, 0.4)/s_5$$

Applying the approach based on the concept of fuzzy optimality, the considered problem can be solved as follows. First, nbF, neF, nwF are calculated:

$$nbF = \begin{bmatrix} 0 & 0.28667 & 0.21 & 0.55667 & 0.42667 \\ 0.02 & 0 & 0.056667 & 0.35667 & 0.28 \\ 0.033333 & 0.14667 & 0 & 0.41333 & 0.31 \\ 0.02 & 0.086667 & 0.053333 & 0 & 0.15667 \\ 0.046667 & 0.16667 & 0.10667 & 0.31333 & 0 \end{bmatrix},$$

$$neF = \begin{bmatrix} 5 & 4.6933 & 4.7567 & 4.4233 & 4.5267 \\ 4.6933 & 5 & 4.7967 & 4.5567 & 4.5533 \\ 4.7567 & 4.7967 & 5 & 4.5333 & 4.5833 \\ 4.4233 & 4.5567 & 4.5333 & 5 & 4.53 \\ 4.5267 & 4.5533 & 4.5833 & 4.53 & 5 \end{bmatrix},$$

$nwF =$

$$\begin{bmatrix} 0 & 0.02 & 0.033333 & 0.02 & 0.046667 \\ 0.28667 & 0 & 0.14667 & 0.086667 & 0.16667 \\ 0.21 & 0.056667 & 0 & 0.053333 & 0.10667 \\ 0.55667 & 0.35667 & 0.41333 & 0 & 0.31333 \\ 0.42667 & 0.28 & 0.31 & 0.15667 & 0 \end{bmatrix}.$$

Then we calculate μ_D and $d(f_i, f_k)$

$$\mu_D = \begin{bmatrix} 0.5 & 0.47333 & 0.48233 & 0.44633 & 0.462 \\ 0.52667 & 0.5 & 0.509 & 0.473 & 0.48867 \\ 0.51767 & 0.491 & 0.5 & 0.464 & 0.47967 \\ 0.55367 & 0.527 & 0.536 & 0.5 & 0.51567 \\ 0.538 & 0.51133 & 0.52033 & 0.48433 & 0.5 \end{bmatrix},$$

$$d(f_i, f_k) = \begin{bmatrix} 0 & 0.93023 & 0.84127 & 0.96407 & 0.89062 \\ 0 & 0 & 0 & 0.75701 & 0.40476 \\ 0 & 0.61364 & 0 & 0.87097 & 0.65591 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{bmatrix}.$$

Finally, we calculate the degree of optimality for each of the considered alternatives:

$$do = \begin{bmatrix} 1 \\ 0.069767 \\ 0.15873 \\ 0.035928 \\ 0.10938 \end{bmatrix}.$$

Thus, the advantages obtained are $\tilde{f}_1 \gtrsim \tilde{f}_3 \gtrsim \tilde{f}_5 \gtrsim \tilde{f}_2 \gtrsim \tilde{f}_4$. The degrees of preference are as follows:

$Deg(\tilde{f}_1 \gtrsim_l \tilde{f}_3) = 0.84, Deg(\tilde{f}_3 \gtrsim_l \tilde{f}_5) = 0.05,$

$Deg(\tilde{f}_5 \gtrsim_l \tilde{f}_2) = 0.04, Deg(\tilde{f}_2 \gtrsim_l \tilde{f}_4) = 0.034.$

The advantage obtained by this method is: $\tilde{f}_1 \gtrsim \tilde{f}_3 \gtrsim \tilde{f}_2 \gtrsim \tilde{f}_5 \gtrsim \tilde{f}_4$. It can be seen from here that the order obtained by our proposed method is almost the same as the result obtained by the method proposed. In addition, the proposed method has several advantages compared to the method. First, our proposed method not only determines the order among alternatives, but also determines how optimal the considered alternative is. This degree is the overall degree to which the alternative under consideration is better than the others. Second, by means of the method proposed, it is possible to separate the considered alternatives into less and optimal ones by determining the corresponding k_F values using our proposed approach. Taking into account these two advantages, the method proposed in the dissertation has almost the same computational complexity as the method.

6.2. Decision making for a multi-criteria marketing problem. Marketing decision making under Z-information.

In this application we intend a problem of marketing decision making in the field of IT. Two new software products were introduced to the market by Techware Incorporated; the company has three alternatives related to these two products: it introduces product 1 only, product 2 only, or introduces both products. The costs for research and development for these two products are \$180,000 and \$150,000, respectively.

The trend of the national economy and the consumers reaction to these products will affect the success of these products in the coming year. If the company introduces product 1, then it will have revenue of \$500,000, \$260,000 and \$120,000 for strong, fair and weak national economy respectively. Similarly, when product 2 is introduced, there will be a revenue of \$420,000, \$230,000 and 110,000 for a strong, fair and weak national economy, respectively.

Finally, when introducing both products 1 and 2 the revenues will be \$820,000, \$390,000, \$200,000 for strong, fair and weak national economy, respectively. The experts of the company are very sure that the probabilities of strong and fair economy are about 0.30 and about 0.50 respectively. The problem is to determine the best decision.

The analyzed data are obtained from Techware Incorporated in [30].

Let us proceed to formal description of the considered decision problem. The partially reliable linguistic decision-relevant

[30] Winston, W. L. and Albright, S.C., Broadie, M. (2002) *Practical management science. Thomas Learning*, 2nd Ed., pp. 496-498.

information in the considered problem will be described by Z-numbers. The set of alternatives:

$$\mathcal{A} = \{f_1, f_2, f_3\},$$

where f_1 denotes introducing product 1, f_2 denotes introducing product 2, f_3 denotes introducing both products (1 and 2). The set of states of nature:

$$\mathcal{S} = \{S_1, S_2, S_3\},$$

where S_1 denotes strong national economy, S_2 denotes fair national economy, S_3 denotes strong national economy. The probabilities of states of nature are

$Z_{P(S_1)} = (about\ 0.3, quite\ sure)$,

$Z_{P(S_2)} = (about\ 0.5, quite\ sure)$.

The set of outcomes:

$\mathcal{X} = \{(low, likely), (more\ than\ low, likely),$

$(medium, likely)\ (below\ than\ high, likely),$

$(high, likely)\}.$

The partially reliable linguistic information for the probabilities of states of nature and the utilities of each alternative taken at different states of nature is shown in Table 2.

Table 2. The values of utilities for different alternatives and probabilities of states of nature

	S_1	S_2	S_3
	(about 0.3, quite sure)	(about 0.5, quite sure)	(about 0.2, quite sure)
f_1	(high; likely)	(medium; likely)	(low; likely)
f_2	(below than high; likely)	(medium; likely)	(low; likely)
f_3	(high; likely)	(more than low; likely)	(low; likely)

The corresponding decision matrix with Z-number-based representation is shown in Table 3.

Table 3. Decision matrix with Z-number

	S_1 Z_{41}	S_2 Z_{42}	S_3 Z_{43}
f_1	Z_{11}	Z_{12}	Z_{13}
f_2	Z_{21}	Z_{22}	Z_{23}
f_3	Z_{31}	Z_{32}	Z_{33}

6.3. Optimal Planning of Company Production by Z-linear Programming.

Consider the following Z-LP problem with two decision variables [31]:

$$Z_{c_1}Z_{x_1} + Z_{c_2}Z_{x_2} \to \max$$

subject to

$$Z_{a_{11}}Z_{x_1} + Z_{a_{12}}Z_{x_2} \leq^Z Z_{b_1},$$
$$Z_{a_{21}}Z_{x_1} + Z_{a_{22}}Z_{x_2} \leq^Z Z_{b_2},$$

$Z_{x_1}, Z_{x_2} \geq^Z Z_0,$

where $Z_{x_1} = (A_{x_1}, B_{x_1})$, $Z_{x_2} = (A_{x_2}, B_{x_2})$ and $Z_{c_1} = (A_{c_1}, B_{c_1})$, $Z_{c_2} = (A_{c_2}, B_{c_2})$, $Z_{a_{11}} = (A_{a_{11}}, B_{a_{11}})$, $Z_{a_{12}} = (A_{a_{12}}, B_{a_{12}})$, $Z_{a_{21}} = (A_{a_{21}}, B_{a_{21}})$, $Z_{a_{22}} = (A_{a_{22}}, B_{a_{22}})$, $Z_{b_1} = (A_{b_1}, B_{b_1})$, $Z_{b_2} = (A_{b_2}, B_{b_2})$. The values of these Z-numbers are given below.

A Z-number $Z_{c_1} = (A_{c_1}, B_{c_1})$:

$$A_{c_1} = {}^{0.0}/_{0} + {}^{0.0}/_{1} + {}^{0.0003}/_{2} + {}^{0.01}/_{3} + {}^{0.14}/_{4} + {}^{0.61}/_{5} + $$
$$+ {}^{1.0}/_{6} + {}^{0.61}/_{7} + {}^{0.14}/_{8} + {}^{0.01}/_{9} + {}^{0.0003}/_{10},$$

$$B_{c_1} = {}^{0}/_{0} + {}^{0}/_{0.1} + {}^{0}/_{0.2} + {}^{0.01}/_{0.3} + {}^{0.14}/_{0.4} + {}^{0.60}/_{0.5} + $$
$$+ {}^{1}/_{0.6} + {}^{0.61}/_{0.7} + {}^{0.14}/_{0.8} + {}^{0.01}/_{0.9} + {}^{0}/_{1}.$$

A Z-number $Z_{c_2} = (A_{c_2}, B_{c_2})$:

[31] Aliev, R. A., Alizadeh, A. V., Huseynov, O. H., Jabbarova, K.I. Z-number based Linear Programming. *Int. J. Intell. Syst.*

$$A_{c_2} = {}^{0.0}/_{0} + {}^{0.0}/_{1} + {}^{0.0}/_{2} + {}^{0.0}/_{3} + {}^{0.0003}/_{4} + {}^{0.01}/_{5} + + {}^{0.14}/_{6} + {}^{0.61}/_{7} + {}^{1.0}/_{8} + {}^{0.61}/_{9} + {}^{0.14}/_{10},$$

$$B_{c_2} = {}^{0}/_{0} + {}^{0}/_{0.1} + {}^{0.01}/_{0.2} + {}^{0.14}/_{0.3} + {}^{0.61}/_{0.4} + + {}^{1}/_{0.5} + {}^{0.61}/_{0.6} + {}^{0.14}/_{0.7} + {}^{0.01}/_{0.8} + {}^{0}/_{0.9} + {}^{0}/_{1}.$$

A Z-number $Z_{a_{11}} = (A_{a_{11}}, B_{a_{11}})$:

$$A_{a_{11}} = {}^{0.01}/_{0} + {}^{0.14}/_{1} + {}^{0.61}/_{2} + {}^{1.0}/_{3} + {}^{0.61}/_{4} + {}^{0.14}/_{5} + + {}^{0.01}/_{6} + {}^{0.001}/_{7} + {}^{0}/_{8} + {}^{0}/_{9} + {}^{0}/_{10},$$

$$B_{a_{11}} = {}^{0}/_{0} + {}^{0}/_{0.1} + {}^{0}/_{0.2} + {}^{0}/_{0.3} + {}^{0}/_{0.4} + {}^{0}/_{0.5} + + {}^{0.01}/_{0.6} + {}^{0.14}/_{0.7} + {}^{0.61}/_{0.8} + {}^{1}/_{0.9} + {}^{0.61}/_{1}.$$

A Z-number$Z_{a_{12}} = (A_{a_{12}}, B_{a_{12}})$:

$$A_{a_{12}} = {}^{0.61}/_{0} + {}^{1}/_{1} + {}^{1.0}/_{2} + {}^{0.61}/_{3} + {}^{0.14}/_{4} + {}^{0.01}/_{5} + + {}^{0.001}/_{6} + {}^{0}/_{7} + {}^{0}/_{8} + {}^{0}/_{9} + {}^{0}/_{10},$$

$$B_{a_{12}} = {}^{0}/_{0} + {}^{0}/_{0.1} + {}^{0}/_{0.2} + {}^{0}/_{0.3} + {}^{0}/_{0.4} + {}^{0}/_{0.5} + + {}^{0.01}/_{0.6} + {}^{0.14}/_{0.7} + {}^{0.61}/_{0.8} + {}^{1}/_{0.9} + {}^{0.61}/_{1}.$$

For simplicity, $Z_{a_{21}} = (A_{a_{21}}, B_{a_{21}})$ and $Z_{a_{22}} = (A_{a_{22}}, B_{a_{22}})$ are chosen as singletons:

$A_{a_{21}} = 1, B_{a_{21}} = 1;$

$A_{a_{22}} = 1, B_{a_{22}} = 1.$

A Z-number $Z_{b_1} = (A_{b_1}, B_{b_1})$:

$$A_{b_1} = {}^{0.14}/_{0} + {}^{0.61}/_{1} + {}^{1}/_{2} + {}^{0.61}/_{3} + {}^{1.0}/_{4} + {}^{0.61}/_{5} + + {}^{0.14}/_{6} + {}^{0.01}/_{7} + {}^{0.001}/_{8} + {}^{0}/_{9} + {}^{0}/_{10},$$

$$B_{b_1} = {}^{0}/_{0} + {}^{0}/_{0.1} + {}^{0}/_{0.2} + {}^{0}/_{0.3} + {}^{0}/_{0.4} + {}^{0}/_{0.5} + + {}^{0.01}/_{0.6} + {}^{0.14}/_{0.7} + {}^{0.61}/_{0.8} + {}^{1}/_{0.9} + {}^{0.61}/_{1}.$$

A Z-number $Z_{b_2} = (A_{b_2}, B_{b_2})$:

$A_{b_2} = {}^{0.01}/_{0} + {}^{0.14}/_{1} + {}^{0.61}/_{2} + {}^{1.0}/_{3} + {}^{0.14}/_{4} + {}^{0.01}/_{5} +$
$+ {}^{0}/_{6} + {}^{0}/_{7} + {}^{0}/_{8} + {}^{0}/_{9} + {}^{0}/_{10},$

$B_{b_2} = {}^{0}/_{0} + {}^{0}/_{0.1} + {}^{0}/_{0.2} + {}^{0}/_{0.3} + {}^{0}/_{0.4} + {}^{0}/_{0.5} +$
$+ {}^{0.01}/_{0.6} + {}^{0.14}/_{0.7} + {}^{0.61}/_{0.8} + {}^{1}/_{0.9} + {}^{0.61}/_{1}.$

By adding Z-valued slack variables, we obtain:

$$Z_{c_1}Z_{x_1} + Z_{c_2}Z_{x_2} \rightarrow \max$$

subject to

$Z_{a_{11}}Z_{x_1} + Z_{a_{12}}Z_{x_2} + Z_{x_3} = Z_{b_1},$
$Z_{a_{21}}Z_{x_1} + Z_{a_{22}}Z_{x_2} + Z_{x_4} = Z_{b_2},$

$Z_{x_1}, Z_{x_2}, Z_{x_3}, Z_{x_4} \geq^{Z} Z_0.$

Then we arrive at the equivalent form:

$$-(Z_{c_1}Z_{x_1} + Z_{c_2}Z_{x_2}) + (Z_{b_1} - (Z_{a_{11}}Z_{x_1} + Z_{a_{12}}Z_{x_2} + Z_{x_3} + Z_{x_4})) +$$
$$+(Z_{b_2} - (Z_{a_{21}}Z_{x_1} + Z_{a_{22}}Z_{x_2} + Z_{x_3} + Z_{x_4})) \rightarrow \min$$

subject to

$Z_{x_1}, Z_{x_2}, Z_{x_3}, Z_{x_4} \geq^{Z} Z_0.$

We applied the suggested DEO algorithm-based method for solving this problem [32]. The following values of the parameters of the DE optimization algorithm were used: mutation rate $F = 0.8$, crossover probability $CR = 0.7$, population size is $PN = 80$. The obtained optimal solution and the optimal value of the objective function are given below.

The first decision variable $Z_{x_1} = (A_{x_1}, B_{x_1})$:

$A_{x_1} = {}^{1.0}/_{0} + {}^{0.61}/_{1} + {}^{0.14}/_{2} + {}^{0.01}/_{3} + {}^{0.0}/_{4} + {}^{0.0}/_{5} +$
$+ {}^{0.0}/_{6} + {}^{0.0}/_{7} + {}^{0.0}/_{8} + {}^{0.0}/_{9} + {}^{0.0}/_{10},$

[32] Aliev, R. A., Alizadeh, A. V., Huseynov, O. H., Jabbarova, K.I. Z-number based Linear Programming. *Int. J. Intell. Syst.*

$$B_{x_1} = {}^{0.14}/_{0} + {}^{0.61}/_{0.1} + {}^{1.0}/_{0.2} + {}^{0.61}/_{0.3} + {}^{0.14}/_{0.4} + $$
$$+ {}^{0.01}/_{0.5} + {}^{0.0}/_{0.6} + {}^{0.0}/_{0.7} + {}^{0.0}/_{0.8} + {}^{0.0}/_{0.9} + {}^{0.0}/_{1}.$$

The second decision variable $Z_{x_2} = (A_{x_2}, B_{x_2})$:

$$A_{x_2} = {}^{0.01}/_{0} + {}^{0.14}/_{1} + {}^{0.61}/_{2} + {}^{1}/_{3} + {}^{0.61}/_{4} + {}^{0.14}/_{5} + $$
$$+ {}^{0.01}/_{6} + {}^{0}/_{7} + {}^{0}/_{8} + {}^{0}/_{9} + {}^{0}/_{10},$$

$$B_{x_2} = {}^{0}/_{0} + {}^{0}/_{0.1} + {}^{0}/_{0.2} + {}^{0}/_{0.3} + {}^{0}/_{0.4} + {}^{0.0001}/_{0.5} + $$
$$+ {}^{0.001}/_{0.6} + {}^{0.01}/_{0.7} + {}^{0.14}/_{0.8} + {}^{0.61}/_{0.9} + {}^{1}/_{1}.$$

The third (slack) decision variable $Z_{x_3} = (A_{x_3}, B_{x_3})$:

$$A_{x_3} = {}^{0}/_{0} + {}^{0.01}/_{1} + {}^{0.14}/_{2} + {}^{0.61}/_{3} + {}^{1}/_{4} + {}^{0.61}/_{5} + $$
$$+ {}^{0.14}/_{6} + {}^{0.01}/_{7} + {}^{0}/_{8} + {}^{0}/_{9} + {}^{0}/_{10},$$

$$B_{x_3} = {}^{0}/_{0} + {}^{0}/_{0.1} + {}^{0}/_{0.2} + {}^{0.001}/_{0.3} + {}^{0.01}/_{0.4} + {}^{0.14}/_{0.5} + $$
$$+ {}^{0.61}/_{0.6} + {}^{1}/_{0.7} + {}^{0.61}/_{0.8} + {}^{0.14}/_{0.9} + {}^{0.01}/_{1}.$$

The fourth (slack) decision variable $Z_{x_4} = (A_{x_4}, B_{x_4})$:

$$A_{x_4} = {}^{0}/_{0} + {}^{0.01}/_{1} + {}^{0.14}/_{2} + {}^{0.61}/_{3} + {}^{1}/_{4} + {}^{0.61}/_{5} + $$
$$+ {}^{0.14}/_{6} + {}^{0.01}/_{7} + {}^{0}/_{8} + {}^{0}/_{9} + {}^{0}/_{10},$$

$$B_{x_4} = {}^{0}/_{0} + {}^{0}/_{0.1} + {}^{0}/_{0.2} + {}^{0.01}/_{0.3} + {}^{0.14}/_{0.4} + $$
$$+ {}^{0.61}/_{0.5} + {}^{1}/_{0.6} + {}^{0.61}/_{0.7} + {}^{0.14}/_{0.8} + {}^{0.01}/_{0.9} + {}^{0}/_{1}.$$

The optimal value of the objective function $Z_f(Z_{x_1}, Z_{x_2}) = (A_f, B_f)$:

$$A_f = {}^{0}/_{0} + {}^{0.61}/_{14} + {}^{1}/_{24} + {}^{0.61}/_{32} + {}^{0.14}/_{43} + $$
$$+ {}^{0.14}/_{66} + {}^{0}/_{86} + {}^{0}/_{120} + {}^{0}/_{160},$$

$$B_f = {}^{0}/_{0.09} + {}^{0}/_{0.12} + {}^{0}/_{0.14} + {}^{0}/_{0.143} + {}^{0}/_{0.17} + {}^{0}/_{0.2} + $$
$$+ {}^{0.0001}/_{0.22} + {}^{0.01}/_{0.224} + {}^{0.14}/_{0.25} + {}^{0.61}/_{0.28} + {}^{1}/_{0.3}.$$

For comparative analysis of results of this Z-LP example, the same LP problem from [33] which is stated in terms of GFNs is considered. A GFN is a modified trapezoidal fuzzy number$(a, b, c, d; w)$, where w is a maximal value of the membership function. The results obtained in are as follows:

$x_1 = 0$, $x_2 = (1,2,4,7; 0.7)$ and $f = (4,12,40,112; 0.5)$.

The core of the first component in $Z_{x_1} = (A_{x_1}, B_{x_1})$ is equal to $core(A_{x_1}) = 0$ and the center of the interval with the highest membership values for the GFN x_1 is 3. The core of the first component in $Z_{x_2} = (A_{x_2}, B_{x_2})$ is $core(A_{x_1}) = 3$ and center of the interval with the highest membership values for the GFN x_2 is 2.45. The core of the first component of a Z-number $Z_f(Z_{x_1}, Z_{x_2}) = (A_f, B_f)$ describing optimal value of the objective function is $core(A_f) = 24$ and the rank of center of the interval with the highest membership values for the GFN $(4,12,40,112; 0.5)$ is 21. At the same time, the difference in the reliability levels of the results obtained by the compared approaches is larger. However, one can see that the results obtained by both the approaches are close to each other. There are two basic differences between the approach suggested in [23] and our approach[34].

The first concerns structures of a Z-number and a generalized fuzzy number (GFN). A GFN is a modified trapezoidal fuzzy number$(a, b, c, d; w)$, where w is a maximal value of the membership function. Thus, in this formalization, belief related to an imprecise estimation is represented as a maximal value of membership function which may be lower than 1. In a Z-number, belief (reliability) to A_x is formalized as a value of probability measure of A_x. That is, a Z-number is a more structured formal construct and has a more expressive power. Moreover, in computation with Z-numbers,

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[33] Kumar, A., Singh P. and Kaur, J. (2010). Generalized Simplex Algorithm to Solve Fuzzy Linear Programming Problems with Ranking of Generalized Fuzzy Numbers. *TJFS,* 1, pp:80-103.

[34] Aliev, R. A., Alizadeh, A. V., Huseynov, O. H., Jabbarova, K.I. Z-number based Linear Programming. *Int. J. Intell. Syst.*, accepted.

propagation of reliability is carried on a more fundamental level, and reliability is computed at each step. In contrast, when using GFNs, propagation of belief is more sketchy (min operation is used). From the other side, in a Z-number belief is described by a fuzzy number, whereas in GFN belief is a precise value which is counterintuitive.

6.4. Multi-criteria decision making based on Z-mathematical programming.

Business Decision Making in Mix-product. A manufacturing company produces products A, B and C and has six processes for production. A decision maker has three objectives: maximizing profit, quality and worker satisfaction. Naturally, the parameters of objective functions and constraints are assigned by Z-numbers. Z-information on manufacturing planning is given in Table 4. Z-information on expected profit, index of quality and worker satisfaction index is given in Table 5.

Table 4. Z-information on manufacturing planning data

Resource type	Product A (Z_{x_1})	Product B (Z_{x_2})	Product C (Z_{x_3})	Maximum available capacity per month (hours) (Z_{b_i})
1	$Z_{a_{11}} = (about\ 12, 0{,}9)$	$Z_{a_{12}} = (about\ 17, 0{,}9)$	$Z_{a_{13}} = (about\ 0, 0{,}9)$	$Z_{b_1} = (about\ 1400{,}0{,}9)$
2	$Z_{a_{21}} = (about\ 2, 0{,}9)$	$Z_{a_{22}} = (about\ 9, 0{,}9)$	$Z_{a_{23}} = (about\ 8, 0{,}9)$	$Z_{b_2} = (about\ 1000, 0{,}9)$
3	$Z_{a_{31}} = (about\ 10, 0{,}9)$	$Z_{a_{32}} = (about\ 13, 0{,}9)$	$Z_{a_{33}} = (about\ 15, 0{,}9)$	$Z_{b_3} = (about\ 1750{,}0{,}9)$
4	$Z_{11} = (about\ 6, 0{,}9)$	$Z_{42} = (about\ 0, 0{,}9)$	$Z_{a_{13}} = (about\ 16, 0{,}9)$	$Z_{b_4} = (about\ 1325, 0{,}9)$
5	$Z_{51} =$	$Z_{52} = (about\ 12{,}0{,}9)$	$Z_{a_{53}} =$	$Z_{b_5} =$

	$(about\ 0, 0{,}9)$		$(about\ 7, 0{,}9)$	$(about\ 900, 0{,}9)$
6	$Z_{61} = (about\ 9{,}5, 0{,}9)$	$Z_{62} = (about\ 9{,}5, 0{,}9)$	$Z_{a_{63}} = (about\ 4, 0{,}9)$	$Z_{b_6} = (about\ 1075, 0{,}9)$

Table 5. Z-information on profits, quality, and worker satisfaction

Type of objectives	Product A	Product B	Product C
Profit	$Z_{C_{11}} = (about\ 0, 0{,}8)$	$Z_{C_{12}} = (about\ 100, 0{,}8)$	$Z_{C_{13}} = (about\ 17{,}5, 0{,}8)$
Quality	$Z_{C_{21}} = (about\ 92, 0{,}8)$	$Z_{C_{22}} = (about\ 75, 0{,}8)$	$Z_{C_{11}} = (about\ 50, 0{,}8)$
Worker satisfaction	$Z_{C_{31}} = (about\ 25, 0{,}8)$	$Z_{C_{32}} = (about\ 100, 0{,}8)$	$Z_{C_{33}} = (about\ 75, 0{,}8)$

Taking into account Z-information given in Tables 4, 5, multi-criteria Z-LP model for multi-criteria planning decision may be formulated as follows.

$$Z_{f_1}(Z_x) = \big[(about\ 50, \quad 0{,}8 \cdot Z_{x_1} + (about\ 100, \quad 0{,}8 \cdot Z_{x_2} + +(about\ 17{,}5, \quad 0{,}8 \cdot Z_{x_3}) \to \max$$

$$Z_{f_2}(Z_x) = \big[(about\ 92, \quad 0{,}8 \cdot Z_{x_1} + (about\ 75, \quad 0{,}8 \cdot Z_{x_2} + +about\ 50, \quad 0{,}8 \cdot Z_{x_3}) \to \max$$

$$Z_{f_3}(Z_x) = \big[(about\ 25, \quad 0{,}8 \cdot Z_{x_1} + (about\ 100, \quad 0{,}8 \cdot Z_{x_2} + +(about\ 75, \quad 0{,}8 \cdot Z_{x_3}) \to \max$$

$$(about\ 12,\quad 0.9)\cdot Z_{x_1} + (about\ 17,\quad 0.9)\cdot Z_{x_2} \le (about\ 1400,\ 0.9)$$

$$\begin{gathered}(about\ 2,\quad 0.9)\cdot Z_{x_1} + (about\ 9,\quad 0.9)\cdot Z_{x_2} + \\ +(about\ 8{,}0.9)\cdot Z_{x_3} \le (about\ 1000,\quad 0.9)\end{gathered}$$

$$\begin{gathered}(about\ 10,\quad 0.9)\cdot Z_{x_1} + (about\ 13,\quad 0.9)\cdot Z_{x_2} + \\ +(about\ 15,\quad 0.9)\cdot Z_{x_3} \le (about\ 1750,\quad 0.9)\end{gathered}$$

$$(about\ 6,\quad 0.9)\cdot Z_{x_1} + (about\ 16,\quad 0.9)\cdot Z_{x_3} \le about\ 1325,\ 0.9)$$

$$(about\ 12,\quad 0.9)\cdot Z_{x_2} + (about\ 7,\quad 0.9)\cdot Z_{x_3} \le about\ 900,\ 0.9)$$

$$\begin{gathered}(about\ 9.5,\quad 0.9)\cdot Z_{x_1} + (about\ 9.5,\quad 0.9)\cdot Z_{x_2} + \\ (about\ 4,\quad 0.9)\cdot Z_{x_3} \le about\ 1075,\quad 0.9)\end{gathered}$$

$$Z_{x_1}, Z_{x_2}, Z_{x_3} \ge 0.$$

6.5. Modeling of cerebrospinal fluid (CSF) flow based on Z-Differential equations.

In [35] , they consider the application of Z-valued differential equations to the modeling of cerebrospinal fluid (CSF) flow. However, Z^{+}-numbers are used instead of Z-numbers. We give the solution of differential equations described entirely by Z-numbers. CSF surrounds the brain and spinal cord. The CSF flow acts as a cushion to protect the brain, as well as being crucial for the removal of waste products from the brain. Hydrocephalus is a brain disease in which abnormalities in CSF flow result in ventricular dilatation and brain

[35] M. Mazandarani and Y. Zhao, "Z-Differential Equations," in *IEEE Transactions on Fuzzy Systems*. doi: 10.1109/TFUZZ.2019.2908131

compression [36,37]. In other words, excessive accumulation of CSF causes the ventricles to dilate abnormally, creating potentially damaging pressure on brain tissue. The mathematical model of such a disorder is as follows:

$$z'(t) = (-\frac{k}{r}) \cdot (z(t))^2 + (k \cdot I_f(t)) \cdot z(t) + \frac{k \cdot p_d}{r} \cdot z(t),$$

$$z(t_0) = z_0, \qquad (32)$$

where $z(t)$ is the CSF pressure in mmH2O, $k > 0$ is the cerebral elasticity, $r > 0$ is the resistance of CSF to absorption, I_f is the rate of CSF formation and p_d is the pressure of the venous system which is usually equal to $z(t_0)$.

Assume that

$$z(t_0) = (z_A, z_B) = ((110{,}115{,}120), (0.5{,}0.65{,}1)).$$

Without loss of generality, we consider exact values for k, $r$and $I_f(t)$ as $k = \frac{1}{0.6}$, $r = 700$, and $I_f(t) = 0.1$.

Solution of (32) is shown in Figs. 4 - 9. For comparison, we consider two cases (Figs. 6 - 9): case 1 - when similarity and dependency is not taken into account, case 2 - when similarity and dependency are taken into account [38].

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[36] Kauffman Justin, Drapaca Corina S. A fractional pressurevolume model of cerebrospinal uid dynamics in hydrocephalus. Mech Biol Syst Mater, vol. 4, 179-84, 2014.

[37] Marmarou A, Shulman K, Rosende RM, A nonlinear analysis of the cerebrospinal uid system and intracranial pressure dynamics. J Neurosurg, vol. 48, 332-344, 1978.

[38] Rafik A. Aliev Z-Differential Equations, Advances in Intelligent Systems and Computing Springer, Cham. https://doi.org/10.1007/978-3-030-35249-3_8, 2019, 69-77, https://link.springer.com/chapter/10.1007%2F978-3-030-35249-3_8#citeas

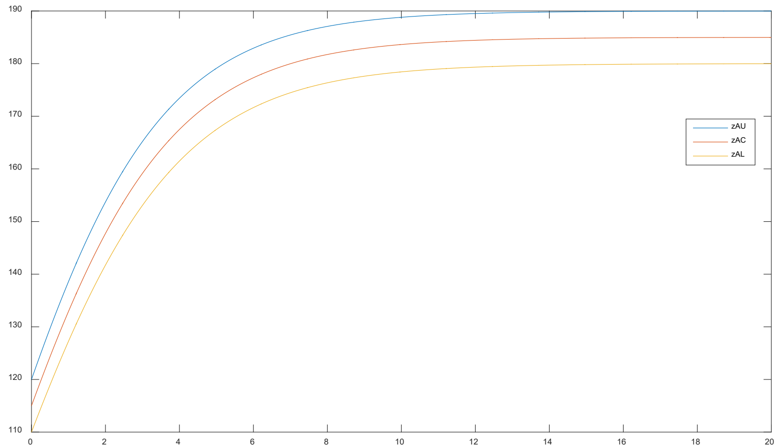


Fig. 4. $Z_A(t)$ component of the solution of equation (32).

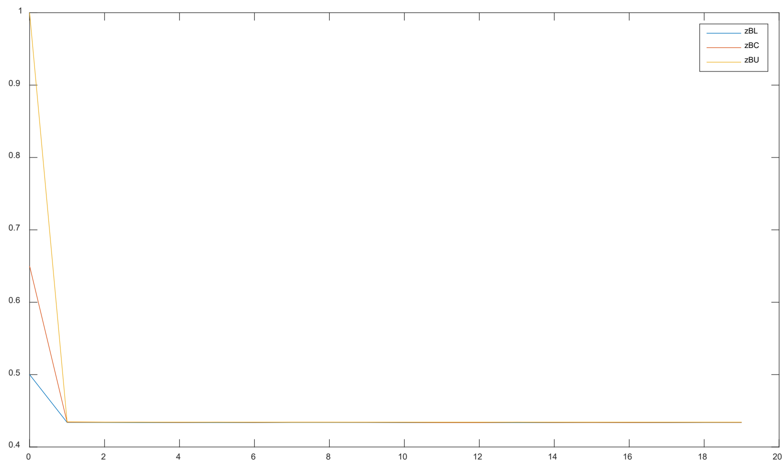


Fig. 5. $Z_B(t)$ component of the solution of equation (32).

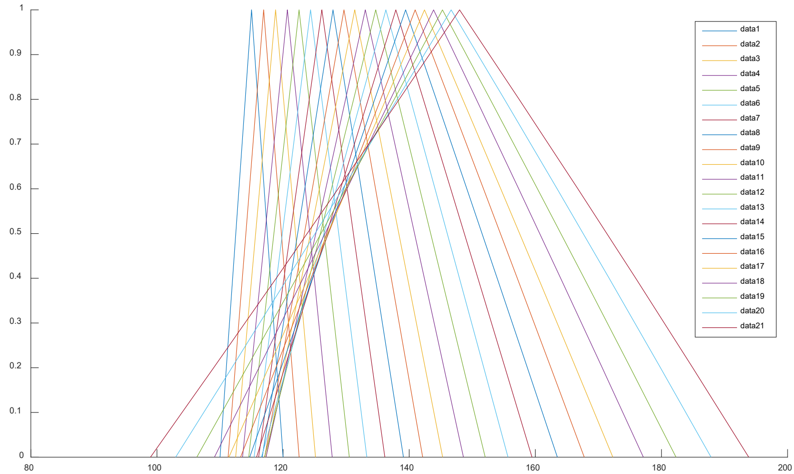


Fig. 6. $Z_A(t)$ component of the solution of equation (32), case 1.

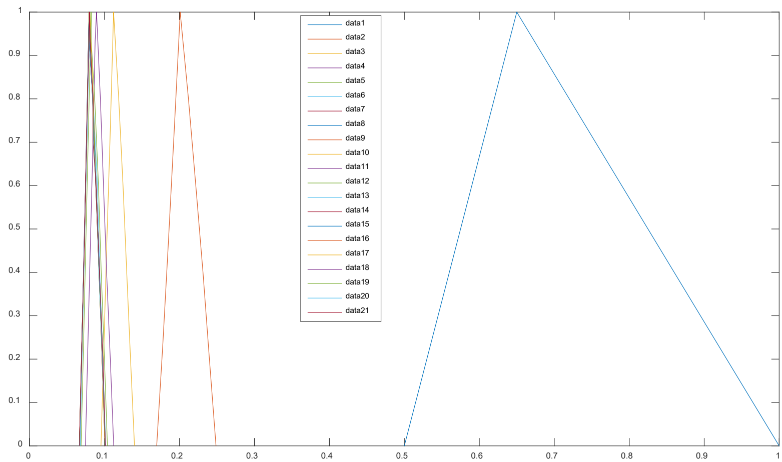


Fig. 7. $Z_B(t)$ component of the solution of equation (32), case 1.

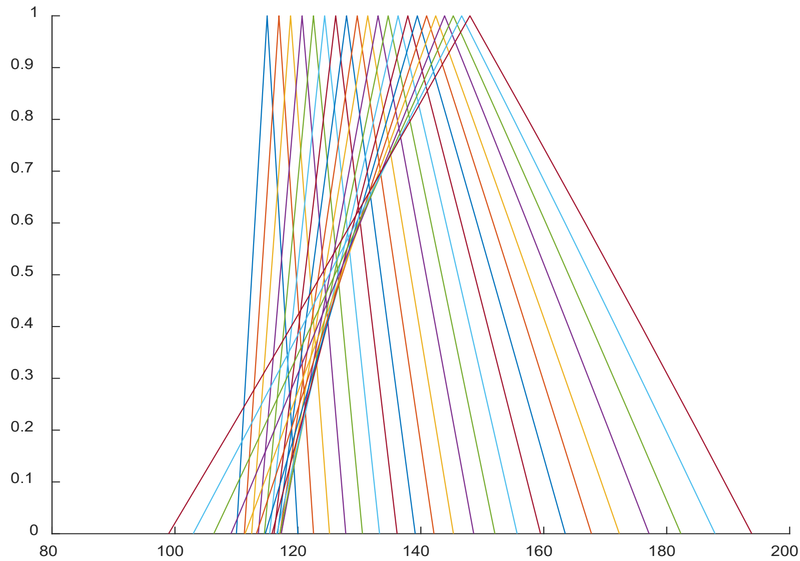
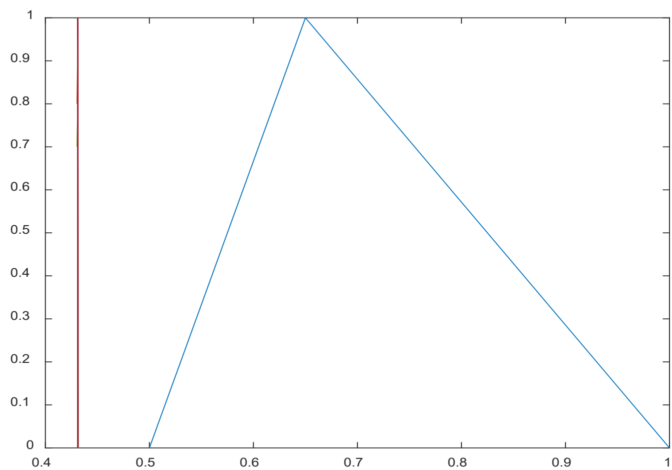


Fig. 8. $Z_A(t)$ component of the solution of equation (32), case 2.



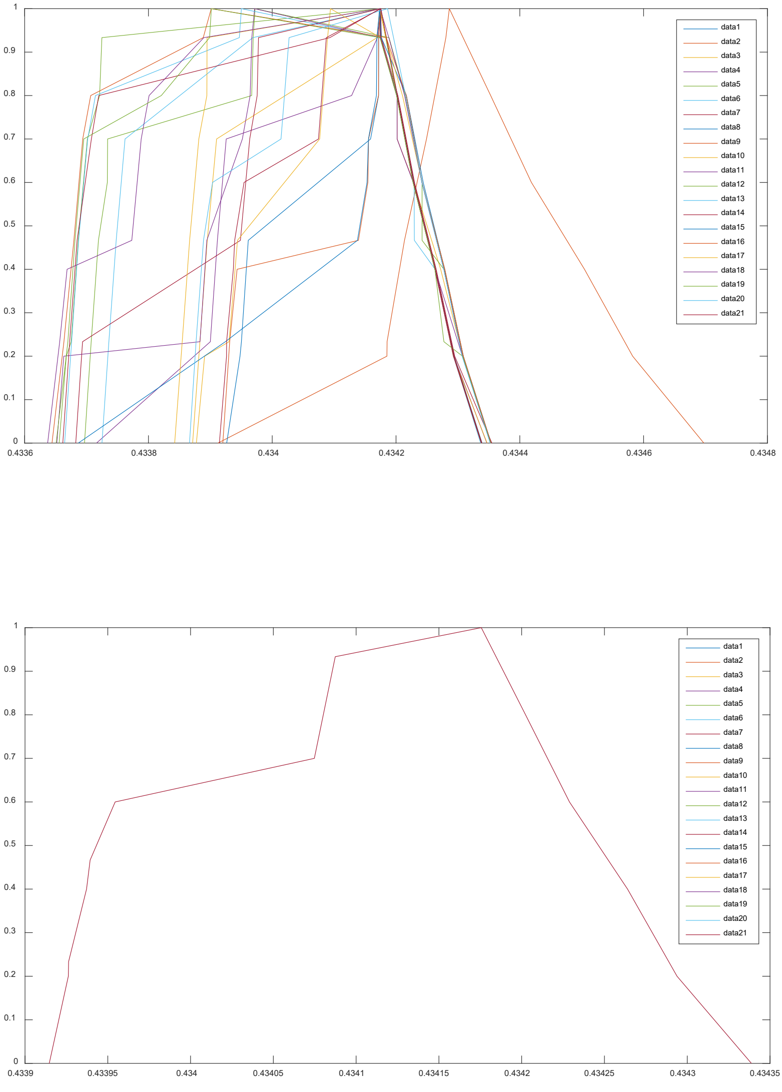


Fig. 9. $Z_B(t)$ component of the solution of equation (32), case 2.

As can be seen, in case 2 $Z_B(t)$ is better preserved during the fuzzy reliability process (it remains above 0.4, Figure 9), in case 1 it is below 0.1, and vice versa (figure 7). If we use the threshold value of (μ, ν)-cuts, we can get a better result. Thus, the proposed approach is an initial step for modeling dependent differential equations with Z-valued variables with improved handling of the combination of two types of uncertainty.

We use the horizontal membership functions (HMF)-based approach to calculations on Z-numbers as a basis for formulating and solving the initial condition problem expressed in Z-numbers. For the first time, Z-variable DTs are solved without introducing fuzzy and Z+-based analogs. At the same time, we propose an algorithm for computation with Z-numbers describing data about dependent random variables. This allows more adequate modeling of real dynamic processes, because the future states of the process naturally depend on the previous state. A new and efficient approach similar to Euler's method is proposed for calculating the Z-valued initial condition problem and numerical solution.

To show the advantage of this approach, an example of the solution of the DT described by Z-valued variables describing the dynamic process in the medical field is shown. The obtained results show that the proposed approach allows better processing of fuzzy reliability of information as one of the main indicators of bimodal data.

MAIN SCIENTIFIC RESULTS

The main scientific results obtained in the dissertation work are as follows:

1. In order to take into account, the reliability of information in decision-making, the problems of developing arithmetic and algebraic operations on Z-numbers were considered;
2. Operations on continuous Z-numbers;
3. Operations on discrete Z-numbers;

4. The solution of the Z-linear programming problem was proposed for the first time;
5. A method of decision-making in the Z-information environment, which is free of shortcomings of the utility function, is given;
6. A method based on the Pareto optimality principle was proposed for ranking alternatives in the Z-information environment;
7. The theoretical propositions proposed in the dissertation have been widely applied in the Zlab software package, created according to the Matlab software package, including operations on Z-numbers, Z-linear programming, Pareto optimality, etc. It is included in the Zlab package and is widely used in different countries of the world.
8. The proposed scientific basics were applied to solving the problems of decision-making: multi-criteria decision-making in marketing problem, optimal planning of the company's production based on Z-line programming, and the obtained results confirmed their effectiveness.

The main content of the dissertation was published in the following works:

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