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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

SYSTEMATIC APPROACH TO THE IMPLEMENTATION OF CONTROL ALGORITHMS FOR MECHANICAL SYSTEMS

Specialty: 3338.01 – Systems analysis, control, and information processing (by fields)

Field of science: Technical sciences

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GENERAL CHARACTERISTICS OF THE WORK

Relevance of the topic. The control of various-purpose mechanical systems is one of the important problems facing modern science. The problems arising during the investigation of mechanical systems can mainly be divided into two parts: the modeling of the kinematics and dynamics of these systems and their control. These two problems are closely related to each other, and in the solution of one of them, the other is widely used.

The construction of mathematical models of the dynamics of mechanical systems is one of the most important steps, as this makes it possible to control these systems by mathematical methods. During the control of complex mechanical systems, the application of a single mathematical model and the corresponding solution algorithm does not always lead to an adequate result. For this reason, it is necessary to regard and investigate that object or process as a complete system. It is necessary to investigate such an object or process by the systems approach method, which, instead of investigating the separate parts of the system individually, involves studying the system as a whole. It should be noted that the systems approach makes it possible to control a complex system or process in accordance with all factors and possibilities according to their degree of significance, which leads to finding the optimal operating mode of the real system. Recently, with the help of the systems approach, various problems have begun to be solved. Moreover, these problems arise and are continuing to arise in various fields of society and the economy.

One of the fields in which the application of the systems approach is necessary is the operation of oil wells. The process of oil extraction is so complex that dividing it into separate parts and solving them individually, then combining them, does not yield the required results. It is more appropriate to regard such complex processes as a single system and to solve many of the issues arising there by using methods of systems analysis and the systems approach.

It is known that the development of oil fields and the production of oil are carried out by various methods. At the initial stage, the operation of oil wells is carried out by the fountain method. The essence of this method is that oil rises from the layer to the surface under its own energy due to the high pressure in the layer. This process does not last for a long time. As the layer pressure gradually decreases, oil extraction by the fountain method is no longer possible, and for this reason, other oil recovery methods are used. These include the gas lift method, the method of extracting oil using rod pump units, and others.

The gas lift method is an exploitation method widely used after the fountain method. The essence of this method is that when it is no longer possible to extract oil by the fountain method due to the decrease in layer pressure, additional gas is injected into the well, creating an oil–gas mixture at the bottom of the well, and as the specific gravity of this mixture decreases, the bottom-hole pressure allows this mixture to rise to the surface. This process is called the gas lift method, which is one of the most widely used exploitation methods for medium-productive wells. The idea of gas lift is based on air lift, which allows lifting a liquid–air mixture upward by injecting air into a vertically lowered pipe.

The theoretical study of the gas lift process begins with its modeling. In this direction, the works of A.Kh. Mirzajanzade, I.M. Muravyov, V.I. Shurov, R.N. Baktizin, and others can be noted. In these works, various models of the gas lift process operating under different conditions have been developed, and based on these models, the issue of optimizing the gas lift has been considered.

One of the most relevant works on the modeling of the gas lift process is the work of F. A. Aliyev et al[1][2]. The main significance of these works is that here the gas lift process is described by a system

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[1] Алиев, Ф.А. Задачи моделирования оптимальной стабилизации газлифтного процесса / Ф.А. Алиев, М.Х. Ильясов, Н.Б. Нуриев // Прикладная механика, -Киев:- 2010. Т. 46. №6,- с.113-122.

[2] Алиев, Ф.А. Моделирование работы газлифтной скважины/ Ф. А. Алиев, М. Х. Ильясов, М. А. Джамалбеков// Доклады НАН Азербайджана, -Баку:- 2008. Т. LXIV. №4, - с. 30 – 41.

of special derivative differential equations, which, after various transformations, are brought into a form such that the resulting new problem can be reduced to the problem of optimal control. In this case, the optimization problem of the gas lift process, that is, obtaining the maximum production rate with minimal gas injection, is solved as a classical linear-quadratic optimal control problem.

Various models describing the gas lift process have been proposed, many of which are described by differential equations. In the dissertation, one of these models was improved by using a delayed argument in the equations describing the process for obtaining the gas-liquid mixture at the bottom of the well, which makes it possible to describe the process more adequately.

The processes occurring at the bottom of the well are among the most complex processes. The mixing of the gas injected around the pipe with the oil and the transfer of this mixture to the lifting pipe take place with a certain time delay, and taking this delay into account in the mathematical model is very important. Taking this delay into account can manifest itself in the differential and difference equations describing the model, which can ultimately lead to the derivation of differential equations with delayed arguments. For this reason, the study of differential equations with delayed arguments, the development of their solution methods, as well as the investigation of the optimal control problems posed through these equations, arise as an actual issue.

Various works have been devoted to the study of initial value problems described by differential equations with delayed arguments and the optimal control problems posed on their basis[3][4][5][6][7]. The study

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[3] Митропольский, Ю.А. Периодические и квази периодические колебания систем с запаздыванием/ Ю.А.Митропольский, Д. И. Мартынюк, - Киев: Вища школа, -1979. -248 с.

[4] Мышкис, А.Д. Линейные дифференциальные уравнения с запаздывающим аргументом/ А.Д. Мышкис.- Москва : Наука, -1972. -352 с.

[5] Пименов, В. Г. Функционально-дифференциальные уравнения в биологии и медицине/ В. Г. Пименов.-Екатеринбург:-2008. -92 с.

[6] Эльсгольц, Л.Э. Введение в теорию дифференциальных уравнений с отклоняющимся аргументом/ Л.Э. Эльсгольц, С.Б. Норкин -Москва: Наука, -1971. -296 с.

of such problems can be found in the works of S.B. Norkin, N.N. Krasovsky, R.F. Qabasov, P.T. Yanushevsky, F.M. Kirillova, and others.

The development of computer engineering and information technologies currently makes it possible to use systems analysis and mathematical modeling as a means of qualitatively solving the control problems of complex technical systems at a new level. The use of modern mathematical packages of applied programs makes it possible to conduct comprehensive research with high accuracy and minimal consumption of the resources of these technical systems.

One of such complex systems is the unmanned aerial vehicles (UAVs), especially quadcopters, which have recently become increasingly popular as a convenient and relatively inexpensive technical means for collecting information remotely, monitoring the environment, delivering small loads, performing combat missions, and carrying out a number of other tasks.

Quadcopters, which are a type of UAV, are used in various fields, and their main advantages are light weight, small size, maneuverability, and simple control system; these features make it possible to use quadcopters in various fields, including the military field. Therefore, the necessity arises to construct an adequate mathematical model of the movement of UAVs. The process of controlling the flight dynamics of a quadcopter must be based on an adequate mathematical model[8]. It should also be noted that among various issues such as the stability and control of UAV movement, the problem of constructing and controlling the mathematical model of quadcopter movement is one of the current problems. Due to being lightweight, the quadcopter, which is a highly maneuverable flying vehicle, has low stability[9]. The control system of the

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[7] Gopalsamy, K. Stability and oscillations in delay differential equations of population dynamics/ K. Gopalsamy.- Netherlands: Dordrecht, -1992, -512 p.

[8] Castillo, P., Lozano R., Dzul, A. Stabilization of a Mini Rotorcraft with Four Rotors// IEEE Control Systems Magazine, december, -vol.25. – 2005. iss.6, –p. 45 – 55.

[9] Aliev, F.A., Mutallimov, M.M., Velieva, N.İ., Huseynova, N.Sh. Mathematical modeling and control of quadcopter motion // Proceedings of the 8th International

quadcopter must perform operations such as ascending to a certain altitude, landing, hovering, and flying along a specified trajectory by carrying out angular and spatial regulation. Thus, the refinement of the mathematical model of quadcopter flight, as well as the problems of its control and regulation, remain current issues.

Another issue of no less importance in the control of UAV motion is the study and development of navigation systems for flying vehicles. The creation of navigation systems for both manned and unmanned aerial vehicles is one of the most complex tasks. It should be noted that one of the methods used to determine the coordinates and other navigation parameters of a flying vehicle is the inertial navigation system (INS), which is employed to determine the navigation parameters of an object in motion relative to the Earth's surface: its position, velocity, and direction.

Simple inertial navigation systems are also used in UAVs. However, long-term autonomous use of such INS may lead to significant errors, as a result of which the required accuracy in determining navigation parameters is not achieved. Therefore, from the perspective of systems analysis, when using an INS it is necessary to integrate it with other channels that serve as sources of navigation data. One such channel is the satellite navigation system (GPS), in which case the INS is considered as part of the GPS/INS navigation complex. In this case, the determination of navigation parameters on the basis of the mathematical model of the vehicle’s motion makes it possible to construct an effective algorithm for quadcopter control synthesis, which remains a relevant issue today.

Thus, the systems approach applied to the solution of the mentioned problems encompasses addressing the above issues and remains a relevant problem in the development of control algorithms for mechanical systems.

Object and subject of the research. The object of the dissertation is the systems approach mathematical models of mechanical systems intended to be controlled, which make it

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possible to develop control methods. The subject of the research is the development and application of new algorithms in the control of such mechanical systems.

Purpose of the work. The aim of the dissertation is to develop control algorithms for mechanical systems through the application of the systems approach, and to apply them to the gas-lift process and the motion of quadcopters.

- Improvement of the mathematical model of the gas-lift process using the systems approach;
- Formulation of the delayed-argument optimal control problem for the operating process using the gas-lift method;
- Development of numerical solution methods for differential equations with delayed arguments;
- Development of a numerical solution method for the continuous optimal control problem with delayed arguments;
- Development of a solution method for the discrete optimal control problem with delayed arguments;
- Development of an optimal regulator for controlling the motion of a quadcopter in space;
- Development of the inertial navigation system of quadcopters;
- Development of optimal controllers and filters for the discrete linear-quadratic Gauss problem in steady-state mode;
- Application of the linear-quadratic Gauss problem to the regulation of a quadcopter’s motion in space.

Research methods. The work uses the theory of systems analysis, methods of the theory of differential equations, methods of optimal control, methods of mathematical modeling, and methods of numerical analysis.

Scientific novelties.

- The mathematical model of the gas-lift process has been improved using the systems approach;

- The delayed-argument optimal control problem describing the operating process using the gas-lift method has been formulated and studied;
- Numerical methods for solving differential equations with delayed arguments have been developed;
- A numerical solution method for the continuous optimal control problem with delayed arguments has been developed;
- A solution method for the discrete optimal control problem with delayed arguments has been developed;
- An optimal regulator for controlling the motion of a quadcopter in space has been developed;
- The inertial navigation system of quadcopters has been developed;
- An algorithm for the development of optimal controllers and filters for the discrete linear-quadratic Gauss problem in steady-state mode has been developed;
- The solution of the linear-quadratic Gauss problem has been applied to the regulation of a quadcopter's motion in space.

Theoretical and practical significance. The results obtained in the dissertation can be used in the solution of the delayed-argument optimal control problem, in the solution of continuous and discrete optimal control problems with delayed arguments, in the improvement of the operating process of oil wells using the gas-lift method, as well as in the development of an optimal regulator for controlling the motion of a quadcopter in space, and in the development of the inertial navigation system of unmanned aerial vehicles.

Reliability of the obtained results. The main results of the dissertation are mathematically well-grounded. The proposed algorithms have been implemented on model examples and, by comparing them with the results of similar works, their advantages and effectiveness have been clearly demonstrated.

Approbation. The results obtained in the dissertation have been presented at the following seminars and conferences:

- The 8th international Conference COIA, 2022, Baku, Azerbaijan;

- Republican Scientific-Practical Conference Dedicated to the 2nd Anniversary of the Victory in the 44-Day Patriotic War, 2022, Baku, Azerbaijan;
- The 7th international Conference COIA, 2020, Baku, Azerbaijan;
- The 6th international Conference COIA, 2018, Baku, Azerbaijan;
- The 5th international Conference COIA, 2015, Baku, Azerbaijan;
- International Conference Dedicated to the 80th Anniversary of Academician Akif Hajiyev, 2017, Baku, Azerbaijan;
- Scientific Seminar of the Institute of Applied Mathematics of BSU;

- Seminar of the “Information Technologies and Systems” Department of Azerbaijan University of Architecture and Construction.

Publications. The main results of the dissertation have been published in **7** articles and **6** theses by the author.

The total volume of the dissertation work, with the volume of each structural section indicated separately: The total volume of the dissertation work is 165,131 characters (title page, table of contents, and introduction – 39,934 characters; Chapter 1 – 35,888 characters; Chapter 2 – 40,974 characters; Chapter 3 – 46,943 characters; conclusion – 1392 characters). The dissertation work consists of an introduction, three chapters, a conclusion, a list of 120 references, and an appendix. The dissertation includes a 143-page text and 13 figures.

MAIN CONTENTS OF THE DISSERTATION

In the introduction, the relevance of the topic has been substantiated, the aim of the work has been defined, the main scientific innovations, the theoretical and practical significance of the work, the main provisions submitted for defense have been stated, and information on the research methods and the approbation of the work has been provided.

The first chapter of the dissertation consists of three subchapters and explains certain aspects of the systems approach to the control of mechanical systems.

In the first subchapter of the first chapter of the dissertation, certain features of the control of mechanical systems have been studied, and the important characteristics of control problems as a system have been examined. Here, the essence of the control problems posed for various mechanical systems has been analyzed in a comparative manner.

In the second subchapter of the first chapter, gas-lift wells have been studied as a mechanical system; the systems approach has been applied to the operating process of oil wells using the gas-lift method, and in several cases their existing mathematical models have been analyzed. Based on these models, various methods for solving posed problems have been demonstrated.

The third subchapter of the first chapter is devoted to the mathematical modeling of the motion of unmanned aerial vehicles as a mechanical system and to methods of their control. The mathematical models in the relevant works have been analyzed, and based on these models, constructive solutions to various problems — optimization and regulation problems — have been described. It has been noted that the construction of the mathematical model of a quadcopter, as an essential part of the process of developing the system for controlling and regulating its motion, is considered one of the most important tasks.

The second chapter of the dissertation is devoted to the application of the systems approach in the control of gas-lift wells.

The first subchapter of the second chapter of the dissertation proposes a new approach to improving the mathematical model of the gas-lift process. The essence of this approach lies in describing the mixing of the injected gas with the oil in the bottom-hole zone and the transfer of the resulting liquid-gas mixture to the lifting pipe with a certain delay, which results in the appearance of a delayed argument term in the differential equations describing the mathematical model. This approach can serve as a means to obtain a more adequate model. It is known that the mathematical model of the operating process of a gas-lift well is described by special derivatives of a system of linear differential equations:

$$\begin{cases} \dfrac{\partial P}{\partial t} = -\dfrac{c^2}{\overline{F}} \dfrac{\partial Q}{\partial x} \\ \dfrac{\partial Q}{\partial t} = -\overline{F} \dfrac{\partial P}{\partial x} - 2aQ \end{cases} \tag{1}$$

where $t \geq 0, x \in [0,2L]$. Then, based on this model, the delayed-argument optimal control problem is formulated. For this purpose, by dividing $l = L/N$ and using the straight-line method, from the above system of differential equations with special derivatives, for N=2, the following system of ordinary differential equations is obtained for the post-pipe (loop) region.

$$\begin{cases} \dot{P}_1(t) = -\dfrac{c_1^{\,2}}{F_1 l} Q_1(t) + \dfrac{c_1^{\,2}}{F_1 l} Q_0(t), \\ \dot{Q}_1(t) = -\dfrac{F_1}{l} P_1(t) + \dfrac{F_1}{l} P_0(t) - 2a_1 Q_1(t), \\ \dot{P}_2(t) = -\dfrac{c_1^{\,2}}{F_1 l} Q_2(t) + \dfrac{c_1^{\,2}}{F_1 l} Q_1(t), \\ \dot{Q}_2(t) = -\dfrac{F_1}{l} P_2(t) + \dfrac{F_1}{l} P_1(t) - 2a_1 Q_2(t). \end{cases} \tag{2}$$

Here, the functions $Q_0(t), P_0(t)$ correspond to the volume flow and pressure of the gas injected into the loop region, by means of which the gas-lift process is controlled. $Q_2(t)$ and $P_2(t)$ correspond to the values at the bottom of the loop region. Unlike other works, we assume here that the pressure and flow rate formed at the bottom of the lifting pipe in the well depend on their respective values at the bottom of the loop region with a certain delay of the τ argument, that is,

$$\begin{aligned} \overline{Q}_2(t) &= \alpha Q_2(t-\tau), \\ \overline{P}_2(t) &= \beta P_2(t-\tau), \end{aligned} \tag{3}$$

relations are satisfied. Taking this into account, for the lifting pipe we get the following equations

$$\begin{cases} \dot{P}_3(t) = -\dfrac{c_2^{\,2}}{F_2 l} Q_3(t) + \dfrac{c_2^{\,2}}{F_2 l} \alpha Q_2(t-\tau), \\ \dot{Q}_3(t) = -\dfrac{F_2}{l} P_3(t) + \dfrac{F_2}{l} \beta P_2(t-\tau) - 2a_2 Q_3(t), \\ \dot{P}_4(t) = -\dfrac{c_2^{\,2}}{F_2 l} Q_4(t) + \dfrac{c_2^{\,2}}{F_2 l} Q_3(t), \\ \dot{Q}_4(t) = -\dfrac{F_2}{l} P_4(t) + \dfrac{F_2}{l} P_3(t) - 2a_2 Q_4(t). \end{cases} \tag{4}$$

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Thus, we obtain the system of differential equations (2), (4) closed with the conditions (3), whose initial conditions can be taken as follows:

$$P_k(t_0) = P_k^0, \quad Q_k(t_0) = Q_k^0 \qquad k = \overline{1,4} \tag{5}$$

Now, if we make the following notations

$$\begin{aligned} x(t) &= \left[P_1(t), Q_1(t), P_2(t), Q_2(t), P_3(t), Q_3(t), P_4(t), Q_4(t)\right]', \\ x^0 &= \left[P_1^0, Q_1^0, P_2^0, Q_2^0, P_3^0, Q_3^0, P_4^0, Q_4^0\right]', \\ u(t) &= \left[Q_0(t), P_0(t)\right]' \end{aligned}$$

from expressions (2), (4), and (5) we obtain in a compact form the

$$\dot{x}(t) = Ax(t) + B(\alpha, \beta)x(t-\tau) + Gu(t), \tag{6}$$

$$x(t) = x^0(t), \ t \in [-\tau, 0] \tag{7}$$

problem. Here A, B, G are matrices formed from the coefficients of equations (2) and (4), and , $u(t)$ is the control function.

$$J = \frac{1}{2}\left[Q_4(T) - \overline{Q}\right]^2 + \frac{1}{2}\int_0^T \{x'(t)Rx(t) + u'(t)Cu(t)\}dt \tag{8}$$

If we require the minimization of the functional we obtain the delayed-argument linear–quadratic optimal control problem. Here

$R' = R > 0, C' = C > 0$. After certain transformations, the functional (8) can be brought to the standard form (9)

$$J = \frac{1}{2}\left(x(T) - \bar{x}\right)' N\left(x(T) - \bar{x}\right) + \frac{1}{2}\int_0^T \{x'(t)Rx(t) + u'(t)Cu(t)\}dt \qquad (9)$$

For the problem (6), (7), (9), if we construct the extended functional and set its gradient equal to zero

$$x'(t)=Ax(t)+Bx(t-\tau)-GC^{-1}G'\lambda(t) \qquad (10)$$

$$\lambda'(t)=-Rx(t)-A'\lambda(t)-B'\lambda(t+\tau) \qquad (11)$$

$$x(0)=x^0 \qquad (12)$$

$$\lambda(T)=N'\left[x(T)-\bar{x}\right] \qquad (13)$$

we obtain the system of Euler–Lagrange equations. It should be noted that in these equations the parameter τ and the coefficients α,β are unknown, and they can be determined by identification methods.

In the second subchapter of the second chapter of the dissertation, the identification problem for the system of delayed-argument differential equations (6)-(7) is considered; that is, an identification problem based on a mathematical model constructed for the gas-lift process. Based on the mathematical model built under the assumption that the pressure and flow rate formed at the bottom of the lifting pipe depend on the pressure and flow rate formed at the bottom of the loop region with a certain time delay and parameters, a method for the identification of the delay and parameters is proposed.

Here, at an arbitrary value of the parameter N, that is, at $N = n$, the straight-line method is applied, and a system corresponding to the system (6)-(7) is obtained. In this case, it is defined as

$$x(t) = \left[P_1(t), Q_1(t), P_2(t), Q_2(t), \ldots, P_n(t), Q_n(t), P_{n+1}(t), Q_{n+1}(t), \ldots, P_{2n}(t), Q_{2n}(t)\right]',$$

$$x^0 = \left[P_1^0, Q_1^0, P_2^0, Q_2^0, \ldots, P_n^0, Q_n^0, P_{n+1}^0, Q_{n+1}^0, \ldots, P_{2n}^0, Q_{2n}^0,\right]',$$

By decomposing the system (6)-(7), we obtain the

$$\begin{cases} x_1'(t) = A_1 x_1(t) + B_1 u(t) \\ x_2'(t) = A_1 x_2(t) + B_1 x_1(t) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n'(t) = A_1 x_n(t) + B_1 x_{n-1}(t) \\ x_{n+1}'(t) = A_2 x_{n+1}(t) + V(\alpha,\beta)\, x_n(t-\tau) \\ x_{n+2}'(t) = A_2 x_{n+2}(t) + B_2 x_{n+1}(t) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_{2n}'(t) = A_2 x_{2n}(t) + B_2 x_{2n-1}(t) \end{cases} \tag{14}$$

$$x_i(0) = x_i{}^0, i = \overline{1,n}, i = \overline{n+2,2n} \tag{15}$$

$$x_{n+1}(t) = x_{n+1}^0(t), t \in [-\tau, 0] \tag{16}$$

system, where $x_i(t) = \begin{bmatrix} P_i(t) \\ Q_i(t) \end{bmatrix}, x_i^0 = \begin{bmatrix} P_i^0 \\ Q_i^0 \end{bmatrix}, i = \overline{1,2n}$ and $A_1, A_2, B_1, B_2, V(\alpha,\beta)$ matrices depend on the $F, c, a, l, \alpha, \beta$ parameters. The solution of the Cauchy problem (14)–(16) depends on the parameters α, β and τ as well as the control u(t) and it can be expressed as follows:

$$x(t) = x[\alpha, \beta, \tau, u(t), t]$$

Then, since the $x_{2n}(t)$ coordinate of this solution is in the form $x_{2n}(t) = \lfloor P_{2n}(t), Q_{2n}(t) \rfloor$ we obtain

$$Q_{2n}(\alpha,\beta,\tau,u,t) = \mathrm{I} \cdot x_{2n}(t), \qquad \mathrm{I} = [0\ 1]$$

Now, let us assume that for a certain time moment from the history of the gas-lift well, the statistical data $\overline{Q}_0^i$ and $\overline{Q}_{2n}^i$ (i=1, k. k – number of observations) are known. $\overline{Q}_0^i$ - is the volume of gas injected at the inlet of the gas-lift well, $\overline{Q}_{2n}^i$- is the flow rate. Now, let us assume that from the solution of the problem (14)-(16) within the statistical data Q_0^i we can find the value of

$$Q_{2n}^i(\alpha, \beta, \tau, Q_0^i, T)$$

Then, the problem of finding the parameters α, β and τ is reduced to the problem of finding the minimum of the

$$\mathrm{I}(\alpha,\beta,\tau)=\sum_{i=1}^{k}\left[Q_{2n}^{i}(\alpha,\beta,\tau,Q_0^i,T)-\overline{Q}_{2n}^{i}\right]^2$$

functional. In the subchapter, both the algorithm for solving the problem (14)-(16) and the algorithm for finding the minimum of the functional $\mathrm{I}(\alpha,\beta,\tau)$ are presented. The considered problem is also solved for the discrete case.

Algorithm.

Step 1. C, F, a, L constants and $Q_0(t)$ and $P_0(t)$ functions are introduced.

$$\begin{cases} \dfrac{\partial P}{\partial t}=-\dfrac{c^2}{\bar{F}}\dfrac{\partial Q}{\partial x} \\ \dfrac{\partial Q}{\partial t}=-\bar{F}\dfrac{\partial P}{\partial x}-2aQ \end{cases}$$

$$Q(L+0,t)=\alpha Q(L-0,t-\tau),$$
$$P(L+0,t)=\beta P(L-0,t-\tau).$$
$$\bar{P}_n(t)=\beta P_n(t-\tau)$$
$$\bar{Q}_n(t)=\alpha Q_n(t-\tau)$$

Step 2. ${Q_0}^i$and $Q_{2n}^i(i=\overline{1,k})$ statistical data are introduced.

Step 3. The following system of equations is solved, and the quantity $Q_{2n}^i(\alpha,\beta,\tau,Q_0^i,T)$ is determined.

$$\begin{cases} \acute{x}_1(t)=A_1x_1(t)+B_1u(t) \\ \acute{x}_2(t)=A_1x_2(t)+B_1x_1(t) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \acute{x}_n(t)=A_1x_n(t)+B_1x_{n-1}(t) \\ \acute{x}_{n+1}(t)=A_2x_{n+1}(t)+V(\alpha,\beta)x_n(t-\tau) \\ \acute{x}_{n+2}(t)=A_2x_{n+2}(t)+B_2x_{n+1}(t) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \acute{x}_{2n}(t)=A_2x_{2n}(t)+B_2x_{2n-1}(t) \end{cases}$$

$$x_i(0)={x_i}^0, i=\overline{1,n}, i=\overline{n+2,2n}$$
$$x_{n+1}(t)=x_{n+1}^0(t), t\in[-\tau,0]$$

Step 4. The following functional is formulated.

$$I(\alpha,\beta,\tau)=\sum_{s=1}^{k}\left[Q_{2n}^{s}(\alpha,\beta,\tau,T)-Q_{2n}^{s,st}\right]^2$$

Step 5. The parameters α, β and τ are determined

$$\frac{\partial I(\alpha,\beta,\tau)}{\partial \beta} \approx \frac{I(\alpha,\beta+h_\beta,\tau) - I(\alpha,\beta,\tau)}{h_\beta}$$

$$\frac{\partial I(\alpha,\beta,\tau)}{\partial \tau} \approx \frac{I(\alpha,\beta,\tau+h_\tau) - I(\alpha,\beta,\tau)}{h_\tau}$$

$$\frac{\partial I(\alpha,\beta,\tau)}{\partial \alpha} = 0$$

$$\frac{\partial I(\alpha,\beta,\tau)}{\partial \beta} = 0$$

$$\frac{\partial I(\alpha,\beta,\tau)}{\partial \tau} = 0$$

Step 6. For a sufficiently small number $\varepsilon > 0$, if the

$$\left|\frac{\partial I(\alpha,\beta,\tau)}{\partial \alpha}\right| < \varepsilon, \left|\frac{\partial I(\alpha,\beta,\tau)}{\partial \beta}\right| < \varepsilon, \left|\frac{\partial I(\alpha,\beta,\tau)}{\partial \tau}\right| < \varepsilon$$

conditions are satisfied, the calculation is stopped, otherwise, the steps $h_\alpha, h_\beta, h_\tau$ are reduced, and the 3rd step is repeated.

Thus, the parameters α, β and τ that make it possible to adequately reflect the history of the gas-lift well are determined, thereby constructing a mathematical model that is adequate for the process.

In the third subchapter of the second chapter, a method is proposed for solving the linear–quadratic optimal control problem. Here, after certain notations and substitutions, the following delayed- and advanced-argument differential equation is obtained in a compact form.

If we make the $z(t) = [x(t), \lambda(t)]^T$ notation then the system (10)–(11) can be written in the form of

$$\begin{bmatrix} x'(t) \\ \lambda'(t) \end{bmatrix} = \begin{bmatrix} A & -Gc^{-1}G^T \\ -R & -A' \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t-\tau) \\ \lambda(t-\tau) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -B' \end{bmatrix} \begin{bmatrix} x(t+\tau) \\ \lambda(t+\tau) \end{bmatrix} \quad (17)$$

and after making certain notations, we obtain in a compact form the following delayed- and advanced-argument

$$\dot{z}(t) = \bar{A}z(t) + \bar{B}z(t-\tau) + \bar{C}z(t+\tau) \quad (18)$$

differential equation. As the solution of system (18), we will look for a function $z(t)$ defined on the $[t_0-\tau, T+\tau]$ segment such that it behaves as if it is differentiable everywhere on the $[t_0, T]$ segment, satisfies the differential equation (18) and the boundary conditions of

$$\begin{cases} z(t) = \varphi(t), & t_0 - \tau \le t \le t_0 \\ z(t) = \psi(t), & T \le t \le T + \tau \end{cases} \tag{19}$$

Let us denote by Z the set of piecewise-continuous vector functions defined on the $[t_0-\tau, T+\tau]$ segment, continuous on the $[t_0, T]$ segment, and having first-kind discontinuities only at the points $t = t_0$ and $t = \tau$. Let us introduce the metric

$$\rho_k(z_1, z_2) = \sup_t \left[e^{-kt} \| z_1(t) - z_2(t) \| | x_0 - \tau \le t < T + \tau \right] \tag{20}$$

in such a set Z for a number $k > 0$. Let us take the subset of the set Z as

$$\Omega = \{ z \in Z | z(t) = \varphi(t), t \in [t_0 - \tau, t_0), x(t) = \psi(t), t \le (T, T + +\tau] \} \tag{21}$$

Now let us consider the $T\Omega \to \Omega$ operator defined by the following equalities:

$$Tz(t) = \varphi(t), \quad t_0 - \tau \le t \le t_0,$$

$$Tz(t) = \psi(t), \quad T \le t \le T + \tau, \tag{22}$$

$$Tz(t) = z_0 + \int_{t_0}^{t} [Az(\xi) + Bz(\xi - \tau) + Cz(\xi + \tau)] d\xi, \quad t_0 \le t \le T.$$

Then, for any two elements z_1 and z_2 from Ω, it can be proved that

$$\rho_k(Tz_1, Tz_2) = \max_{t_0 \le t \le T} \left[e^{-kt} \| Tz_1(t) - Tz_2(t) \| \right] \le \delta_k \rho_k(z_1, z_2) \tag{23}$$

It can be shown that, for a certain value of k, $\delta_k < 1$ is satisfied. Then, according to relation (23), the operator T defined by (22) is a contraction operator in the set Ω. Since Ω is a complete metric space, the operator T has a unique fixed point [28, p. 543]. This fixed point z(t) will be the unique solution of the boundary value problem (18), (19).

(Let us use an approximate method for solving the boundary value problem (18), (19). To discretize the problem, let us use the explicit Euler scheme. Here, for simplicity, let us assume that there exists a natural number $l \in N$ such that $\frac{T-t_0}{\tau}$. On the other hand, let us divide the number τ into $m \in N$ equal parts and denote it as: $\Delta t = \frac{\tau}{m}$. Then, by dividing the $[t_0 - \tau, T + \tau]$ segment with a step of Δt we obtain the following grid.

$$t_i = t_0 + \Delta t \cdot i, i = \overline{-m,\ p + m} \tag{24}$$

Using this grid and the Δt step in the Euler scheme, we obtain the following finite difference equation from (18), (19):

$$\begin{cases} u^i = \varphi^i, i = -m, \ldots, -1 \\ u^{i+1} = u^i + \Delta t \cdot \left[Au^i + Bu^{i-m} + cu^{i+m}\right], i = 0, \ldots, p-1 \\ u^i = \psi^i, i = p+1, \ldots, p+m \end{cases} \tag{25}$$

To find the unknowns $u^i(i = \overline{-m, p+m})$ the obtained equation (25) is a $p+2m+1$ -dimensional system of linear algebraic equations and is the approximate solution of the boundary value problem $u^i \approx z^i = z(t_i)$ (18), (19).

The third chapter of the dissertation is devoted to the control of the motion of unmanned aerial vehicles and the design of optimal regulators for them.

In the first subsection of this chapter, the design of optimal regulators for controlling the motion of quadcopters in space has been carried out. For this purpose, first, the known mathematical model was linearized and improved, and based on it, a linear-quadratic optimal control problem was formulated. Then, based on the linearized mathematical model of the quadcopter's dynamics, optimal regulators were designed to control the motion of the quadcopter in space when it is simultaneously lifted to a given height in the horizontal plane under a given roll angle.

Let us consider a quadcopter with given physical parameters (Fig. 1), which can be controlled by changing the rotational speed of its propellers. The spacecraft moves relative to a fixed inertial reference system given by the Ox, Oy, and Oz coordinate axes, which are perpendicular to each other and related to the Earth, and the Oz axis is directed opposite to the gravity vector. The task in this work is that the quadcopter moves from the initial point (x_0, y_0, z_0) with initial angles $(\theta_0, \varphi_0, \psi_0)$ to the given point $(0,0,z_d)$ with the angle $(0,0,\psi_d)$, which means raising the quadcopter to the given altitude and rotating along the course angle. Experimental verification of this assumption has shown that the error of such an approximation is quite small and can be neglected for practical applications

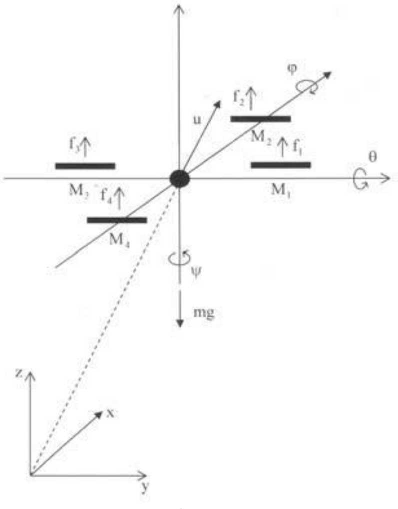

Figure 1. Scheme of the quadcopter

It is known that the motion of the quadcopter is described by the following equations:

$$m\ddot{x} = -u\sin\theta, \tag{26}$$

$$m\ddot{y} = u\cos\theta\sin\varphi, \tag{27}$$

$$m\ddot{z} = u\cos\theta\cos\varphi - mg, \tag{28}$$

$$\ddot{\psi} = \tilde{\tau}_\psi, \tag{29}$$

$$\ddot{\theta} = \tilde{\tau}_\theta, \tag{30}$$

$$\ddot{\phi} = \tilde{\tau}_\varphi. \tag{31}$$

In equalities (26)-(31) m is the mass of the quadcopter, $g = 9{,}8$ m/s^2 is the acceleration due to gravity, and, u, as well as $\tilde{\tau}_\psi$, $\tilde{\tau}_\theta$, $\tilde{\tau}_\varphi$ are the control functions depending on f_i. Taking into account that u, is used for controlling the ascent of the vehicle, $\tilde{\tau}_\psi$ control allows adjustment of the course angle.

Now, we adopt the assumption $\cos\theta\cos\varphi \neq 0$, and accordingly define the u controller, which determines the vehicle's ascent altitude, by the following relation:

$$u = (r_1 + mg)\frac{1}{\cos\theta\cos\varphi} \tag{32}$$

In this case for r_1 we obtain the following

$$r_1 = a_1\dot{z} + a_2 z + a_3 \tag{33}$$

In formula (28), taking into account relations (32) and (33):

$$m\ddot{z} = a_1\dot{z} + a_2 z + a_3.$$

By the same principle, to control the course angle we obtain the following

$$\tilde{\tau}_\psi = b_1\dot{\psi} + b_2\psi + b_3 \tag{34}$$

Then, taking condition $\cos\theta\cos\varphi \neq 0$ into account along with (32)–(34), we obtain it from formulas (28) and (29):

$$\ddot{z} = \frac{1}{m}(a_1\dot{z} + a_2 z + a_3), \tag{35}$$

$$\ddot{\psi} = b_1\dot{\psi} + b_2\psi + b_3. \tag{36}$$

It should be noted that the unknown coefficients a_1, a_2, a_3 and b_1, b_2, b_3 in these equations must be chosen according to the asymptotic stability condition in the vertical direction and in the course angle, which in turn will ensure the fulfillment of the relation $z \to z_d$, $\psi \to \psi_d$ for the given z_d, ψ_d.

It is known from the theory of differential equations that for the differential equation (35), the roots of the characteristic equation

$$k^2 - \frac{a_1}{m}k - \frac{a_2}{m} = 0 \tag{37}$$

lie in the left half-plane of the complex plane if the coefficients a_1, a_2 are negative, meaning $\operatorname{Re} k_i < 0, i = 1,2$. Thus, since the function $z = -\frac{a_3}{a_2}$ f is a particular solution of the non-homogeneous equation (35), the function

$$z = c_1 e^{k_1 t} + c_2 e^{k_2 t} - \frac{a_3}{a_2}, \tag{38}$$

is the general solution of this equation, and under the condition $t \to \infty$, the relation $z(t) \to -\frac{a_3}{a_2}$ is also satisfied. If we denote $a_1 = -a_{z_1}, a_2 = -a_{z_2}$ and $-\frac{a_3}{a_2} = z_d$ then from formula (33), we obtain the following relation for r_1:

$$r_1 = a_1\dot{z} + a_2 z + a_3 = -a_{z_1}\dot{z} - a_{z_1} z + a_{z_1} z_d = -a_{z_1}\dot{z} - a_{z_1}(z - z_d) \tag{39}$$

In this case, (35) will transform into the following equation:

$$\ddot{z} = \frac{1}{m}\left(-a_{z_1}\dot{z} - a_{z_2}(z - z_d)\right). \tag{40}$$

Similarly, from equation (36) we obtain equation

$$\ddot{\psi} = -a_{\psi_1}\dot{\psi} - a_{\psi_2}(\psi - \psi_d) \tag{41}$$

Let us note that for positive values of the coefficients a_{ψ_1}, a_{ψ_2}, a_{z_1}, a_{z_2} in equations (40) and (41), the asymptotic stability condition of these systems is ensured, which in turn will guarantee the fulfillment of the $\psi \to \psi_d$, $z \to z_d$ relations.

It should be noted here that by applying the vertical motion regulation algorithm we will achieve the same result. That is, let us construct the

$$p_1 = [(z - z_d), \frac{d(z - z_d)}{dt}]'$$

vector. Then applying the

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, u_1 = \frac{r_1}{m} \tag{42}$$

notations, we obtain from formulas (39) and (40):

$$\begin{aligned} \dot{p}_1 &= A_1 p_1 + B_1 u_1 \\ p_1(0) &= p_1^0 \end{aligned}, \tag{43}$$

In this case, let us use the

$$J_1 = \int_0^\infty \left(p_1' Q_1 p_1 + u_1' R_1 u_1\right) dt, \tag{44}$$

quality criterion. Let us assume that the matrices Q_1 and R_1 ensure the solution of the synthesis problem controlling the motion of the quadcopter along the z -axis. Then, if we apply the standard LQR synthesis procedure, for the state feedback gains we have $u_1 = -K_z p_1$ or $K_z = \frac{1}{m}[a_{z_1}, a_{z_2}]$ or $u_1 = \frac{r_1}{m} = \frac{1}{m}[-a_{z_1}\dot{z} - a_{z_2}(z - z_d)]$, in this case under the condition $t \to \infty$, $z \to z_d$ is ensured.

The synthesis problem for controlling the course angle is solved according to this rule. By using this procedure, we can carry out the control of both the (φ, y) and (θ, x) coordinates. Thus, as a result of the synthesis of the quadcopter's control, we obtain such $x(t), y(t), z(t)$ and $\theta(t), \varphi(t), \psi(t)$ solutions that under the condition $t \to \infty$ for them, respectively, $\theta \to 0,\ x \to 0,\ y \to 0,\ \varphi \to 0, \psi \to \psi_d$, $z \to z_d$ is ensured.

The second subsection of the third chapter of the dissertation is devoted to the development of the inertial navigation system of quadcopters and the creation of its correction algorithm in integration with GPS, magnetometer, and altimeter devices.

Let us assume that the accelerometer device measures the acceleration $w_a(t)$ of the flying apparatus, and the gyroscope device provides the values of the Euler angles $\psi(t), \vartheta(t), \varphi(t)$. In this case, the absolute acceleration is calculated using the following formula:

$$w(t) = w_a(t) - g \tag{45}$$

Here g- is the acceleration due to gravity. he projection of the angular velocity vector of the flying apparatus relative to the UAV coordinate axes $\omega(t) = [\omega_1(t), \omega_2(t), \omega_3(t)]^T$, will be in the following form:

$$\begin{cases} \omega_1(t) = \dot{\psi}(t) sin\vartheta(t) sin\varphi(t) + \dot{\vartheta}(t)\cos\varphi(t), \\ \omega_2(t) = \psi(t) sin\vartheta(t) cos\varphi(t) + \dot{\vartheta}(t)\sin\varphi(t), \\ \omega_3(t) = \psi(t) cos\theta(t) + \dot{\varphi}(t). \end{cases} \tag{46}$$

If the initial state of the UAV is known, then when the value of the function $\omega(t)$ is measured, the Rodrigues–Hamilton parameters vector $\lambda(t) = [\lambda_0(t), \lambda_1(t), \lambda_2(t), \lambda_3(t)]^T$ (quaternion) is found from the equation

$$\dot{\lambda} = \frac{1}{2\Omega\lambda} \tag{47}$$

Here

$$\Omega = \begin{bmatrix} 0 - \omega_1 - \omega_2 - \omega_3 \\ \omega_1 \quad 0 \quad \omega_3 \quad -\omega_2 \\ \omega_2 - \quad \omega_3 \quad 0 \quad \omega_1 \\ \omega_3 \quad \omega_2 \quad -\omega_1 \quad 0 \end{bmatrix}, \ \|\lambda\|^2 = \lambda^T\lambda\,, \tag{48}$$

The initial condition is as shown below

$$\lambda(t_0) = \lambda^0 = [\lambda_0^0 \lambda_1^0 \lambda_2^0 \lambda\ {}_3^0]^T \tag{49}$$

The equation for calculating the change in the velocity coordinates of the UAV is taken in the form

$$\frac{dv}{dt} = w(t) - 2\Omega_z \times v - \Omega_z \times \Omega_z \times R \tag{50}$$

where ν s the relative velocity of the object, Ω_z s the angular velocity of the Earth's rotation, and r is the radius vector of the point in the geocentric coordinate system. To reproject the accelerometer’s readings from the inertial coordinate system relative to the Earth $(Oxyz)$ to the moving coordinate system of the UAV, we find the transformation matrix A using the coordinates of the vector of the Rodrig-Hamilton parameters:

$$A(\lambda) = \begin{bmatrix} \lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2 & 2(\lambda_1\lambda_2 + \lambda_0\lambda_3) & 2(\lambda_1\lambda_3 + \lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2 + \lambda_0\lambda_3) & \lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2 & 2(\lambda_2\lambda_3 + \lambda_0\lambda_1) \\ 2(\lambda_1\lambda_3 + \lambda_0\lambda_2) & 2(\lambda_2\lambda_3 + \lambda_0\lambda_1) & \lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2 \end{bmatrix} \qquad (51)$$

The matrix (51) is also called the cosine matrix. Then in formula (50), instead of the function $w(t)$ we will use the function $\widetilde{w}(t)$ defined by the formula

$$\widetilde{w}(t) = A\big(\lambda(t)\big)\, w(t) \qquad (52)$$

As noted earlier, if the INS Inertial Navigation System operates for a long time without correction, it leads to large errors. Therefore, to correct the navigation parameters obtained with the help of the INS, it is necessary to integrate it with other sources of navigation data. One such channel can be the use of the GPS satellite navigation system.

As seen from the scheme above, the problem of determining the navigation parameters using the INS (Inertial Navigation System) is related to solving the system of differential equations (50). On the other hand, measurements from the gyroscope and accelerometer are taken in discrete form, i.e., after equal time intervals Δt. Therefore, it is reasonable to calculate the sought navigation parameters after each Δt time interval. At this point, after determining the $\omega(t)$ vector according to the gyroscope’s measurements, we can calculate the quaternion increments $\nabla\theta_i$ using the formula

$$\nabla\theta_i = \int_{t_i-1}^{t_i} \omega(t)dt \approx \frac{\Delta t}{12}[5\omega(t_i) + 8\omega(t_{i-1}) - \omega(t_{i-1})] (53)$$

After that, using formula (48), the Ω matrix is determined, and over the time interval Δt, the elemental quaternions $\delta\lambda(t_i)$ can be found as the solution of equation (47) with the initial condition $\delta\lambda(t_0) = [1\ 0\ 0\ 0\,]^T$

$$\delta\lambda(t_i) = \begin{bmatrix} 1 - \frac{1}{12}\|\nabla\theta_i\|^2 \\ \frac{1}{2}\nabla\theta_i - \frac{1}{24}(\nabla\theta_i \times \nabla\theta_{i-1}) \end{bmatrix} \tag{54}$$

The Rodriquez–Hamilton parameters will be defined as follows:

$$\lambda(t_i) = \begin{bmatrix} \lambda_0(t_i) \\ \lambda_1(t_i) \\ \lambda_2(t_i) \\ \lambda_3(t_i) \end{bmatrix} = \begin{bmatrix} \delta\lambda_0(t_i) - \delta\lambda_1(t_i) & -\delta\lambda_2(t_i) - \delta\lambda_3(t_i) \\ \delta\lambda_1(t_i)\delta\lambda_0(t_i) \;\; \delta\lambda_3(t_i) & -\delta\lambda_2(t_i) \\ \delta\lambda_2(t_i) \;\; -\delta\lambda_3(t_i) & \delta\lambda_0(t_i)\delta\lambda_1(t_i) \\ \delta\lambda_3(t_i) \;\; \delta\lambda_2(t_i) & -\delta\lambda_1(t_i) \;\; \delta\lambda_0(t_i) \end{bmatrix} \begin{bmatrix} \lambda_0(t_{i-1}) \\ \lambda_1(t_{i-1}) \\ \lambda_2(t_{i-1}) \\ \lambda_3(t_{i-1}) \end{bmatrix} \tag{55}$$

Now let us examine the determination of the navigation parameters of the flying vehicle - its velocity $v(t)$ and coordinates $r(t)$. First, as noted earlier, based on the accelerometer readings and the cosine matrix (51), let us define the function $\widetilde{w}(t)$ using formulas (45) and (52). Then, using the values of the function $\widetilde{\omega}(t)$ at the moment t_{i-2}, t_{i-1}, t_i we obtain the following relations for determining the navigation parameters

$$v(t_i) = [5w(t_i) + 8\widetilde{w}(t_{i-1}) - \widetilde{w}(t_{i-2})]\frac{\Delta t}{12} + v(t_{i-1}) \tag{56}$$

$$r(t_i) = [3\widetilde{w}(t_i) + 10\widetilde{w}(t_{i-1}) - \widetilde{w}(t_{i-2})]\frac{\Delta t^2}{24} + \Delta t \cdot v(t_{i-1}) + r(t_{i-1}) \tag{57}$$

Let $\mu, \delta v, \delta r$ denote the error vectors of the INS, where μ is the error vector of the calculation of the object's angular velocity, $\delta v, \delta r$- are the error vectors of the object's velocity and coordinates, and, δc is the systematic error vector of the gyroscope.

If $w = [w_1, w_2, w_3]^T$ is the absolute acceleration vector and A is the direction cosine matrix obtained from formula (51), then the error equation of the INS will have the following form:

$$\dot{x} = Fx + n, \tag{58}$$

Here

$$x = \begin{bmatrix} \mu \\ \delta v \\ \delta r \\ \delta c \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 0 & 0 & A \\ c & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \tag{59}$$

n vector white noise. 0 – is the zero matrix, I is the identity matrix of the corresponding dimension. Here, it is clear that the vectors x and n are 12-dimensional.

The discrete analogue of equation (58) will take the form

$$x_{k+1} = \Phi_k x_k + n_k \tag{60}$$

Here it take the form

$$\Phi_k = I + F\Delta t + \frac{(\Delta t)^2}{2} F^2 = \begin{bmatrix} I & 0 & 0 & A^T \Delta t \\ c\Delta t & I & 0 & c\,A^T (\frac{\Delta t}{2})^2 \\ c\left(\frac{\Delta t}{2}\right)^2 & I\Delta t & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \tag{61}$$

n_k is the random error vector of the INS operation. Here, the lower index k corresponds to the time interval $k\Delta t$. Suppose that at the k-nth step of the INS operation, GPS information from the satellite is received, i.e., the observation

$$z_k = Hx_k + \xi_k \tag{62}$$

Is performed, where $H = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix}$, ξ_k is the measurement error.

This shows that, by using relation (62), the INS correction problem can be regarded as an optimal filtering problem. In that case, the solution of this optimal filtering problem (the Kalman filter) will take the following form:

$$\hat{x}_k = \bar{x}_k + K_k(z_k - H\bar{x}_k); \quad \bar{x}_{k+1} = \Phi_k \hat{x}_k, \tag{63}$$

where K_k matrix will be determined by means of the following relation.

$$K_k = M_k H^T (HM_k H^T + R_k)^{-1}, \tag{64}$$

$$M_{k+1} = \Phi_k S_k \Phi_k^T + Q_k^T, \tag{65}$$

$$S_k = M_k - K_k(HM_k H^T + R_k)K_k^T \tag{66}$$

Here the matrices $Q_{k,} R_{k,}$ used in formulas (64)–(66) are the covariance matrices of the noises u_k, ξ_k. The matrix M_0 is given.

Thus, by solving the filter equation (63), we determine the vector $\bar{x}_k$, whose first 9 components are the estimates of the error vectors $\mu_k, \delta v_k, \delta r_k$. From this, the corrected navigation parameters of the PUA are found by the formulas:

$$\omega_{kor}(t_k) = \omega(t_k) - \mu_k, \tag{67}$$

$$v_{\ kor}(t_k) = v(t_k) - \delta v_k, \tag{68}$$

$$r_{kor}(t_k) = r(t_k) - \delta r_k \tag{69}$$

The matrix H is of size 9×12 and is found as follows:

$$H = \left[H_{9,9};\ O_{9,3}\right], \qquad H_{9,9} = \begin{bmatrix} O_{8,1} & I_{8,8} \\ O_{8,1}^{T} & 1 \end{bmatrix} \tag{70}$$

Here $O_{9,3}$ is a 9×3 zero matrix, $O_{8,1}$ is an 8×1 zero matrix, and $I_{8,8}$ is 8×8 identity matrix.

In addition to GPS signals, the issue of correcting the INS operation using the device readings of the magnetometer and the altimeter barometer has also been solved in an appropriate manner.

Algorithm.

1. The flight time T and the time interval Δt are entered.
2. $M = \frac{T}{\Delta t}$ is calculated. It is accepted that $t_0 = 0$. $k = 0$ is taken.
3. The value of the accelerometer $W_a(t_k)$ is taken.
4. The absolute acceleration is calculated using formula (45).
5. he gyroscope readings are taken and $\psi(t), \vartheta(t), \varphi(t)$ Euler angles are determined.
6. The angular velocity vector $\omega(t_k)$ is calculated using formula (46).
7. $n_c = [\tau_1, \tau_2, \tau_3]^T$systematic error vector and $n_\omega = [\tau_1^\omega, \tau_2^\omega, \tau_3^\omega]^T$ random error vector are given.
8. The $\text{quasi} - \text{coordinates}\ \Delta\theta_k$ are calculated using formula (58).
9. The elemental quaternions are found using formula (54)
10. The Rodrigues–Hamilton parameters are calculated using formula (55).
11. The cosine matrix $A(\lambda(t_k))$ is found using formula (51)
12. The function $\widetilde{w}(t)$ is determined using formula (52).
13. The navigation parameters $v(t_k), r(t_k)$ are found using formulas (55) and (57).
14. H matrix is determined using formula (70).
15. The covariance matrix M_0 is given
16. The matrices Q_k, R_k are defined as the covariance matrices of the noises n_k, ξ_k n formulas (60) and (62).
17. The matrix K_k is found using formula (64).
18. The vector $\hat{x}_k$ and the estimate $\bar{x}_{k+1}$ are found using formula (63).

19. The matrix S_k is found from formula (66).
20. The matrix M_{k+1} s found from formula (65).
21. The values of the navigation parameters are corrected using formulas (67), (68), and (69)
22. $k = k + 1$ is assigned a new value. When $k > M$ the process stops; otherwise, steps 3–21 are repeated.

As a result of the algorithm's operation, at values $t_k = t_0 + \Delta t \cdot k$ the corrected navigation parameters $\omega_{kor}(t_k), v_{kor}(t_k,)$ and $r_{kor}(t_k,)$ are found.

In the third subsection of this chapter, the problem of constructing optimal controllers and filters for the discrete linear-quadratic Gauss (DLQG) problem in steady-state regime is considered. To determine the control action, two problems are solved - the linear-quadratic deterministic problem and the linear-quadratic Gauss estimation obtained with the Kalman-Bucy filter. In both cases, the coefficients of the deterministic controller and the filter are determined using the positive-definite solutions of the corresponding algebraic Riccati equations.

Let us consider the following discrete linear–quadratic Gauss problem. The description of the object's motion equation is

$$x_{i+1} = \Phi x_i + \Gamma u_i + \omega_i, \tag{71}$$

and at time i, measurements z_i linearly related to the state of the trajectory x_i are performed:

$$z_i = Hx_i + v_i \tag{72}$$

Here ω_i - is the vector of random external errors, and υ_i - is the vector of random measurement errors accepted as a Gaussian random quantity of the “white noise” type. Furthermore, let us assume that for the random variables x_0, ω_i and υ_i the mathematical expectations are

$$E(\omega_i) = E(v_i) = E(x_0) = 0 \tag{73}$$

and the correlation matrices are defined as follows:

$$E\left\{\begin{bmatrix}\omega_i \\ v_i\end{bmatrix}\begin{bmatrix}\omega_j^T v_j^T\end{bmatrix}\right\} = \begin{bmatrix}Q & 0 \\ 0 & R\end{bmatrix}\delta_{ij} \tag{74}$$

$$E\{x_0 x^T\} = P_0,\ E\{x_0\, v_j^T\} = E\{x_0 \omega_j^T\} = 0 \tag{75}$$

it is required to find the controller u_i as a function of the observation z_i so that,

$$J = \frac{1}{2}\sum_{i=1}^{\infty}\left\{\left[x_i^T\, u_i^T\right]\begin{bmatrix} A & N \\ N & B \end{bmatrix}\begin{bmatrix} x_i \\ u_i \end{bmatrix}\right\} \tag{76}$$

minimizes the quadratic functional.

In the solution of the discrete linear–quadratic Gauss (DLQG) problem (62)–(67), let us look for the controller in the form

$$u_i = -C\widehat{x}_i \tag{77}$$

Where the matrix

$$C = (\Gamma^T S \Gamma + B)^{-1}(\Gamma^T S \Phi + N^T) \tag{78}$$

$$S = \Phi^T S \Phi - C^T(B + \Gamma^T S \Gamma)C + A \tag{79}$$

is the solution of the system of algebraic matrix Riccati equations. Next, let us assume that the following

$$K = MH^T(HMH^T + R)^{-1} \tag{80}$$

$$M = \Phi P \Phi^T + Q \tag{81}$$

$$P = M - K(HMH^T + R)K^T \tag{82}$$

is the solution of the system of matrix equations. If we substitute (80)–(82), we get

$$P = M - MH^T(HMH^T + R)^{-1}(HMH^T + R)(HMH^T + R)^{-1}HM^T =$$

$$= M - MH^T(HMH^T + R)^{-1}HM^T$$

Then, substituting this expression for P into equation (81), we obtain the following Riccati matrix equation for M :

$$M = \Phi[M - MH^T(HMH^T + R)^{-1}HM^T]\Phi^T + Q$$

or

$$M = \Phi M \Phi^T - \Phi M H^T(HMH^T + R)^{-1}HM^T\Phi^T + Q \tag{83}$$

After that, equations (79) and (83) are solved to determine S and M , and then C and K are determined. After u_i is determined, using the Kalman–Bucy filter we can obtain the estimate of $\hat{x}_i$ as follows:

$$\hat{x}_i = \bar{x}_i + K(z_i - H\bar{x}_i) \tag{84}$$

$$\bar{x}_{i+1} = \Phi\hat{x}_i + \Gamma u_i \tag{85}$$

Substituting expression (85) into (84), we get

$$\hat{x}_i = P\hat{x}_{i-1} + \Gamma u_{i-1} + K[z_i - H(\Phi\hat{x}_{i-1} + \Gamma u_{i-1})] =$$

$$= \Phi\hat{x}_{i-1} + \Gamma u_{i-1} + Kz_i - KH\Phi\hat{x}_{i-1} - KH\Gamma u_{i-1} =$$

$$= [\Phi - KH\Phi]\hat{x}_{i-1} + [\Gamma - KH\Gamma]u_{i-1} + Kz_i$$

In other words, if for estimation we take into account

$$\hat{x}_{i+1} = [\Phi - KH\Phi]\hat{x}_i + [\Gamma - KH\Gamma]u_i + Kz_{i+1}, \ \ \hat{x}_0 = 0 \tag{86}$$

As well as (77) and (71) we get

$$x_{i+1} = \Phi x_i - \Gamma C\hat{x}_i \tag{87}$$

And from (86)

$$\begin{aligned} \hat{x}_{i+1} &= [\Phi - KH\Phi]\hat{x}_i + (\Gamma - KH\Gamma)u_i + K[Hx_{i+1} + v_{i+1}] = \\ &= [\Phi - KH\Phi]\hat{x}_i + [\Gamma - KH\Gamma]u_i + KH[\Phi x_i + \Gamma u_i + w_i] + \\ &+ Kv_{i+1} = [\Phi - KH\Phi]\hat{x}_i + \Gamma u_i + KH\Phi x_i + KH\omega_i + Kv_{i+1} = \\ &= [\Phi - KH\Phi]\hat{x}_i + KH\Phi x_i + KH\omega_i + Kv_{i+1} \end{aligned} \tag{88}$$

Then let us rewrite (71) (78) as follows:

$$x_{i+1} = \Phi x_i - \Gamma C\hat{x}_i + \omega_i \tag{89}$$

$$\hat{x}_{i+1} = [\Phi - KH\Phi - \Gamma C]\hat{x}_i + KH\Phi x_i + KH\omega_i + Kv_{i+1} \tag{90}$$

and from here we obtain the closed system of difference equations:

$$\begin{cases} x_{i+1} - \Phi x_i + \Gamma C\hat{x}_i = \omega_i \\ \hat{x}_{i+1} - [\Phi - KH\Phi - \Gamma C]x_i - KH\Phi x_i = KH\omega_i + Kv_{i+1} \end{cases} \tag{91}$$

In this case $\Delta(z)$ characteristic determinant of the closed system (91) will be in the following form:

$$\begin{aligned} \Delta(z) &= det\begin{bmatrix} Ez - \Phi & \Gamma C \\ -KH\Phi & Ez - (\Phi - KH\Phi - \Gamma C) \end{bmatrix} = \\ det&\begin{bmatrix} Ez - \Phi + \Gamma C & \Gamma C \\ Ez - \Phi + \Gamma C & Ez - \Phi + KH\Phi + \Gamma C \end{bmatrix} = \\ &= det\begin{bmatrix} Ez - \Phi + \Gamma C & \Gamma C \\ 0 & Ez - \Phi + KH\Phi \end{bmatrix} = \\ &= det(Ez - \Phi + \Gamma C) \cdot det(Ez - \Phi + KH\Phi) \end{aligned} \tag{92}$$

Here, the main goal is to find C and K such that the closed system is asymptotically stable. In other words, the roots z of the equation $\Delta(z) = 0$ must lie inside the unit circle.

Thus, the following algorithm is proposed.

Algorithm.

1. The matrices Φ, Γ, H,Q, R, A,B, N are formed.
2. The matrix algebraic Riccati equation (MCRT) in (79) is solved, and its positive-definite solution matrix is found.

3. The MCRT in (83) is solved, and its positive-definite solution matrix is found.
4. From (78), the matrix C and from (80) the matrix K are formed.
5. From (86), the optimal estimation of $\hat{x}_i$ is determined, and from (77), the optimal regulator u_i is defined.

As seen from the algorithm, the main step in this method is to find the positive-definite solutions of the MCRT, for which there are numerous methods.

After that, using the above-mentioned algorithm for constructing optimal regulators and filters for the steady-state discrete linear-quadratic Gauss problem, a control algorithm for the flight of quadcopters based on data obtained from the GPS satellite navigation system has been developed.

CONCLUSION

The presented dissertation work is devoted to the development of effective methods and corresponding algorithms for the control of certain mechanical systems.

As a result of the research carried out in the dissertation, the following main findings have been obtained:

1. The mathematical model of the gas-lift process has been improved using a systems approach[1,2,5,6,7,9,11,13];
2. A delayed-argument optimal control problem describing the operation process by the gas-lift method has been formulated and studied[2,3,6,9,11,13];
3. Numerical methods for solving delayed-argument differential equations have been developed[3,4,5];
4. A numerical solution method for the continuous delayed-argument optimal control problem has been developed[3];
5. A solution method for the discrete delayed-argument optimal control problem has been developed[13];
6. An optimal controller for controlling the motion of a quadcopter in space has been developed[8,10];

7. An inertial navigation system for quadcopters has been developed[8,10];
8. An algorithm for constructing optimal controllers and filters for the discrete linear–quadratic Gaussian (LQG) problem in steady-state has been developed[12];
9. The solution to the linear–quadratic Gaussian (LQG) problem has been applied to the regulation of a quadcopter's motion in space[8,10,12].

The results obtained in the dissertation can be applied in solving delayed-argument optimal control problems, in solving continuous and discrete delayed-argument optimal control problems, in improving the exploitation process of oil wells by the gas-lift method, as well as in designing optimal controllers for regulating the motion of a quadcopter in space and developing inertial navigation systems for unmanned aerial vehicles.

The following scientific articles and theses have been published on the main scientific results of the dissertation:

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